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EQUATIONS AUX DERIVEES PARTIELLES

THE DISTRIBUTION OF EIGENVALUES OF PARTIAL DIFFERENTIAL OPERATORS

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THE DISTRIBUTION OF EIGENVALUES OF PARTIAL DIFFERENTIAL OPERATORS

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§1. Main results

In this work we consider the asymptotic behavior of eigenvalues of scalar selfadjoint differential operators acting on compact manifolds.

1. Let us introduce the necessary notation.

Let M be a compact n-dimensional ($n \geq 2$) C^∞-manifold (with boundary Γ or without boundary); without loss of generality, M is assumed to be connected (otherwise the problem decomposes into a finite number of independent subproblems). By T^*M we denote the cotangent bundle on M. Local coordinates on M are denoted by $x = (x_1, x_2, \dots, x_n)$ or by $y = (y_1, y_2, \dots, y_n)$; covectors over the point x are denoted by $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, covectors over the point y are denoted by $\eta = (\eta_1, \eta_2, \dots, \eta_n)$. Then (x, ξ) or (y, η) can be considered as local coordinates on T^*M. In a neighborhood of Γ we will use only special coordinate systems of the type $x = (x', x_n)$, $\xi = (\xi', \xi_n)$, where $x' = (x_1, x_2, \dots, x_{n-1})$ are coordinates on Γ (the boundary coordinates), $x_n \geq 0$ is the "normal" coordinate, $\Gamma = \{x_n = 0\}$, $\xi' = (\xi_1, \xi_2, \dots, \xi_{n-1})$, and ξ_n is the conormal component of the covector ξ.

Let A be a formally selfadjoint elliptic differential operator on M of order $2m$ with complex C^∞-coefficients acting on half-densities. If M is a manifold with boundary, then the operator A will be considered together with some regularly elliptic formally selfadjoint boundary conditions. Formal selfadjointness of the operator and of the boundary conditions means that $(Au, v) = (u, Av)$ for all C^∞-"functions" (more precisely, half-densities) $u(x), v(x)$ satisfying the boundary conditions; here

$$(u, v) = \int_M u(x)\overline{v(x)}\, dx$$

is the scalar product in the space $L_2(M)$, $dx = dx_1 dx_2 \dots dx_n$. The definition of regular ellipticity can be found in [9].

For simplicity we also assume that A is positive, i.e., $(Au, u) > 0$ for all C^∞-"functions" $u \not\equiv 0$ satisfying the boundary conditions.

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Under the above-mentioned assumptions the differential operator A generates a positive selfadjoint unbounded operator $\mathcal{A}$ in $L_2(M)$ with a discrete spectrum accumulating to $+\infty$. Eigenvalues ν of this operator can be found from the differential equation

$$Au(x) = \nu u(x), \tag{1.1}$$

with the boundary conditions (in the case of a manifold with boundary)

$$(B_j u(x))_\Gamma = 0, \qquad j = 1, 2, \ldots, m, \tag{1.2}$$

where the B_j are "boundary" differential operators. Let us enumerate the eigenvalues of the problem (1.1) or (1.1), (1.2), taking account of multiplicity: $0 < \nu_1 \leq \nu_2 \leq \cdots \leq \nu_k \leq \cdots$. To simplify the notation of the spectral asymptotic formulas, we will use below $\lambda_k = \nu_k^{1/(2m)}$ instead of ν_k (λ_k are the eigenvalues of the operator $\mathcal{A}^{1/(2m)}$).

In physical and mechanical applications the mathematical problem (1.1) or (1.1), (1.2) usually describes free oscillations of some system (an elastic body, a resonator, etc.), with ν being the frequency parameter proportional to some power of the natural frequency. Selfadjointness of the problem means conservation of the full energy in the oscillating system, and positivity means stability (i.e., absence of movements with the amplitude exponentially growing in time).

The number of eigenvalues λ_k smaller than a given λ is called the *eigenvalue distribution function*:

$$N(\lambda) = \{k : \lambda_k < \lambda\}.$$

The aim of this paper is to describe the asymptotic behavior of $N(\lambda)$ as $\lambda \to +\infty$.

2. Let $A_{2m}(x, \xi)$ denote the principal symbol of the differential operator A. Let $S^*M = \{(x, \xi) : A_{2m}(x, \xi) = 1\}$. There exists a standard symplectic measure on T^*M with an element written in local coordinates as $dx\, d\xi = dx_1 \cdots dx_n\, d\xi_1 \cdots d\xi_n$. This measure generates a standard measure on S^*M with the element $dx\, d\tilde{\xi}$ satisfying the equality $dx\, d\xi = (2m)^{-1}\, dx\, d\tilde{\xi}\, dA_{2m}$.

The principal symbol of the operator determines the first term in the asymptotics of the eigenvalue distribution function.

Theorem 1.1 ([13], [17], [15], [3]).

$$N(\lambda) = c_0 \lambda^n + O(\lambda^{n-1}), \qquad \lambda \to +\infty, \tag{1.3}$$

where

$$\begin{aligned} c_0 &= (2\pi)^{-n} \operatorname{vol}\{(x, \xi) : A_{2m}(x, \xi) \leq 1\} \\ &= (2\pi)^{-n} \int_{\{A_{2m} \leq 1\}} dx\, d\xi. \end{aligned} \tag{1.4}$$

The estimate of the remainder term in (1.3) is precise. This means that O cannot be replaced by o without some additional assumptions about the operator. Moreover, we will see that the refined asymptotic formula (for example, for a manifold with boundary) contains a second term of order λ^{n-1}.

3. Singling out the second term in the asymptotics of $N(\lambda)$ requires the study of some global geometric characteristics of the problem.

