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EQUATIONS AUX DERIVEES PARTIELLES

# LARGE TIME BEHAVIOUR OF MOMENTS OF SOLUTION OF PARABOLIC DIFFERENTIAL EQUATIONS WITH RANDOM COEFFICIENTS

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## I. Introduction

In this paper we discuss some results on the large time asymptotics of the fundamental solution $K(t,x,y), t \geq 0, x,y \in \mathbf{R}^d$, of the equation

$$\frac{\partial K}{\partial t} = \mathcal{D}\Delta_x K - V(t,x)K \ , \ K|_{t=0} = \delta(x-y) \tag{1}$$

where the function $V(x)$ (potential) is a random ergodic field in $\mathbf{R}^d$. For time independent $V$ these results were mainly obtained in the context of the spectral theory of random Schrödinger operator

$$H = -\mathcal{D}\Delta + V \ , \tag{2}$$

acting in $L^2(\mathbf{R}^d)$ (see book [1] and referencies therein). They, however, can also be interpreted in the terms of diffusion in random media. Namely, random nonnegative $V(x)$ models traps randomly distributed in the medium, nonpositive $V(x)$ models sources of diffusing particles and $V(x)$ which assumes values of both signes can be used to describe, for instance, the evolution of biological species in a random environments both of nutrients and inhibitors (see e.g. [2]).

## II. First Moment

Consider $E\{K(t,x,y)\}$ where the symbol $E\{.\}$ denotes the expectation with respect to the probability measure generated by $V(x)$. Due to the ergodicity of this random field the expectation depends only on $x-y$. Thus we can write that

$$E\{K(t,x,y)\} = Q(t,x-y) \tag{3}$$

In particular,

$$E\{K(t,x,x)\} = Q(t,0) \equiv Q(t) \tag{4}$$

We derive now an upper and a lower bound for $Q(t)$.

Upper bound.

By the Trotter formula for $x,y$ varying in any compact domain of $\mathbf{R}^d$

$$K(t,x,y) = (e^{-tH})(x,y) = \lim_{n\to\infty}((e^{\frac{t}{n}D\Delta}e^{\frac{-tV}{n}})^n)(x,y)$$

or

$$K(t,x,y) = \lim_{n\to\infty}\int k(\frac{t}{n},x-x_1)\cdots k(\frac{t}{n},x_{n-1}-y)e^{-\frac{t}{n}\sum_1^n V(x_i)}dx_1\cdots dx_{n-1} \tag{5}$$

where $x_n = y$ and

$$k(t,x) = (4\pi\mathcal{D}t)^{-d/2}e^{-x^2/4\mathcal{D}t} \tag{6}$$

is the fundamental solution of (1) with $V \equiv 0$.

By the Jensen inequality we have

$$e^{-\frac{t}{n}\sum_1^n V(x_i)} \leq \frac{1}{n}\sum_1^n e^{-tV(x_i)}$$

Now plugging this inequality in (5) and using the ergodicity of $V(x)$ we obtain the upper bound

$$Q(t, x-y) \leq k(t, x-y)F(t) \tag{7}$$

where

$$F(t) = E\{e^{-tV(0)}\} \tag{8}$$

Lower bound.

Since $K(t,x,y)$ is a positive defined kernel

$$K^{1/2}(t,x,x)K^{1/2}(t,y,y) \geq K(t,x,y) \tag{9}$$

Take a nonnegative smooth function $\psi(x)$ with a compact support and unit $L^2$-norm, multiply (9) by $\psi(x)\psi(y)$ and integrate over $x$ and $y$. We obtain

$$Q(t) \geq \|\psi\|_1^{-2}E\{(e^{-tH}\psi,\psi)\} \tag{10}$$

where

$$\|\psi_1\|_1 = \int_{\mathbf{R}^d} \psi dx$$

and we have used the Schwarz inequality in the l .h.s. of (10). By spectral theorem and the Jensen inequality we have

$$(e^{-tH}\psi,\psi) \geq e^{-t(H\psi,\psi)}$$

This inequality and (10) yield the lower bound

$$Q(t) \geq \|\psi\|_1^{-2}e^{-\mathcal{D}t\int_{\mathbf{R}^d}|\nabla\psi|^2dx}\Phi[t\psi^2(.)] \tag{11}$$

where

$$\Phi[\chi(.)] \equiv E\{\exp[-t\int_{\mathbf{R}^d} V(x)\chi(x)dx]\} \tag{12}$$

is known as the characteristic functional of a random field $V(x)$.

