Séminaire Équations aux dérivées partielles - École Polytechnique

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Séminaire Équations aux dérivées partielles (Polytechnique) (1991-1992), exp. nº 15, p. 1-11

<http://www.numdam.org/item?id=SEDP_1991-1992____A15_0>

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Séminaire 1991-1992

EQUATIONS AUX DERIVEES PARTIELLES

CONJOINT SPECTRAL ASYMPTOTICS FOR THE FAMILIES OF COMMUTING OPERATORS AND FOR OPERATORS WITH THE PERIODIC HAMILTONIAN FLOW

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Exposé n° XV

17 Mars 1992

1. General theory

Let $A_j = A_j(x, hD, h)$ $(j = 1, ..., l)$ be h-pseudo-differential operators on manifold X. We assume that

(1.1) Either $X = \mathbf{R}^d$ or X is a compact closed C^K-manifold or X is a compact C^K-manifold with a boundary $\partial X \in C^K$ with large enough $K = K(d)$, $d = \dim X$. In the third case A_j are differential operators and their original domains are given by boundary value conditions $B_i u|_{\partial X} = 0$ for $i = 1, ..., \mu$ (originally A_j are given on smooth functions and then we take their closures).

Moreover, let us assume that

(1.2) Operators A_j are self-adjoint operators in $L^2(X, \mathsf{H})$ where H is a Hermitian space[1)] and commute (i.e. their spectral projectors commute).

Then

$$A_j = \int_{\mathbf{R}^l} \tau_j d_\tau E(\tau) \tag{1.3}$$

where $\tau = (\tau_1, ..., \tau_l)$, $E(\tau) = E_1(\tau_1)...E_l(\tau_l)$, $E_j(\tau_j)$ is the spectral projector of A_j. We are interested in the semi-classical asymptotics of $\operatorname{Tr} QE(\tau'; \tau)$ where $E(\tau'; \tau) = (E_1(\tau_1) - E_1(\tau'_1))...(E_l(\tau_l) - E_l(\tau'_l))$ and Q is an appropriate h-pseudo-differential operator.

To give more precise description of operators in question we assume that

(1.4) Either A_j are h-differential operators of orders m_j or $A_j \in \tilde{\Psi}^{m_j}_{\rho,\gamma,h}$

[1)]Surely, one can consider operators acting in the Hermitian bundles.

where classes of h-pseudo-differential operators $\Psi_{\rho,\gamma,h}$ and $\tilde{\Psi}^m_{\rho,\gamma,h}$ are introduced in [1] for $\rho,\gamma\in(0,1]$, $\rho\gamma\geq h^{1-\delta}$, $\delta>0$ arbitrarily small.

The main result of this section is

THEOREM 1.1. *Let operators $A_1,...,A_l$ satisfy* (1.1),(1.2),(1.4). *Let*

$$\Omega\subset\{(x,\xi)\in T^*X, |x|\leq c, |\xi|\leq c\}, \operatorname{dist}(\Omega,\partial X)\geq\epsilon_0 \tag{1.5}$$

and

$$|D^\alpha_{x,\xi,h}A_j(x,\xi,h)|\leq c \qquad \forall(x,\xi)\in\Omega\quad\forall h\in(0,h_0)\quad\forall\alpha:|\alpha|\leq K \tag{1.6}$$

with $K=K(d,l,s)$ where s is arbitrary and $\epsilon_0>0$, $h_0>0$ are arbitrarily small constants.

Moreover let us assume that the following microhyperbolicity condition is fulfilled:

(1.7) *For every $\zeta\in\mathbb{S}^l$ and every $(x,\xi)\in\Omega$ there exists $\mathcal{T}\in T_{(x,\xi)}T^*X$ with $|\mathcal{T}|\leq 1$ such that*

$$\langle\mathcal{T}a_\zeta(x,\xi)v,v\rangle\geq\epsilon_0|v|^2-c\sum_{1\leq j\leq l}|a_j(x,\xi)v|^2\quad\forall v$$