First let M be a manifold without boundary. We will consider the Hamiltonian system on $T^*M\backslash 0$,

$$\dot{x} = \partial h/\partial \xi, \qquad \dot{\xi} = -\partial h/\partial x, \tag{1.5}$$

where $h(x,\xi) = (A_{2m}(x,\xi))^{1/(2m)} > 0$ and the dot denotes differentiation with respect to time t. The system (1.5) generates the Hamiltonian flow Φ^t in the phase space T^*M. Trajectories of this flow are also called *bicharacteristics* of the operator A. We will denote the trajectory starting at a point (y,η) by $(x^t(y,\eta),\xi^t(y,\eta))$. Let us call the trajectory *T-periodic* ($T \neq 0$) if $(x^T(y,\eta),\xi^T(y,\eta)) = (y,\eta)$, and *absolutely T-periodic* if in some coordinate system (and, consequently, in any coordinate system) the quantity

$$|x^T(\overline{y},\overline{\eta}) - \overline{y}|^2 + |\xi^T(\overline{y},\overline{\eta}) - \overline{\eta}|^2,$$

considered as a function of $(\overline{y},\overline{\eta})$, has a zero of infinite order at the point (y,η). Note that in terms of geometric optics absolute periodicity means infinite order focusing of rays returning to a neighborhood of the initial point. In the analytic case absolute periodicity means that not only the trajectory itself is T-periodic, but all the sufficiently close trajectories are also T-periodic and with the same period T. Of course, the existence of an absolutely periodic trajectory is a rather rare event, usually connected with a very strong (obvious or hidden) symmetry of the problem.

Now let A be a second-order operator on a manifold with boundary. We will reflect trajectories from the boundary according to the law of geometric optics: the angle of incidence equals the angle of reflection (the angle between curves which intersect at a point x_0 is calculated in the local coordinate system in which $A_2(x_0,\xi) = |\xi|^2$). The arising construction is called a *geodesic billiards*, and the trajectories are called *billiard trajectories*. Note that a trajectory cannot always be reflected according to this rule (in the case when the angle of incidence equals zero). Moreover, it can happen that in a finite time a billiard trajectory experiences an infinite number of reflections from the boundary, hence it cannot be defined for all t. In the first case we call the trajectory *tangent*, in the second case we call the trajectory *dead-end*. It can be shown [7] that the measure of the set of points $(y,\eta) \in T^*M$, which are the starting points of tangent or dead-end trajectories, equals zero; thus the billiard flow Φ^t is naturally defined on a subset of full measure in T^*M.

For higher-order operators the reflection law does not allow a simple geometric interpretation; moreover, a reflected trajectory is determined, generally speaking, ambiguously. If the trajectory intersects with the boundary at a point (x,ξ), then we call *reflected* those trajectories that emanate from any point $(x,\overline{\xi})$ such that $h(x,\xi) = h(x,\overline{\xi})$ and the covector $\overline{\xi} - \xi$ is conormal to Γ at the point x. In other words, the reflection law is determined by the conditions of continuity of x and ξ' and by the condition of preservation of the value of the Hamiltonian h. It is clear that $\overline{\xi}$ is determined from the above-mentioned conditions, generally speaking,

ambiguously: several ($\leq m$) reflected trajectories can correspond to one incident trajectory. This construction is called the *branching Hamiltonian billiards*. We will still denote its trajectories by $(x^t(y,\eta), \xi^t(y,\eta))$ (we assume that some trajectory is chosen from all the trajectories outgoing from the point (y,η)). The concepts of periodic, absolutely periodic, and dead-end trajectories are naturally introduced for the branching Hamiltonian billiards. We will call a trajectory *tangent* if the vector $\dot{x} = \partial h/\partial \xi$ is tangent to Γ at the point of incidence or the point of reflection. We will call the point $(y,\eta) \in T^*M$ *periodic* (*absolutely periodic, dead-end, tangent*) if at least one periodic

(absolutely periodic, dead-end, tangent) billiard trajectory originates from this point. It can be shown analogously [7] that the measure of the set of tangent points is always equal to zero. However, the measure of the set of dead-end points of a branching Hamiltonian billiards can differ from zero [5].

Theorem 1.2 ([12], [6], [16], [3]). *If the measure of the set of absolutely periodic and dead-end points of the branching Hamiltonian billiards (or the flow Φ^t) equals zero, then*

$$N(\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + o(\lambda^{n-1}), \qquad \lambda \to +\infty\,. \tag{1.6}$$

In Theorem 1.2 the constant c_1 takes into account the boundary conditions and is expressed as some integral over $T^*\Gamma$; for details see §2. Note that for a manifold without boundary $c_1 = 0$.

Numerical examples, which illustrate the high effectiveness of the two-term asymptotic formula (1.6), can be found in [3], [4].

4. There exist different sets of sufficient conditions for Theorem 1.2 [1], [3, §1.2], [5]. Here we describe one of them.

Definition 1.3. We will say that a Hamiltonian $h(x,\xi)$ satisfies the *simple reflection condition* if only one reflected trajectory corresponds to every trajectory arriving at the boundary, i.e., for any point $(x',\xi') \in T^*\Gamma\backslash 0$ the equation $\partial h(x',0,\xi',\xi_n)/\partial\xi_n = 0$ has only one real root $\xi_n = \xi_*(x',\xi')$. If this root is simple, then we will say that the Hamiltonian satisfies the *strong simple reflection condition*.

Under the simple reflection condition the billiard flow Φ^t is defined on a subset of full measure in T^*M.

Definition 1.4. We will say that the manifold M is *Hamilton convex* if $\Gamma \neq \varnothing$, the strong simple reflection condition is fulfilled, and

$$\{\partial h/\partial\xi_n, h\}_{x=(x',0),\xi=(\xi',\xi_*(x',\xi'))} \geq 0, \quad \not\equiv 0$$

on $T^*\Gamma\backslash 0$.

Here $\{f,g\} = f'_\xi g'_x - f'_x g'_\xi$ is the Poisson bracket.

Example 1.5. If M is a domain in $\mathbb{R}^n$ and $A_{2m}(x,\xi) = |\xi|^{2m}$, then Hamilton convexity is equivalent to convexity in the usual sense.

The above-mentioned set of sufficient conditions is contained in the following assertions.

Proposition 1.6. *If the simple reflection condition is fulfilled, then the measure of the set of dead-end points is equal to zero.*

Thus, under the simple reflection condition the billiard flow Φ^t is defined on a subset of full measure in T^*M.