Summarizing (7) and (10) we can write the following two side bound for $Q(t)$ :

$$\|\psi\|_1^{-2}e^{-\mathcal{D}t\int_{\mathbf{R}^d}|\nabla\psi|^2dx}\Phi[t\psi^2(.)] \leq Q(t) \leq (4\pi\mathcal{D}t)^{-d/2}\Phi[t\delta(.)] \tag{13}$$

where $\delta(x)$ is the Dirac $\delta$-function.

We discuss now results, that can be obtained basing on these bounds.

(i) Gaussian random field.

This random field is uniquely determined by first two moments. Since $E\{V(x)\}$ does not depend on $x$ we can set it to be zero without loss of generality. Denote

$$E\{V(x)V(y)\} = B(x-y)\ . \tag{14}$$

This is the correlation function of $V(x)$. Since any linear combination of Gaussian random variables is again Gaussian one and since for any such random variable $\xi$ with $E\{\xi\} = 0$

$$E\{e^{\xi}\} = \exp\{\frac{1}{2}E\{\xi^2\}\} \tag{15}$$

we have for (12) :

$$\Phi[\chi(.)] = \exp\{\frac{t^2}{2}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} B(x-y)\chi(x)\chi(y)dxdy\} \tag{16}$$

Thus the r.h.s. of (13) is

$$\exp\{\frac{t^2 B(0)}{2}(1+o(1))\}, t \to +\infty \tag{17}$$

To obtain the asymptoticaly same lower bound we choose

$$\psi(x) = R^{-d/2}\varphi(\frac{x}{R}) \tag{18}$$

where $\varphi(x)$ is a nonegative smooth function with the unit ball as a support and unit $L^2$-norm and

$$R = \text{const.}\ t^{-\alpha} \tag{19}$$

where $0 < \alpha < 1/2$. With this choice of a "trial" function in the l.h.s of (13) and in (16) we obtain finally that for the Gaussian $V(x)$ with $B(x)$ continuous at $x = 0$

$$\log Q(t) = \frac{t^2 B(0)}{2}(1 + 0(1)), t \to \infty \tag{20}$$

If $B(x)$ is $C^2$-smooth at zero we can replace $0(1)$ in (20) by $0(t^{-1/2})$ with $\alpha = 1/4$ in (19).

Thus, the leading term of $\log Q(t)$ is determined by the contribution of the neighbourhood of $x = 0$ having the size $R \sim t^{-1/4}$ while the range of free diffusion is known to be $t^{1/2}$. In other words, the mean concentration of diffusing particles in the random environment is controlled by a strongest source in the $t^{-1/4}$-neighbourhood of origin.

(ii) Poisson random field.

This is the random field of the form

$$V(x) = \sum_i u(x - x_j) \tag{21}$$

where $u(x)$ is a smooth integrable function and $\{x_i\}$ are Poisson points in $\mathbf{R}^d$.

This means that if $\nu(\Delta) = \#\{x_j \in \Delta\}$ for a Borel set $\Delta$ in on $\mathbf{R}^d$, then for any $k$ $\nu(\Delta_1), \cdots, \nu(\Delta_k)$ are independent random variables if $\Delta_1, \cdots, \Delta_k$ do not intersect and

$$Pr\{\nu(\Delta) = n\} = e^{-\rho|\Delta|}\frac{(\rho|\Delta|)^n}{n!}$$

where $\rho$ is the density of Poisson points. This random function models chaotically distributed traps ($u \geq 0$) or sources ($u \leq 0$) in the diffusion problem and the random potential generated by repulsive ($u \geq 0$) or attractive ($u \leq 0$) impurities in the spectral problem.

The characteristic functional here is Γ

$$\Phi[\chi] = \exp\{\rho \int_{\mathbf{R}^d} (e^{\int_{\mathbf{R}^d} u(x-y)\chi(y)dy} - 1)dx \tag{22}$$

and by using similar "trial" function (18) in (13) we obtain that

(a) if $\min_{x\in\mathbf{R}^d} u(x) \equiv u(0) < 0$

$$\log Q(t) = \rho(\frac{2\pi}{t})^{d/2}(\det h)^{1/2}e^{t|u(0)|}(1 + o(1)), t \to +\infty \tag{23}$$

where $h$ is the Hessian of $u(x)$ at $x = 0$ ;

(b) if $u(x) \geq 0$ , $u(x) = \text{const } |x|^{-a}, |x| \to \infty$ $d < a \leq d + 2$

$$\log Q(t) = -C_{a,d}\, t^{d/a}(1 + o(1)), t \to +\infty \tag{24}$$

where $C_{a,d}$ depends on $a$ and $d$ only.