where $a_\zeta=\sum_{1\leq j\leq l}\zeta_j a_j(x,\xi)$ and $a_j=A_j(x,\xi,0)$ are principal symbols of A_j. Let $Q=Q(x,hD,h)$ be h-pseudo-differential operator with the symbol satisfying (1.6) *and supported in $\Omega_{\epsilon_0}=\{(x,\xi),\operatorname{dist}((x,\xi),\complement\Omega)\geq\epsilon_0$. Let $\phi_j\in C^K([-1,1])$ and*

$$|D^p\phi_j|\leq c\quad\forall p\leq K\quad\forall j=1,...,l.$$

Then estimates

$$\mathcal{R}_1=|\operatorname{Tr}E(\tau';\tau)Q-h^{-d}\int\varkappa_0(\tau';\tau)|\leq Ch^{-d+l+1}\sum_{1\leq i\leq l}\prod_{1\leq j\leq l,j\neq i}(1+|\tau_j-\tau'_j|/h)\quad\forall\tau'_1,...,\tau_l\in[-\varepsilon,\varepsilon], \tag{1.8}$$

$$\mathcal{R}_1=|\int_{\mathbb{R}^l}\phi_1(\tau'_1/L_1)...\phi_l(\tau'_l/L_l)d_{\tau'}E(\tau'',\tau')-\sum_{0\leq n\leq N}h^{-d+n}\int_{\mathbb{R}^l}\phi_1(\tau'_1/L_1)...\phi_l(\tau'_l/L_l)\varkappa'_n(\tau')d\tau'|\leq Ch^{-d}(h/\min_{1\leq j\leq l}L_j)^s\quad\forall L_j\in(h,\varepsilon)\quad j=1,...,l \tag{1.9}$$

hold for small enough constant $\varepsilon > 0$.

REMARK 1.2. (i) Even for $l = 1$ estimate (1.9) of this theorem is stronger than theorem 4.3.9 from [2].

(ii) We leave to the reader to formulate the similar assertion in the case when Ω isn't disjoint from the boundary. In this case all the conditions of §5.3 [3] are assumed to be fulfilled and we assume that locally $X = \{x_1 \geq 0\}$, $Q = Q(x, hD', h)$, $x' = (x_2, ..., x_d)$; then $\Omega \cap \{x_1 = 0\}$ is a cylindrical domain with respect to ξ_1). In this case we assume that the modified standard microhyperbolicity (in an appropriate multidirection condition)of §5.3 [3] is fulfilled for boundary value problems (A_ζ, B) where $A_\zeta = \sum_j \zeta_j A_j$ and the modification is the same as above: in the right-hand expression we replace $|a_\zeta(x,\xi)v|^2$ or $\|a_\zeta(x', D_1, \xi')v\|^2_+$ by $\sum_j |a_j(x,\xi)v|^2$ or $\sum_j \|a_j(x', D_1, \xi')v\|^2_+$ respectively.

(iii) Moreover, if

$$\sum_{1 \leq j \leq l} |a_j(x,\xi)v|^2 \geq \epsilon_0 |v|^2 \quad \forall v \quad \forall (x,\xi) \in \Omega \tag{1.10}$$

then $\varkappa_0 = 0$ and the right-hand expressions of estimates (1.8),(1.9) contain additional factor h.

(iv) Moreover, let us assume in frames of theorem or (ii),(iii) that $L_j \leq h^{1-\delta}$ with $\delta > 0$ for $j = k, ..., l$ (with $k = 1, ..., l$). Then for estimate (1.9) the microhyperbolicity condition should be checked only for ζ with $\zeta_k = ... = \zeta_l = 0$. In this case an estimate similar to (1.8) with mollification on $(\tau_k, ..., \tau_l)$ and with $k-1$ instead of l in the right-hand expression is true.

(v) Coefficients $\varkappa'_n(\tau_1, ..., \tau_l)$ are the formal calculus coefficients of

$$\operatorname{Tr}\Big(\prod_{1 \leq j \leq l} \operatorname{Res}_{\mathbb{R}} (\tau_j - A_j)^{-1} \times Q\Big)$$

where $\operatorname{Res}_{\mathbb{R}} f(\tau) = \frac{1}{2\pi i}(f(\tau - i0) - f(\tau + i0))$. In particular

$$\varkappa_0(\tau'; \tau) = (2\pi)^{-d} \int \operatorname{tr} \mathcal{E}(x,\xi; \tau', \tau) q(x,\xi) dx d\xi \tag{1.11}$$

where $\mathcal{E}(x,\xi;\tau',\tau)$ is the corresponding projector for commuting symbols $a_1(x,\xi), ..., a_l(x,\xi)$.