Proposition 1.7. *If the problem is analytic and the manifold M is Hamilton convex, then the measure of the set of absolutely periodic points is equal to zero.*

Example 1.8. Let M be a convex domain in $\mathbb{R}^n$ with an analytic boundary and $A_{2m}(x,\xi) = |\xi|^{2m}$. Then (1.6) is valid.

Example 1.9. Let M be a part of the sphere $S^n \subset \mathbb{R}^{n+1}$, cut off by a plane in $\mathbb{R}^{n+1}$. Let A be a power of the Laplacian, corresponding to the standard metric on the sphere. If M is strictly less than a hemisphere, then M is Hamilton convex and (1.6) is valid. If M is greater than or equal to a hemisphere, then M is not Hamilton convex and one cannot guarantee (1.6). Moreover, it can be shown that, if M is strictly greater than a hemisphere, then the formula (1.6) is necessarily false. In the intermediate case, when M is a hemisphere, the validity of (1.6) probably depends on the boundary conditions.

5. Let us discuss the realtionship between the concepts of periodicity and absolute periodicity. Let Π_T be the set of T-periodic points in S^*M, let $\Pi_T^a \subset \Pi_T$ be the set of absolutely T-periodic points, and let $\Pi = \bigcup\limits_{T\neq 0} \Pi_T$, $\Pi^a = \bigcup\limits_{T\neq 0} \Pi_T^a$. ¿From the point of view of applications, it is more convenient to deal with absolutely periodic points (see [3] and subsection 4 above) because in analytic problems absolute periodicity means existence of an open set of periodic trajectories, and therefore Π^a is organized simpler than Π is. This fact simplifies the formulation of effective sufficient conditions. The set of periodic trajectories (as shown by simplest examples) is significantly larger and more complicated. Nevertheless, in some works (for example, by V. Ya. Ivriĭ) a condition is imposed on the measure of the set of periodic trajectories ($\operatorname{vol}\Pi = 0$). Such an approach allows one to simplify somewhat the proofs of the main results. It is remarkable that there exists a simple relationship between the sets of periodic points and absolutely periodic points, which is established (if we assume that the measure of the set of dead-end points equals zero) by the following

Proposition 1.10 [5]. $\operatorname{vol} \bigcup\limits_{T\neq 0} (\Pi_T \backslash \Pi_T^a) = 0$, *and consequently* $\operatorname{vol}(\Pi\backslash\Pi^a) = 0$.

Sets of measure zero in S^*M (or in T^*M) do not influence the first and second terms of the asymptotic expansion of $N(\lambda)$. Therefore, by virtue of Proposition 1.10, it makes no difference which set (Π or Π^a) is used.

§2. The coefficient c_1

In this section we describe how to compute effectively the coefficient c_1 from the asymptotics (1.6).

Let us retain in the operators A, B_j only the terms with top order differentiations, replace all the differentiations along the boundary $D_{x'}$ $(= -i\partial/\partial x')$ by ξ', and set $x = (x', 0)$ in the coefficients. We denote the resulting operators by A^Γ, B_j^Γ. These operators depend on the point $(x', \xi') \in T^*\Gamma\backslash 0$ as on a parameter.

Substituting the operators A^Γ, B_j^Γ for A, B_j in the original partial-differential spectral problem (1.1), (1.2), we arrive at a one-dimensional selfadjoint positive spectral problem on the half line $0 \le x_n < +\infty$: an ordinary differential equation with constant coefficients

$$A^\Gamma u(x_n) = \nu u(x_n) \tag{2.1}$$

with boundary conditions at zero

$$(B_j^\Gamma u(x_n))_{x_n=0} = 0, \qquad j = 1, 2, \dots, m\,. \tag{2.2}$$

For convenience, in this part of the paper we use the original spectral parameter $\nu = \lambda^{2m}$.

Let ν be a positive real number. Let us denote

$$\operatorname{trace}(\nu) = (-2\pi i)^{-1} \int_{L(\nu)} \left(\int_0^{+\infty} (r_B(x_n, x_n, \kappa) - r(x_n, x_n, \kappa))\, dx_n \right) d\kappa, \tag{2.3}$$

where $r_B(x_n, y_n, \kappa)$ and $r(x_n, y_n, \kappa)$ are the integral kernels of the resolvent $(\mathcal{A}^\Gamma - \kappa I)^{-1}$ of the problem (2.1), (2.2) on the half line $0 \le x_n < +\infty$ and the problem (2.1) on the whole axis $-\infty < x_n < +\infty$ (without boundary conditions at zero), respectively, and $L(\nu)$ is an oriented smooth curve in the complex κ-plane with a cut along the nonnegative real axis; $L(\nu)$ connects the points $\nu + i0$ and $\nu - i0$ moving counterclockwise. Completing the definition of the function $\operatorname{trace}(\nu)$ at possible discontinuity points by left continuity, let us call this function the *regularized trace of the spectral projection* of the problem (2.1), (2.2). We will also use the notation $\operatorname{trace}(x', \xi', \nu) \equiv \operatorname{trace}(\nu)$, recalling the dependence of the problem (2.1), (2.2) on the point $(x', \xi') \in T^*\Gamma\backslash 0$ as on a parameter.

The coefficient c_1 in the asymptotics (1.6) is defined by the formula

$$c_1 = (2\pi)^{1-n} \int_{T^*\Gamma} \operatorname{trace}(x', \xi', 1)\, dx'\, d\xi'\,. \tag{2.4}$$

The inequality

$$|\operatorname{trace}(x', \xi', \nu)| \le m \tag{2.5}$$

is proved ([3, §2.1]) from variational considerations. By virtue of the positiveness of the spectral problem (2.1), (2.2) we also have

$$\operatorname{trace}(x', \xi', \nu) = 0 \quad \text{when } |\xi'|^{2m} \ge \text{const}\, \nu\,. \tag{2.6}$$

¿From (2.5), (2.6) we conclude that the integral (2.4) is finite.