It is easy to check that asymptotic formulae (20), (23) and (24) can be written in following common form

$$\log E\{(e^{-t(-\mathcal{D}\Delta+V)})(0,0)\} = \log E\{e^{-tV(0)}\}(1 + o(1)), t \to +\infty \tag{25}$$

showing that the quantum mechanical kinetic energy $-\mathcal{D}\Delta$ does not contribute to the leading term of $\log Q(t)$ for $t \to +\infty$. Thus we call these asymptotic formulae classical.

An example of the quantum asymptotic formula in which both operators $-\mathcal{D}\Delta$ and $V$ give contribution of the same order of magnitude provide (21) with nonnegative fastly decreasing $u(x)$, such that $u(x) \leq \text{const}|x|^{-a}, |x| \to \infty$ , $a \geq d+2$. In this case the best lower bound can be obtained from (11) with the trial function of the form (18) in which $R = \text{const. } t^{1/d+2}, t \to +\infty$. This lower bound is

$$\log Q(t) \geq -\ell(t)(1 + 0(1)), t \to +\infty \tag{26}$$

where

$$\ell(t) = (\frac{\mathcal{D}t\gamma_d}{d})^{\frac{d}{d+2}}(\frac{\rho\omega_d}{2})^{\frac{2}{d+2}}(d+2), \tag{27}$$

$\gamma_d$ is the lowest eigenvalue of the Dirichlet problem for $-\Delta$ in the unit ball of $\mathbf{R}^d$ and $\omega_d$ is the volume of this ball.

However the upper bound (7) is not precise enough to coincide asymptotically with (26), (27) for $t \to +\infty$. Corresponding upper bound was obtained by Donsker and Varadhan [3] as the result of their deep study of large deviations of the Wiener process. The connection of our problem with the Wiener process is provided by the Feynman-Kac formula [4] for $K(t, x, y)$ :

$$K(t, x, y) = k(t, x - y)W_{0,x}^{t,y}\{e^{-\int_0^t V(x(s))ds}\} \tag{28}$$

where $W_{0,x}^{t,y}$ denotes the conditional expectation with respect to the Wiener measure, concentrated on the set of trajectories $x(s), 0 \leq s \leq t$ , $x(0) = x$ , $x(t) = y$.

Another example of quantum asymptotic corresponds to (1) for $d = 1$ in which $V(x)$ is the Gaussian white noise, i.e. generalized Gaussian process with $B(x) = B_0\delta(x)$. In this case the asymptoticaly exact lower bound is given again by (11) but respective upper bound is to be extracted from some results of the spectral theory of the Schrödinger operator with the same potential (see [5]). The final result is (cf.(20))

$$\log Q(t) = \frac{\mathcal{D}t^3}{48}(1 + o(1)), t \to +\infty \tag{29}$$

We mention also one more quantum asymptotic for the discrete analog of (1), in which $x \in \mathbf{Z}^d, \Delta$ is the finite-difference Laplacian and $V(x)$ are independent identically distributed and nonnegative random variables. By using the same strategy as in obtaining (29), i.e. (11) for a lower bound and spectral theory for an upper bound [1] we can obtain that if

$$\lim_{t\to\infty} \frac{\log(-\log F(t))}{\log t} = a \geq 0 \ ,$$

then

$$\lim_{t\to\infty} \frac{\log(-\log Q(t))}{\log t} = \frac{d+2a}{d+2a+2} \tag{30}$$

By the way, the classical asymptotics in the discrete case can be obtained very easily. Indeed, since for the discrete Laplacian $\|\Delta\| = 2d < \infty$, we have

$$e^{-\mathcal{D}t\|\Delta\|}F(t) \leq E\{(e^{-tH})(0,0)\} \leq e^{\mathcal{D}t\|\Delta\|}F(t)$$