(vi) Under conditions to Hamiltonian trajectories the above remainder estimates can be improved.

(vii) Certain results can be obtained for the Riesz means. □

SKETCH OF THE PROOF. The proof of theorem 1.1 (as well as proofs of the similar assertions near the boundary) uses the same arguments as in [2,3]. We consider the multiparametrical unitary group

$$U(t) = \exp ih^{-1}(t_1A_1 + ... + t_lA_l) = U_1(t_1)...U_l(t_l),$$
$$U_j(t_j) = \exp ih^{-1}t_jA_j.$$

If operator A_ζ is microhyperbolic at supp Q in an appropriate direction then results of [1] yield immediately that Tr $U(t)Q$ is negligible uniformly with respect to $t = \zeta t'$, $h^{1-\delta} \le t' \le T_0$ where $T_0 > 0$ is an appropriate constant. Therefore, if A_ζ is microhyperbolic at supp Q for every $\zeta \in \mathbb{S}^{l-1}$ then Tr $U(t)Q$ is negligible uniformly with respect to $h^{1-\delta} \le |t| \le T_0$. On the other hand for $|t| \le h^{1/2+\delta}$ the Schwartz kernels of $U_j(t_j)$ are constructed by the successive approximation method in [2] and therefore under the microhyperbolicity condition we have Tr $U(t)Q$ in $B(0, T_0) \subset \mathbb{R}^l$. Then using Tauberian arguments modified for "multi-time" case we obtain estimates

$$\text{Tr } E(\tau', \tau)Q \le Ch^{-d+l} \quad \forall \tau, \tau' : |\tau -' \tau| \le c$$

and

$$\text{(1.12)} \qquad \text{Tr } E(\tau', \tau)Q \le Ch^{-d}(|\tau_1 - \tau_1'| + h)...(|t_l - \tau_l'| + h).$$

Let us apply operator $\phi_1(hD_{t_1}/L_1)...\phi_k(hD_{t_l}/L_l)$ to $\bar\chi_1(t_1)...\bar\chi_l(t_l)U(t)$ and set $t_1 = ... = t_l = 0$ where $\bar\chi_j \in C_0^K([-1,1])$ equal 1 in $[-1/2, 1/2]$. We obtain (1.9). Moreover, one can replace here $\phi_j(\tau_j'/L_j)$ by $\phi_j((\tau_j' - \tau_j)/L_j)$ with $|\tau_j| \le \varepsilon$ and integrate by parts. Replacing τ' by τ'' and taking appropriate partition of unity in the box $\{\tau_j' + h \le \tau_j'' \le \tau_j - h \quad j = 1, ..., l\}$ (without loss of generality one can assume that $\tau_j' \le \tau_j \ \forall j$) we obtain estimate (1.8) modified by the following way: we replace τ', τ by λ', λ, multiply by

$$h^{-2l}\phi_1((\lambda_1' - \tau_1')/h)...\phi_l((\lambda_l' - \tau_l')/h)\phi_1((\lambda_1 - \tau_1)/h)...\phi_l((\lambda_l - \tau_l)/h)$$

and integrate on λ', λ (inside of $|.|$). However (1.11) yields that the same estimate for difference of this modified expression and the original one provided $\int \phi_j(\tau_j)d\tau_j = 1$ for all $j = 1, ..., l$.