In practice, it is not very convenient to calculate the regularized trace trace(ν) by the formula (2.3), because one has to calculate a double integral. However, it turns out that trace(ν) can be calculated without any integration by the formula

$$\text{trace}(\nu) = N_B(\nu) + \varphi_B(\nu)/(2\pi), \tag{2.7}$$

where $N_B(\nu)$ and $\varphi_B(\nu)$ are the eigenvalue distribution function and the scattering phase, respectively, of the problem (2.1), (2.2); the subscript $_B$ emphasizes the dependence of the eigenvalue distribution function and the scattering phase on the boundary conditions (2.2).

That the regularized trace of the spectral projection is expressed in terms of scattering data is a sufficiently well-known fact in mathematical literature; see, e.g., Buslaev's work [18], where a very general case of a partial differential operator with variable coefficients was considered (certainly, the operator acts in an unbounded domain). Our case of the one-dimensional spectral problem with constant coefficients is remarkable only in that all the constructions here are very simple and are actually of a purely algebraic character. For proofs and details see [3, §§1.1 and 4.1].

Let us define the functions $N_B(\nu)$ and $\varphi_B(\nu)$.

A number ν is called an *eigenvalue* of the problem (2.1), (2.2) if for this ν the problem has a solution $u(x_n) \not\equiv 0$ tending to zero as $x_n \to +\infty$; such a solution is called an *eigenfunction*, and the number p of linearly independent eigenfunctions corresponding to a given ν is called the *multiplicity* of the eigenvalue ν.

A number ν is called a *point of the continuous spectrum* of the problem (2.1), (2.2) if for this ν the problem has a solution $u(x_n) \not\equiv 0$ which does not tend to zero as $x_n \to +\infty$, but is majorized by some polynomial in x_n; such a solution is called an *eigenfunction of the continuous spectrum*, and the number q of linearly independent eigenfunctions of the continuous spectrum corresponding to a given ν is called the *multiplicity of the continuous spectrum at the point* ν. The linear independence of a set of eigenfunctions of the continuous spectrum is understood modulo ordinary eigenfunctions, i.e., any nontrivial linear combination of eigenfunctions of the continuous spectrum from the collection indicated must be a function that does not tend to zero as $x_n \to +\infty$.

Note that a point of the continuous spectrum of multiplicity q may at the same time be an eigenvalue of multiplicity p.

The number of eigenvalues of the problem (2.1), (2.2) is finite (as shown in [3, §4.1], it does not exceed $(3m-1)((2m)!)/(2(m!)^2)$). Let us enumerate the eigenvalues in order of their growth taking multiplicities into account, and define the eigenvalue distribution function $N_B(\nu)$ in the standard way as the number of eigenvalues less than a given ν. Here we count both eigenvalues outside the continuous spectrum and those lying on the continuous spectrum.

The continuous spectrum of the problem (2.1), (2.2) fills the semi-infinite interval

$$[\nu_*^{(1)}, +\infty), \quad \text{where } \nu_*^{(1)} = \min A_{2m}(\xi_n), \ -\infty < \xi_n < +\infty$$

($A_{2m}(\xi_n) \equiv A_{2m}(x', 0, \xi', \xi_n)$ is the total symbol of the operator A^Γ). We will call a number ν_* a *singular point of the continuous spectrum* if the equation $A_{2m}(\xi_n) = \nu_*$

has a multiple real ξ_n-root; as will be clear from the following, such points ν_* are "singular" in the sense that the multiplicity q of the continuous spectrum may change at these points. Let us enumerate the singular points of the continuous spectrum:

$$\nu_*^{(1)} < \nu_*^{(2)} < \cdots < \nu_*^{(s)}$$

(certainly, $1 \le s \le 2m-1$). After the removal of the singular points, the continuous spectrum splits into the zones

$$(\nu_*^{(1)}, \nu_*^{(2)}), (\nu_*^{(2)}, \nu_*^{(3)}), \ldots, (\nu_*^{(s-1)}, \nu_*^{(s)}), (\nu_*^{(s)}, +\infty)\,. \tag{2.8}$$

Let ν belong to one of the zones (2.8), and let $2q$ be the number of real ξ_n-roots of the equation $A_{2m}(\xi_n) = \nu$ (certainly, $1 \le q \le m$). Let us enumerate these ξ_n-roots in order of their growth, taking the sign of $A'_{2m} \equiv dA_{2m}/d\xi_n$ into account:

$$\xi_1^-(\nu) < \xi_1^+(\nu) < \xi_2^-(\nu) < \xi_2^+(\nu) < \cdots < \xi_q^-(\nu) < \xi_q^+(\nu), \tag{2.9}$$

$$A_{2m}(\xi_l^-(\nu)) < 0, \quad A_{2m}(\xi_l^+(\nu)) > 0, \qquad l = 1, 2, \ldots, q\,. \tag{2.10}$$

We will look for the eigenfunctions of the continuous spectrum corresponding to the given ν, i.e., for the solutions of the problem (2.1), (2.2) of the type

$$u(x_n) = \sum_{\pm} \sum_{l=1}^{q} \frac{c_l^{\pm} \exp(ix_n \xi_l^{\pm}(\nu))}{(\pm 2\pi A'_{2m}(\xi_l^{\pm}(\nu)))^{1/2}} + \sum_{l=1}^{m-q} d_l v_l(x_n, \nu), \tag{2.11}$$

where $c_l^{\pm}, d_l$ are unknown constants, and $v_l(x_n, \nu)$ are linearly independent solutions of the equation (2.1), which decrease as $x_n \to +\infty$; the normalizing factors $(\pm 2\pi A'_{2m}(\xi_l^{\pm}(\nu)))^{1/2}$ in the first sum in (2.11) are introduced for convenience to make the matrix $S(\nu)$ defined below unitary. It can be proved that the linear space of the eigenfunctions of the continuous spectrum (2.11) is q-dimensional and that each eigenfunction of the continuous spectrum is uniquely determined by the nontrivial set of q constants c_l^- or q constants c_l^+. (Here dimensionality and uniqueness are understood modulo ordinary eigenfunctions. This remark is necessary in the case when the point of the continuous spectrum simultaneously happens to be an eigenvalue.) Thus, the number q (half the number of real ξ_n-roots of the equation $A_{2m}(\xi_n) = 0$) at nonsingular points of the continuous spectrum equals the multiplicity of the continuous spectrum. Finally, it can be proved that the columns of coefficients c_l^+ and c_l^- are related by the proportion

$$\mathbf{c}^+ = S(\nu)\mathbf{c}^-, \tag{2.12}$$

where $S(\nu)$ is a unitary $q \times q$ matrix, regularly dependent on ν in each of the zones (2.8).