This inequality and the relation $\lim_{t\to\infty} t^{-1}. \log F(t) = \infty$ which is valid for any unbounded (say Gaussian) $V(x)$ imply (25) for all such $V's$

Summarising we note that all known rigorous and nonrigorous (see [1,5]) large time asymptotics for $\log Q(t)$ can be written in the universal form

$$\log Q(t) = -\inf_{\psi\geq 0\, \int_{\mathbf{R}^q} \psi^2 dx = 1} \{\int_{\mathbf{R}^d} |\nabla\psi|^2 dx - \log \Phi[t\psi^2(.)]\} \tag{31}$$

## III. Higher moments.

Since the study of the large time behaviour of higher moments is more complicated we restrict ourselves to the Gaussian $V(x)$, for which the correlation function $B(x)$ is continuous at zero.

Consider

$$Q_\ell(t) \equiv E\{K^\ell(t,0,0)\} \tag{32}$$

By using the Feynman-Kac formula (28) and identity (15) we obtain

$$Q_\ell(t) = (4\pi\mathcal{D}t)^{-\ell d/2} W_1 \cdots W_\ell\{\exp[\frac{1}{2}\sum_{i,j=1}^{\ell} \int_0^t \int_0^t B(x_i(s_1) - x_j(s_2))ds_1 ds_2]\} \tag{32}$$

where $W_i, i = 1, \cdots, \ell$, is the conditional Wiener expectaction with respect to trajectories $x_i(s) x_i(0) = x_i(t) = 0$. In other words (32) is the multiple Wiener integral over loops.

Upper bound.

Since $B(x)$ is a positive defined function, $|B(x)| \leq B(0)$. Using this inequality in (32) we have (cf.(13))

$$Q_e(t) \leq (4\pi\mathcal{D}t)^{-d\ell/2} e^{\frac{t^2\ell^2}{2}} B(0) \tag{33}$$

Lower bound.

Given $\varepsilon > 0$ choose $\delta$ such that $B(x) \geq B(0) - \delta, |x| \leq \varepsilon$. Restrict each Wiener integral in (32) to trajectories $|x_i(s)| \leq \varepsilon$, $0 \leq s \leq t$. Then

$$Q_\ell(t) \geq (4\pi\mathcal{D}t)^{-d\ell/2} e^{\frac{t^2\ell^2}{2}(B(0)-\delta)}(k_\delta(t))^\ell \tag{34}$$

where $k_\delta(t)$ is the Wiener measure of trajectories satisfying the condition $|x(s)| \leq \varepsilon, 0 \leq s \leq t, x(0) = x(t) = 0$.

If $k_\delta(t,x,y)$ is the fundamental solution of the heat equation (eq.(1) with $V \equiv 0$) with the Dirichlet boundary condition, at $|x| = \delta$ then $k_\delta(t) = k_\delta(t,0,0)$, This representation yields the bound $k_\delta(t) \geq C_1 \exp\{-C_2 t/\varepsilon^2\}$ where $C_1$ and $C_2$ are independent of $t$ and $\varepsilon$. Combining the latter bound with (34) and (33) we obtain the asymptotic relation

$$\log Q_\ell(t) = \frac{\ell^2 t^2}{2} B(0)(1 + o(1)), t \to +\infty \tag{35}$$

As in (20), the main contribution in (35) is due to trajectories that "live" all the time $0 \leq s \leq t$ in the small neighbourhood of the origin. Here is an example of moment for which relevant trajectories are different.

$$A(t) = E\{(e^{-tH_+} e^{-tH_-})(0,0)\} = \int E\{K_+(t,0,x)K_-(t,x,0)\}dx \tag{36}$$

where $H_\pm = -\mathcal{D}\Delta \pm V$ , $K_\pm(t,x,y) = (e^{-tH_\pm})(x,y)$.

This quantity arises in the semiconductor physics. According to [6], for the Gaussian $V(x)$ with continuous $B(x)$

$$\log A(t) = t^2[B(0) - B(x_0)](1 + o(1)), t \to \infty \tag{37}$$

where $B(x_0) = \min_{x\in\mathbf{R}^d} B(x)$.