On the other hand if condition (1.10) is fulfilled on supp Q then standard elliptic arguments yield that

$$T^{-l}|F_{t\to h^{-1}\tau}\chi_T(t)\text{Tr } U(t)Q|$$

is negligible uniformly with respect to $|\tau| \le 2\epsilon$ and $T \ge h^{1-\delta}$ where $\epsilon > 0$ is a sufficiently small constant, $\chi \in C_0^K(B(0,1))$ is an arbitrary fixed function. Then standard Tauberian arguments yield estimate

$$\|E(\tau';\tau)Q\| \le Ch^s \quad \forall \tau',\tau : |\tau'| \le \epsilon, |\tau| \le \epsilon.$$

Taking small partition of unity we obtain estimates (1.8),(1.9) under microhyperbolicity condition (1.7). □

2. Corollaries. Degenerate case

In this section we discuss different corollaries of theorem 1.1. First of all let us assume that symbols $a_1, ..., a_r$ are scalar with $r = 1, ..., l$. Then condition (1.7) is equivalent to the pair of conditions

$$(2.1) \qquad |a_1| + ... + |a_r| + |da_1| + ... + |da_r| \ge \epsilon_1 \quad \forall (x,\xi) \in \Omega$$

for some constant $\epsilon_1 > 0$ and

(2.2) At $\Sigma = \{(x,\xi) \in \Omega, a_1 = ... = a_r = 0\}$ condition (1.7) is fulfilled for symbols $a_{r+1}, ..., a_l$ with $\mathcal{T}$ tangent to Σ.

The second condition is empty for $r = l$.

Let us assume now that

(2.3) All the symbols $a_1, ..., a_l$ are scalar.

Then instead of (1.7) we have (2.1) with $r = l$. Let us reject this condition assuming however that (2.1) with $r = l - 1$ is fulfilled. In this case without loss of generality one can assume microlocally that $\sigma(A_j) = \xi_j$ for $j = 1, ..., l - 1$ where $\sigma(A_j)$ are complete symbols of A_j. Moreover, then $\sigma(A_l)$ doesn't depend on $x_1, ..., x_r$. Let us consider symbol $b = \sigma(A_l)|_\Lambda$ with $\Lambda = \{x_1 = ... = x_r = \xi_1 = ... = \xi_r = 0\}$. Let us introduce scaling function γ at Λ linked with this symbol:

$$(2.4) \qquad \gamma = \epsilon(|b| + |db|^2)^{1/2} + h^{1/2}$$

(see §4.3 of [2]). Applying method partition-dilatation (see §4.3 [2]) with scaling function 1 on $x_1, ..., x_r$ (and γ^2 with respect to $\xi_1, ..., \xi_r$) and γ with respect to $x_{r+1}, ..., x_d$ (and γ with respect to $\xi_{r+1}, ..., \xi_d$) we obtain

THEOREM 2.1. *Let all the conditions of theorem 1.1 excluding condition* (1.7) *be fulfilled. Moreover let conditions* (2.3) *and* (2.1) *with* $r = l - 1$ *be fulfilled. Let symbol* $a' = a_l|_\Sigma$ *satisfy condition*

(2.5) $|a'| + |d_\Sigma a'| \le \epsilon \Rightarrow \mathrm{Hess}_\Sigma a'$ has two eigenvalues f_1 and f_2 with $|f_1| \ge \epsilon, |f_2| \ge \epsilon$ with constant $\epsilon > 0$.

Then estimates (1.8) *with additional factor* $(|\log h|+1)$ *in the right-hand expression holds.*

Moreover under condition

$(2.5)_+$ $|a'| + |d_\Sigma a'| \le \epsilon \Rightarrow \mathrm{Hess}_\Sigma a'$ has two eigenvalues of the same sign f_1 and f_2 with $|f_1| \ge \epsilon, |f_2| \ge \epsilon$ with constant $\epsilon > 0$

estimate (1.8) *holds.*

REMARK 2.2. (i) Applying more delicate arguments of [4] one can under condition

$(2.5)'$ $|a'| + |d_\Sigma a'| + |\mathrm{Hess}_\Sigma a'| \ge \epsilon$ with some constant $\epsilon > 0$

obtain estimate (1.8) with additional factor $h^{-\delta}$ in the right-hand expression with arbitrary small $\delta > 0$.

(ii) I think that applying propagation singularities arguments (see theorem 4.3.14 [2]) under condition (2.5) one can prove estimate (1.8).