The coefficients c_l^- and c_l^+ are called the *complex amplitudes* of waves coming from $x_n = +\infty$ and going to $x_n = +\infty$, respectively. The matrix $S(\nu)$ is called the *scattering matrix*. The motivation of these names is the following. Let us introduce a time factor $\exp(-it\lambda)$, $\lambda = \nu^{1/(2m)} > 0$, into (2.11) and assume that

the coefficients c_l^- and c_l^+ smoothly depend on λ. Let us denote the resulting function by $u(x_n, t, \lambda)$. We will consider the wave packet

$$u(x_n, t) = \int u(x_n, t, \lambda)\rho(\lambda)\, d\lambda \tag{2.13}$$

($\rho(\lambda) \in C_0^\infty$ is a spectral density with a sufficiently small support), which satisfies the nonstationary problem (2.1), (2.2), with D_t^{2m} $(= (-i\partial/\partial t)^{2m})$ substituted for ν ("the generalized wave equation"). Let us take an arbitrary time $|t| \gg 1$, and determine, taking into account the presence of the asymptotic parameter t, which terms of (2.11), and for which x_n, do not vanish asymptotically. Then, calculating the stationary (with respect to λ) points of the phase functions in the standard way, we will obtain that only the terms with $c_l^\pm$ can give an asymptotically nonvanishing contribution and only when the following equations hold,

$$x_n\, d\bar{\xi}_l^{\pm}(\lambda)/\, d\lambda = t, \qquad \bar{\xi}_l^{\pm}(\lambda) \stackrel{\text{def}}{=} \xi_l^{\pm}(\lambda^{2m})\,. \tag{2.14}$$

We have the identity $A_{2m}(\xi_l^\pm(\nu)) = \nu$; extracting the root of the power $2m$ and denoting $h(\xi_n) = (A_{2m}(\xi_n))^{1/(2m)} > 0$ (the same Hamiltonian was introduced in §1, subsection 3), we will obtain an equivalent identity $h(\xi_l^\pm(\lambda)) = \lambda$. Differentiating it with respect to λ, we get

$$h'(\bar{\xi}_l^{\pm}(\lambda))\, d\bar{\xi}_l^{\pm}(\lambda)/\, d\lambda = 1, \tag{2.15}$$

$h' \equiv dh/\, d\xi_n$. By virtue of (2.15), the conditions (2.14) can be rewritten in the final form

$$x_n = h'(\bar{\xi}_l^{\pm}(\lambda))t\,. \tag{2.16}$$

It only remains to note that $A'_{2m} = 2mh^{2m-1}h'$, so that the signs of A'_{2m} and h' coincide. From (2.10) and the obvious inequality $x_n \geq 0$ (we are solving a problem on the half line) it follows that for $t < 0$ the conditions (2.16) are fulfilled only for the terms with the index "minus," and these points move to the left along the x_n half line with time. For $t > 0$ the conditions (2.16) are fulfilled only for the terms with the index "plus," and these points move to the right along the x_n half line with time. Thus, we are actually dealing with the reflection of components of the wave packet, with h' playing the role of the group velocity.

It can be shown that the scattering matrix $S(\nu)$ is invariant with respect to the choice of local coordinates $x = (x', x_n)$.

The scattering phase $\varphi_B(\nu)$ is the main spectral characteristic of the continuous spectrum. It plays the same role for the continuous spectrum as the eigenvalue distribution function for the discrete spectrum, i.e., modulo the factor $(2\pi)^{-1}$ the function $\varphi_B(\nu)$ is in fact the distribution function of the continuous spectrum. Concretely, for $\nu < \nu_*^{(1)}$ the scattering phase is taken to be zero, and in the zones (2.8) of the continuous spectrum it is an infinitely smooth function, which is expressed in terms of the scattering matrix by the formula

$$\varphi_B(\nu) = \operatorname{Arg}(\det S(\nu)) + \text{const}\,. \tag{2.17}$$

Here const is a real normalizing term. It is constant in each of the zones (2.8), but, generally speaking, is different for different zones.

The missing normalization is given by a formula for the scattering-phase jumps at the singular points of the continuous spectrum. We will write down this formula for the case of a simple singular point $\nu_*^{(k)}$, i.e., when the equation $A_{2m}(\xi_n) = \nu_*^{(k)}$ has only one real multiple root $\xi_n = \xi_*$ and this root is a double one (it is the general situation). In this case

$$\varphi_B(\nu_*^{(k)} + 0) - \varphi_B(\nu_*^{(k)} - 0) = \pm\pi/2, \tag{2.18}$$

i.e., a simple singular point of the continuous spectrum gives a jump of the regularized trace of the spectral projection equal to 1/4 of the jump given by an eigenvalue (see (2.7)), only with an undefined sign. In (2.18) the plus or minus is taken according to whether the singular point $\nu_*^{(k)}$ is soft or rigid, respectively. We say that the point $\nu_*^{(k)}$ is *soft* if for $\nu = \nu_*^{(k)}$ the problem (2.1), (2.2) has a solution of the form

$$u(x_n) = \exp(ix_n\xi_*) + v(x_n), \quad \text{where } v(x_n) \to 0 \text{ as } x_n \to +\infty,$$

and *rigid* if it has no solution of this form. The choice of such a terminology is connected with the classical equation of membrane oscillations $-\Delta u = \lambda^2 u$. Here the boundary condition of a fixed edge $u|_\Gamma = 0$ leads to (2.18) with the minus sign, and the boundary condition of a free edge $\partial u/\partial n|_\Gamma = 0$ (the condition of the absence of forces normal to the membrane surface) leads to (2.18) with the plus sign.