Denote by $\beta_\varepsilon(x)$ the ball of radius $\varepsilon$ centered at $x$. The relevant trajectories for (37) are : $x_1(s)$ seats inside $\beta_\varepsilon(0)$ for $0 \leq s \leq t - \delta$ and jumps from $\beta_\varepsilon(0)$ to $x_0$ during $t - \delta \leq s \leq t$, while $x_2(s)$ seats inside $\beta_\varepsilon(x_0)$ during $0 \leq s \leq t - \delta$ and jumps to 0 during the same period $t - \delta \leq s \leq t$ where $\varepsilon, \delta \to 0$ as $t \to \infty$.

## IV Time dependent coefficients.

Here we demonstrate that for time dependent random $V$ in (1) the large time behaviour of moments of the fundamental solution may be rather different. Subsequent arguments are not completely rigorous but widely accepted in the theoretical physics literature.

We consider the "opposite" case very short correlated in time random Gaussian function $V(t,x)$, in (1) specified by the relations :

$$E\{V(t,x)\} = 0 \ , E\{V(t,x)V(t',x')\} = \delta(t-t')B(x-x') \tag{38}$$

Take the expectation of (1) :

$$\frac{\partial Q}{\partial t} = \mathcal{D}\Delta Q - E\{V(t,x)K(t,x,y)\} \tag{39}$$

Now make use the formula valid for a functional $\mathcal{F}[V(.,.)]$ :

$$E\{V(t,x)\mathcal{F}\} = \int_{-\infty}^{\infty} dt' \int_{\mathbf{R}^d} dx' E\{V(t,x)V(t',x')\}.E\{\frac{\delta\mathcal{F}}{\delta V(t',x')}\}. \tag{40}$$

This formula is a natural continuous analog of the following formula valid for a function $\mathcal{F}(V_1,\cdots,V_n)$ of a family of the Gaussian random variables $V_1,\cdots,V_n, E\{V_i\} = 0$ , $E\{V_iV_k\} = B_{ik}$ :

$$E\{V_i\mathcal{F}\} = \sum_k B_{ik}E\{\frac{\partial\mathcal{F}}{\partial V_k}\}$$

The proof of his formula can be easily obtained by integration by parts.

Using (40) in the second term of the r.h.s. of (39) and taking into account that

$$\frac{\delta K(t,x,y)}{\delta V(t',x')} = \begin{cases} -K(t-t',x,x')K(t',x',y), 0 \le t' \le t \\ 0 \ , \ t' > t \end{cases} \tag{41}$$

we obtain the closed equation for $Q(t,x-y)$ :

$$\frac{\partial Q}{\partial t} = \mathcal{D}\Delta Q + B(0)Q \ , \ Q(0,x) = \delta(x)$$

Thus

$$Q(t,0) = (4\pi\mathcal{D}t)^{-d/2}e^{tB(0)}$$

Comparing this expression with (6) we conclude that for the $\delta$-correlated in time Gaussian $V$ defined by (38) the growth of $E\{K(t,x,x)\}$ is much slower than for time independent Gaussian $V$, defined by (14).

Similar arguments can be applied to

$$Q_\ell(t,x_1,\cdots x_\ell,y_1,\cdots,y_\ell) \equiv E\{\Pi_{i=1}^{\ell}K(t,x_i,y_i)\} \tag{42}$$

Namely, by using (40) and (41) we can obtain for this function the closed equation

$$\frac{\partial Q}{\partial t} = \mathcal{D}\sum_{i=1}^{\ell}\Delta_{x_i}Q + \sum_{i,j=1}^{\ell} B(x_i - x_j)Q = H_\ell Q$$

where

$$H_\ell = -\mathcal{D}\sum_{i=1}^{\ell}\Delta_i - \frac{1}{2}\sum_{i,j=1}^{\ell} B(x_i - x_j)$$

is the $\ell$-body Schrödinger operator with the pair interaction potential $-B(x)$.

Thus the large time behavior of (42) as determined by the lowest eigenvalue $E_\ell$ of his operator. Applying the variational principle one can show that

$$-\frac{\ell^2}{2}(B(0)-\delta) \geq E_\ell \geq -\frac{\ell^2}{2}B(0)$$

where $\delta = o(1), \ell \to \infty$.

Thus, for large $t$ and $\ell$ the behaviour of $Q_\ell$ is given by relation

$$\log Q_\ell = \frac{t\ell^2}{2}B(0)(1+o(1), t \to \infty \ , \ell \to \infty$$

which is apparently, different from (35).

The same relation can also be obtained in the framework of the Wiener integral technique.

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