(iii) Starting from theorem 2.1 and applying the same arguments one can prove estimate (1.8) with additional factor $(|\log h| + 1)^{l-r}$ under conditions (2.3) and (2.1) (with $r \le l-1$) and under certain restriction to symbols $a'_k = a_{r+k}|_\Sigma$, $k = 1, ..., l-k = p$. In the non-uniform form (i.e. for fixed $a_1, ..., a_l$) this condition is given in terms of fundamental matrices (skew-Hessians) of $a'_1, ..., a'_l$ at point where $a'_1, ..., a'_p, da'_1, ..., da'_p$ vanish (provided Q is supported in the small neighborhood of this point); these matrices F_i commute (due to equalities $\{a_i, a_j\} = 0$).

3.Operators with the periodic Hamiltonian flow.I

Now we discuss how to apply the results of sections 1,2 to operators with the periodic Hamiltonian flows of the principal symbols. So let us consider now one scalar operator A with the principal symbol a and let us assume that

(3.1) Either X is a compact closed manifold and $A \in \tilde{\Psi}^{(1)}$ or $X = \mathbb{R}^d$ and symbol A satisfies inequalities

$$|D^\alpha_{(x,\xi)} a| \le C(1 + |x| + |\xi|)^{2-|\alpha|} \quad \forall \alpha : |\alpha| \le K.$$

In this case one says that operator A' is negligible if $h^{-s}\Lambda_s A' \Lambda_s$ is uniformly bounded for large enough s where $\Lambda_s = (1 + |x|^2 + h^2|D|^2)^{s/2}$.

Let us assume that

$$\Phi_1(x,\xi) = (x,\xi) \tag{3.2}$$

where Φ_t is the Hamiltonian flow generated by a. Then

$$e^{ih^{-1}A} \equiv e^{iB}$$

for appropriate h-pseudo-differential operator operator B. Under certain condition to A operator B is really "smaller" than I and can be replaced by ηB with $\eta = h^r$. On the other hand, perturbing A by $\mu A'$ with $h^{r+1} \le \mu \le h^\delta$ we obtain equality

$$e^{ih^{-1}A} \equiv e^{i\eta B} \tag{3.3}$$

with $\eta = \mu h^{-1}$. So let us assume *that (3.3) is fulfilled with* $\eta \in [h^n, h^{\delta-1}]$ *with arbitrarily small exponent* $\delta > 0$. More precisely this initial job is discussed in §4.5 of [2]. It is easy to see that (3.3) is also fulfilled with B replaced by $B' + (h+\eta)B''$ where

$$B' = \int_0^1 e^{-ih^{-1}tA} B e^{ih^{-1}tA} dt$$

and B' is an uniformly bounded h-pdo. Continuing this process one can replace B in (3.3) by operator which commutes with A modulo $O(h^s)$; moreover, at every step one can replace B by $(B+B^*)/2$; so one can assume without loss of generality that

(3.4) B is symmetric operator and $\|[B,A]\| \le Ch^s$.

We need first to treat the case when

$$e^{ih^{-1}A} \equiv I. \tag{3.5}$$

In this case Spec $A \subset \cup_{n\in\mathbb{Z}} I_n$ where $I_n = [(2\pi n - C_0\eta)h, (2\pi n + C_0\eta)h]$ and $\eta \le Ch^s$.

Treating Tr $\Phi(D_t)U(t)|_{t=0}$ one can prove easily

THEOREM 3.1. *Let conditions* (3.1), (3.2), (3.5) *be fulfilled and moreover let us assume that*

$$\mathrm{dist}((x,\xi), \Phi_t(x,\xi)) \ge \epsilon t(1-t) \quad \forall t \in (0,1) \quad \forall (x,\xi) \in \Sigma_0 \tag{3.6}$$

where $\Sigma_\tau = \{a = \tau\}$ *is assumed to be compact; this condition means that there is no subperiodic trajectory. Then* $\forall\tau = 2\pi nh$ *with* $|\tau| \le \varepsilon$ *the Schwartz kernel of* $E(\tau - \zeta, \tau + \zeta)$ *is the Lagrangian distribution with the Lagrangian manifold* $(x, \xi, x', \xi') \in \Sigma_\tau, \exists t(x, \xi) = \Phi_t(x', -\xi')$; $\zeta = C_0 h^s$ *here*[2]*.*

Let B be self-adjoint operator commuting with A. Then one can treat distribution of eigenvalues of B on Ran $(\tau - \zeta, \tau + \zeta)$. The usual analysis of [2] modified by rather obvious way yields