An important special case is when the simple reflection condition (see Definition 1.3) is fulfilled. In this case the continuous spectrum has multiplicity 1, and the scattering matrix $S(\nu)$ is of size 1×1, i.e., it is simply the number c^+/c^-. The quantity $\beta(\nu) = \operatorname{Arg}(c^+/c^-)$ is called the phase shift; see §1. The phase shift $\beta(\nu)$ differs from the scattering phase $\varphi_B(\nu)$ by a normalizing constant. If the strong simple reflection condition is satisfied, then

$$\varphi_B(\nu) = \beta(\nu) + \pi/2, \qquad \varphi_B(\nu_*^{(1)}) = \pm\pi/2$$

(see [1]).

Substituting (2.7) into (2.4) we obtain the final formula for the coefficient c_1 in the asymptotics (1.6):

$$c_1 = (2\pi)^{1-n} \int_{T^*\Gamma} \left(N_B(x', \xi', 1) + \frac{\varphi_B(x', \xi', 1)}{2\pi} \right) dx'\, d\xi'. \tag{2.19}$$

Formula (2.19) is very convenient for concrete calculations in applications. On the basis of this formula it proves possible to compute explicitly the coefficient c_1 for all the main problems of the theory of elasticity up to the most complicated problems of shell theory; a summary of results is given in [4].

It is also interesting that the formula (2.19) has a clear mechanical sense. Indeed, for some types of boundary conditions in the original problem (1.1), (1.2) a subseries of eigenvalues can appear, with the eigenfunctions localized in a very small

($x_n \lesssim \lambda^{-1}$, i.e., approximately a wavelength) neighborhood of the boundary Γ. For example, for a three-dimensional elastic body with a free boundary such a subseries is generated by the so-called Rayleigh surface waves. It is clear that the eigenvalues of the auxiliary problem (2.1), (2.2) correspond exactly to such boundary waves. The first term in (2.19), being multiplied by λ^{n-1}, describes asymptotically the distribution of eigenvalues from this subseries. The mechanical meaning of the second term in (2.19) consists, roughly speaking, in the following. The jump being experienced by the phase of a wave at the moment of reflection from the boundary is in fact equivalent to a variation of the distance passed by the wave by a small quantity (of the order of a wavelength). The corresponding small increment of the main term $c_0\lambda^n$ of the asymptotics due to such a displacement of the boundary of the cotangent bundle T^*M (see the integral (1.4)) is described by the second term in (2.19).

It is also interesting that there exists a deep analogy between the formulas (1.4) and (2.19), which determine the coefficients c_0 and c_1 in the asymptotics (1.6). To put it in evidence, we rewrite (1.4) in the equivalent form

$$c_0 = (2\pi)^{-n} \int_{T^*M} N_A(x,\xi,1)\, dx\, d\xi, \tag{2.20}$$

where

$$N_A(x,\xi,\nu) = \begin{cases} 0 & \text{when } \nu \le A_{2m}(x,\xi), \\ 1 & \text{when } \nu > A_{2m}(x,\xi) \end{cases} \tag{2.21}$$

(the subscript $_A$ emphasizes that this function depends only on the differential operator A and is independent of boundary conditions). For fixed x and ξ the function (2.21) is the eigenvalue distribution function of the operator of multiplication by a fixed constant $A_{2m}(x,\xi)$ in $\mathbb{R}^1$, i.e., the eigenvalue distribution function of the (1×1)-matrix $A_{2m}(x,\xi)$. The trivial problem

$$A_{2m}u = \nu u \tag{2.22}$$

has only one eigenvalue $\nu = A_{2m}$, and the distribution function (2.21) can be written down explicitly. The comparison of the formulas (2.20) and (2.19) shows that the coefficients c_0 and c_1 are organized roughly in the same way. In both cases integrals are taken over a cotangent bundle: in (2.20) it is the cotangent bundle over the manifold M, in (2.19) it is the cotangent bundle over its boundary Γ. In both cases the integrand is the eigenvalue distribution function of a certain auxiliary problem. In (2.20) it is the problem (2.22), which comes from the original one as the result of retaining the top order differentiations, freezing the coefficients and taking a formal Fourier transform along all the coordinates without taking the boundary conditions into account. In (2.19) it is the problem (2.1), (2.2), which comes from the original one in the same way, the only difference being that the Fourier transform with respect to the "normal" coordinate x_n is omitted and the boundary conditions are taken into account. The specific characteristic of the formula (2.19) in comparison with (2.20) is that (2.19) includes also the scattering phase $\varphi_B(\nu)$ in addition to the eigenvalue distribution function $N_B(\nu)$. As has been mentioned above, the function

$\varphi_B(\nu)/(2\pi)$ plays the role of the distribution function of the continuous spectrum of the auxiliary problem (2.1), (2.2).

We must note that the expression for the coefficient c_1 has to include, in principle, an additional term with an integral of the subprincipal symbol of the operator A over S^*M (cf. [12]). Actually, we allow coefficients to be complex-valued, thereby the subprincipal symbol of the selfadjoint operator A is, generally speaking, not zero. However, in our case the indicated additional integral turns out to be zero because the subprincipal symbol is odd with respect to ξ (our operator A is differential, not pseudodifferential).

§3. The Levitan wave method

Theorems 1.1 and 1.2 are proved by the *wave equation method* introduced by B. M. Levitan [8] and extensively developed later by L.Hörmander, V.Ivriĭ and other authors. This method is based on the study of singularities of the distribution

$$\sigma(t) = \operatorname{Tr}\exp(-it\mathcal{A}^{1/(2m)}) \equiv \sum_{k=1}^{+\infty}\exp(-it\lambda_k) = \mathcal{F}_{\lambda\to t}(N'), \tag{3.1}$$

(here $\mathcal{F}_{\lambda\to t}(\cdot) = \int_{-\infty}^{+\infty}\exp(-it\lambda)(\cdot)\,d\lambda$ denotes the Fourier transform and $\mathcal{F}^{-1}_{t\to\lambda}(\cdot) = (2\pi)^{-1}\int_{-\infty}^{+\infty}\exp(it\lambda)(\cdot)\,dt$ denotes the inverse Fourier transform). It is clear that the right-hand side in (3.1) is not determined in the usual sense because the unitary exponent

$$\exp(-it\mathcal{A}^{1/(2m)}) = \mathcal{F}_{\lambda\to t}(\delta(\mathcal{A}^{1/(2m)} - \lambda I)) \tag{3.2}$$

is not an operator of the trace class for any t. However, $\sigma(t)$ can be considered as a distribution, if we, for example, set

$$\sigma(t) = \partial_t^p \operatorname{Tr}(-i\mathcal{A}^{1/(2m)})^{-p}\exp(-it\mathcal{A}^{1/(2m)})$$

with sufficiently large p.

The unitary exponent $\exp(-it\mathcal{A}^{1/(2m)})$ is effectively constructed using the theory of Fourier integral operators [13], [14], [10]. Techically this is done by solving the nonstationary equation

$$(D_t^{2m} - A)u = 0 \tag{3.3}$$

with boundary conditions (1.2) (the equation and boundary conditions which the Schwartz kernel of the unitary exponent (3.2) satisfies). With the aid of the theory of Fourier integral operators one can represent the Schwartz kernel of the operator $\exp(-it\mathcal{A}^{1/(2m)})$ outside a small neighborhood of the boundary Γ as a finite sum of oscillatory integrals. Then the singularities of these integrals are calculated. In

a neighborhood of Γ another (less refined) method is used. This method takes into account the smallness of the boundary zone and reduces our analysis to that of an auxiliary problem with "frozen" coefficients. As a result we obtain an effective description of the singularities of $\sigma(t)$.

Knowing the singularities, it is easy to reconstruct the asymptotics of the functions

$$\mathcal{F}^{-1}_{t\to\lambda}(\hat{\rho}\sigma) = (\rho * N')(\lambda)$$

and

$$(\rho * N)(\lambda) = \int_{-\infty}^{\lambda} (\rho * N')(\tau)\, d\tau,$$

where $\hat{\rho} \in C_0^\infty$. Then one can retrieve the asymptotics of $N(\lambda)$ with the help of the Tauberian theorem 3.1 formulated below (which is close in essence to the Tauberian theorem used implicitly in [12]).

Let us denote by ρ a nonnegative even function on $\mathbb{R}^1$, which possesses the following properties: $\rho(\lambda) \in S(\mathbb{R}^1)$, $\hat{\rho}(t) = \mathcal{F}_{\lambda\to t}(\rho) \in C_0^\infty(\mathbb{R}^1)$, $\rho(\lambda) > 0$ for $|\lambda| \le 1$, $\operatorname{supp}\hat{\rho} \subset [-1,1]$ and is sufficiently small, $\hat{\rho}(0) = 1$ (such a function exists [11], [12], [13]).

Theorem 3.1. *Let $N(\lambda)$ be a nondecreasing function, equal to zero for $\lambda \le 0$, and growing at most polynomially at $+\infty$. Let*

$$(\rho * N')(\lambda) = O(\lambda^{n-1}),$$

as $\lambda \to +\infty$. Then

$$N(\lambda) = (\rho * N)(\lambda) + O(\lambda^{n-1}).$$

If, in addition,

$$(\gamma * N')(\lambda) = o(\lambda^{n-1})$$

for any function $\gamma(\lambda) \in S(\mathbb{R}^1)$ such that $\hat{\gamma}(t) = \mathcal{F}_{\lambda\to t}(\gamma) \in C_0^\infty(\mathbb{R}^1)$, $\{0\} \notin \operatorname{supp}\hat{\gamma}$, then

$$N(\lambda) = (\rho * N)(\lambda) + o(\lambda^{n-1}).$$

Remark 3.2. In Theorem 3.1, n can be regarded as an arbitrary real (in particular, negative) number.

It is useful to discuss the physical meaning of Levitan's wave method. As already pointed out, eigenvalue problems of the type (1.1), (1.2) usually appear when one searches for natural frequencies of elastic bodies or resonators, i.e., when one searches for motions harmonically dependent on time. Therefore, the transition from (1.1), (1.2) to (3.3), (1.2) is a return to the initial nonstationary dynamical problem. The nonstationary dynamical problem (3.3), (1.2) (we omit for now the question of initial conditions, see §. 4 below for more details) contains all the information on the spectral problem (1.1), (1.2), being at the same time essentially more complex. It may seem senseless to replace the simpler problem (1.1), (1.2) by the more complicated problem (3.3), (1.2). However, for the purpose of finding spectral asymptotics it is not necessary to solve the nonstationary dynamical problem (3.3),

(1.2) precisely—it is enough to be able to trace the propagation of the wave front sets, which move with a finite speed due to the equal order of differentiations with respect to t and x. The information on the propagation speed of wave front sets allows one to determine the spectral asymptotics uniquely. Moreover, the longer the time interval is, on which we can trace the propagation of wave front sets, the more precise the spectral asymptotics are. A rough mechanical interpretation: the greater the wave propagation speed, the higher the body stiffness, and the sparser the distribution of the natural frequencies. The Tauberian theorem 3.1 formulated above adds a quantitative sense to these qualitative considerations.

The Levitan wave method can also be interpreted as a method of reconstruction of the asymptotics of the discontinuous function $N(\lambda)$ (which is inconvenient to deal with) from the asymptotics of the infinitely smooth function $(\rho_T * N)(\lambda)$. In this connection, the transition to Fourier images is perfectly natural because it greatly simplifies the operation of convolution by transforming it to a product.

§4. Characteristic properties of distributions associated with the wave group

One of the main difficulties in the application of the Levitan wave method to higher order ($m \geq 2$) operators is the singling out the unitary exponent (3.3) out of all the solutions of the "generalized wave equation" (3.3). A fundamental role in the proof of Theorems 1.1, 1.2 is played by the abstract lemma formulated below which allows us to avoid the consideration of an ill-posed Cauchy problem (in the variable t) for the equation (3.3).

Let us define precisely the functional spaces of our distributions.