THEOREM 3.2. *Let* A *satisfy conditions of theorem 3.2 and let* B *be self-adjoint operator commuting with* A. *Let us assume that condition* (2.1) *is fulfilled for* $a_1 = a$ *and* $a_2 = b$ *at* $\Sigma_{\tau,0}$; $|\tau| \le \varepsilon$. *Then* $\forall\tau = 2\pi nh$ *with* $|\tau| \le \varepsilon$

$$(3.7)\qquad |\mathrm{Tr}\ E_A(\tau - \zeta, \tau + \zeta)E_B(\lambda', \lambda)Q - \varkappa_0 h^{1-d}(\tau; \lambda', \lambda)| \le Ch^{2-d} \quad \forall\lambda', \lambda : |\lambda'| \le \varepsilon, |\lambda| \le \varepsilon$$

and moreover estimate similar to (1.9) *also has place. Here*

$$(3.8)\qquad \varkappa_0 = (2\pi)^{-d} \int_{\{\Sigma_\tau, \lambda' \le b \le \lambda\}} q(x, \xi) dx d\xi : da.$$

REMARK 3.3. (i) Moreover, similar results hold for a family of commuting operators B_j (commuting also with A). Moreover, A can replaced be by a family of commuting operators A_i also.

(ii) In frames of theorem 3.2 method of partition and dilatation can be applied. Therefore one can replace condition (1.7) by condition (2.5) or $(2.5)_+$.

(iii) Let us consider A satisfying (3.1)-(3.3). Without loss of generality one can assume that (3.4) is fulfilled also. Then operator $A^0 = A - B$ satisfies conditions of theorem 3.1. This yields that there exists $B' \equiv B$ commuting with A^0 and then under condition (3.6) theorem 3.2 provides all the results of §4.5[2] and even slightly better (under (2.5)-type conditions) for operator $A' = A^0 + B'$; these results obviously remains true for A.

(iv) Let the periodicity condition (3.2) be fulfilled with $\Omega = T^*X$. Then under conditions (3.5) and (3.6) fulfilled at the energy level $\{a(x, \xi) = \tau^0\}$ it is easy to prove that formula

$$(3.9)\qquad \Delta\mathrm{N}(\tau) = \nu(\tau) = \frac{2\pi h}{T}\mathrm{Tr}\ \mathrm{Res}_{\mathbb{R}}(\tau - A)^{-1}$$

[2]However $E(\tau - \zeta; \tau + \zeta)$ is not Fourier integral operator because $\dim\mathrm{Ker}\ d\pi_X = \dim\mathrm{Ker}\ d\pi_Y = 1 (\ne 0)$ for projectors $\pi_X : \Lambda \to T^*X$, $\pi_Y : \Lambda \to T^*Y$.

holds for the number of the eigenvalues lying in the cluster lying near $\tau = \tau_m = 2\pi mh/T$ with $m \in \mathbf{Z}$ such that $|\tau - \tau^0| \le \epsilon$. This cluster lies in the interval $J_m = (\tau_m - \zeta, \tau_m + \zeta)$ and

$$\mathrm{Spec}\, A \cap (\tau^0 - \epsilon, \tau^0 + \epsilon) \setminus \bigcup_m J_m = \emptyset.$$

Here $T = 1$ is the period of the Hamiltonian flow, $\zeta = h^s$ and we assume that $h < h_1$ where $\epsilon > 0$ and $h_1 > 0$ are small enough constants. The more complicated formula holds under condition (3.3) with $\eta < \epsilon$ instead of (3.5); in this case $\zeta = C_0 h\eta$. □

4.Operators with the periodic Hamiltonian flow.II

On the other hand, let us reject condition (3.6) and assume that in the neighborhood of the energy level $\{a(x,\xi) = \tau^0\}$

$$(4.1) \qquad |H_a| \ge \epsilon_0 > 0.$$

Then $\Phi_t(x,\xi) = (x,\xi)$ for $t \in (0,T)$ yields that $t = Ti/q$ where $i, q \in \mathbf{N}$, $i = 1, ..., q-1$. Taking ratio i/q in its lowest terms we obtain that the same is true for $i = 1$. Let us assume that for every $q \in Z^+$ in the neighborhood of the energy level $\{a(x,\xi) = \tau^0\}$