Following Schwartz [19, §3.7] and Hörmander [20, §2.3] we denote by $\mathcal{E}(M)$ the vector space of infinitely differentiable (up to the boundary!) complex-valued half-densities $v(x)$ on M equipped with the usual C^∞-topology defined by the semi-norms

$$v \to \sum_{p=1}^{q} \sum_{|\alpha|\leq k} \max_{x\in M^{(p)}} |\partial_x^\alpha(\chi_p v)|$$

where k ranges over all integers ≥ 0 and $1 = \sum_{p=1}^{q} \chi_p(x)$, $\chi_p \in C^\infty(M)$, $\operatorname{supp}\chi_p \subset M^{(p)}$, is some partition of unity on M with local coordinates x in coordinate maps $M^{(p)}$. We denote by $\mathcal{E}_B(M)$ the subspace of $\mathcal{E}(M)$ consisting of all the half-densities which satisfy the boundary conditions

$$(B^{(j)}A^r\overline{v})\Big|_{\partial M} = 0, \qquad j = 1,2,\ldots,m, \qquad r = 0,1,2,\ldots;$$

the topology on $\mathcal{E}_B(M)$ is taken to be the same as on $\mathcal{E}(M)$. By $\mathcal{E}'(M)$, $\mathcal{E}'_B(M)$ we denote the dual spaces of $\mathcal{E}(M)$, $\mathcal{E}_B(M)$ respectively (i.e. spaces of linear continuous functionals on $\mathcal{E}(M)$, $\mathcal{E}_B(M)$) equipped with the dual (strong) topology generated by the initial topology on $\mathcal{E}(M)$, $\mathcal{E}_B(M)$, see [19, §§3.2, 3.3 and 3.7].

Obviously, $\mathcal{E}'(M) \subset \mathcal{E}'_B(M)$ because $\mathcal{E}_B(M) \subset \mathcal{E}(M)$. The value of the functional (distribution) u on the test half-density v will be denoted by $\langle u, v\rangle_x$ with the subscript x emphasizing the variable in which the distribution is acting.

Let $T_- < T_+$ be real numbers, finite or $\pm\infty$. In accordance with traditional notation we denote by $\mathcal{D}'(T_-, T_+)$ the vector space of linear continuous functionals on $C_0^\infty(T_-, T_+)$. The value of the distribution $f \in \mathcal{D}'(T_-, T_+)$ on the test function $g \in C_0^\infty(T_-, T_+)$ will be denoted by $\langle f, g\rangle_t$. For the sake of simplicity we, following Hörmander [20, §2.1], equip $\mathcal{D}'(T_-, T_+)$ with the weak* topology defined by the semi-norms

$$\mathcal{D}'(T_-, T_+) \ni f \to |\langle f, g\rangle_t|$$

where g is any fixed function from $C_0^\infty(T_-, T_+)$.

By $C^\infty((T_-, T_+) \times M_y; \mathcal{E}'(M_x))$, $C^\infty((T_-, T_+) \times M_y; \mathcal{E}'_B(M_x))$ we shall denote the class of distributions from $\mathcal{E}'(M_x)$, $\mathcal{E}'_B(M_x)$ respectively infinitely differentiably depending on $(t, y) \in (T_-, T_+) \times M_y$ as on a parameter (the subscripts x and y are used to distinguish the two copies of the manifold M). By $C^\infty(M_x \times M_y; \mathcal{D}'(T_-, T_+))$ we shall denote the class of distributions from $\mathcal{D}'(T_-, T_+)$ infinitely differentiably depending on $(x, y) \in M_x \times M_y$ as on a parameter. Here infinite differentiability is understood in the strong (Fréchet) sense with account of the respective topologies in $\mathcal{E}'(M_x)$, $\mathcal{E}'_B(M_x)$, $\mathcal{D}'(T_-, T_+)$.

We shall use the notation $f = O(|\lambda|^{-\infty})$ to describe the fact that the function $f(\lambda) \in C^\infty(\mathbb{R})$ vanishes faster than any given negative power of $|\lambda|$ as $\lambda \to -\infty$. More generally, we shall use this notation for "functions" $f(\lambda, x, y)$ depending on additional parameters $x \in M_x$, $y \in M_y$ to describe the fact that f as well as any given derivative of f with respect to x, y vanishes faster than any given negative power of $|\lambda|$ as $\lambda \to -\infty$ uniformly over $M_x \times M_y$.

Lemma 4.1. *Let $T_- < 0 < T_+$ be real numbers, finite or $\pm\infty$, and let*

$$u(t, x, y) \in C^\infty((T_-, T_+) \times M_y; \mathcal{E}'_B(M_x)) \cap C^\infty(M_x \times M_y; \mathcal{D}'(T_-, T_+))$$

be a distribution which behaves as a function in the variable t and as a half-density in the variables x, y.

If

$$u(t, x, y) - \exp(-it\mathcal{A}_x^{1/(2m)})u(0, x, y) \in C^\infty((T_-, T_+) \times M_x \times M_y) \tag{4.1}$$

then

$$D_t^{2m}u - A_x u \in C^\infty((T_-, T_+) \times M_x \times M_y), \tag{4.2}$$

$$(B_x^{(j)}u)\Big|_{\partial M_x} \in C^\infty((T_-, T_+) \times \partial M_x \times M_y), \qquad j = 1, 2, \ldots, m, \tag{4.3}$$

$$\mathcal{F}^{-1}_{t\to\lambda}[gu] = O(|\lambda|^{-\infty}) \quad \text{as} \quad \lambda \to -\infty \tag{4.4}$$

for any $g(t) \in C_0^\infty(T_-, T_+)$.

Inversly, if (4.2), (4.3) hold, (4.4) holds for some $g(t) \in C_0^\infty(T_-, T_+)$, $g \not\equiv 0$, *and, in addition,*

$$(B_x^{(j)} A_x^r u)\Big|_{\partial M_x} = 0 \quad at \quad t = 0, \qquad j = 1, 2, \ldots, m, \quad r = 0, 1, 2, \ldots,$$

then (4.1) is fulfilled.

Detailed proofs of Theorems 1.1, 1.2 and of Lemma 4.1 (in a slightly different version) can be found in [3].

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