(4.2) $\Phi_t(x,\xi) = (x,\xi)$ for $t = T/q$ if and only if $(x,\xi) \in \cup_{j=1,\dots,j_q} \Xi_{q,j}$ where Ξ_{jq} are symplectic manifolds, $\Xi_{pl} \cap \Xi_{qj} = \emptyset$ for $(p,i) \ne (q,j)$, and for every $(x,\xi) \in \Xi_{qj}$ $H_a \in T_{(x,\xi)}\Xi$ and for every $t = Ti/q$ with $i = 1, ..., q-1$ $\mathrm{Ker}\,(D\Phi_t - I) = T_{(x,\xi)}\Xi$.

Then one can prove formula $\Delta\mathrm{N}(\tau) = \nu(\tau) + \nu'(\tau)$ for $\tau = 2\pi mh/T$ where $\nu(\tau)$ is given in (iv) and $\nu'(\tau) = \sum_{q,j} \nu_{qj}$ and $\nu_{qj} = O(h^{1-D_{qj}})$ are linked with Ξ_{qj}, $D_{qj} = \dim \Xi_{qj}/2$.

Both this formula and (3.9) were obtained in the paper of Y.Colin de Verdiere [6] in different form and coefficients in the asymptotics expansions of $\nu(\tau)$ and $\nu'(\tau)$ were calculated in the case of asymptotic with respect to spectral parameter. All his arguments remain true in the semi-classical case[3)] and in fact, my proof almost completely coincides with some part of his[4)].

[3)] I didn't check arguments linked with the statements which were not formulated here.

[4)] The stupid thing: I forgot about his paper and read it only after L.Boutet de Monvel drew my attention.

Let us treat the influence of subperiodic trajectories to asymptotics of $\mathbf{N}(\tau)$ when τ lies inside of the cluster. It is not difficult to prove that under condition (4.2) and condition

$$|d_{\Sigma_\tau} b| \geq \epsilon\eta \tag{4.3}$$

asymptotics

$$\mathbf{N}(\tau) = \varkappa_0(\tau)h^{-d} + \varkappa_1(\tau)h^{1-d} + F(\tau, h^{-1}\tau) + O(h^{2-d}) \tag{4.4}$$

remains true where d_{Σ_τ} means gradient along Σ_τ and function $F(\tau,\lambda)$ was introduced in §4.5 of [2][5]. The heuristic explanation is very simple: $\nu'(\tau)$ surely doesn't exceed the remainder estimate. However, for the Riesz means the situation changes: the remainder estimate under conditions (4.2),(4.3) and

$$|d_{\Sigma_\tau \cap \Xi_{qj}} b| \geq \epsilon\eta \tag{4.5}$$

(which should be checked only for Ξ_{qj} with large enough D_{qj} the remainder estimate in the asymptotics formula of (4.4)-type (with a lot of Weylian and non-Weylian terms) should be $O(h^{2+2\vartheta-d}\eta^{\vartheta})$ and contribution of subperiodic trajectories is $\sim h^{1-D+\vartheta}$ where ϑ is the order of the Riesz mean and $D = \max D_{qj}$.

Under conditions (4.2),(4.3) and (4.5) I can prove the asymptotic formula for $\mathbf{N}_\vartheta(\tau)$ with the indicated remainder estimate and with the Weylian and non-Weylian terms (including terms generated by subperiodic trajectories). I can weaken condition (4.3) and I hope to weaken condition (4.5) in future and to calculate these terms.

Moreover, under conditions (4.2) and (4.5) one can derive complete asymptotics of $\mathbf{N} * \phi_L$ with $\phi_L(\tau) = \phi(\tau/L)$, $\phi \in C_0^K(\mathbf{R})$ and $L \geq \eta h^{2-\delta}$. This asymptotics (for $L \leq h$) surely contains non-Weylian terms including terms generated by subperiodic trajectories). I hope to rid condition (4.5).

References

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[5] Moreover, one can weaken condition (4.3) and replace it by condition $(2.5)_+$ or even $(2.5)'$ which can be fulfilled for any τ while condition (4.3) surely fails for some τ.