Séminaire Équations aux dérivées partielles – École Polytechnique

A. Bahri
P. H. Rabinowitz

Periodic solutions of some problems of 3-body type

Séminaire Équations aux dérivées partielles (Polytechnique) (1988-1989), exp. nº 21, p. 1-11
<http://www.numdam.org/item?id=SEDP_1988-1989____A22_0>

CENTRE
DE
MATHEMATIQUES
Unité de Recherche Associée D 0169

ECOLE POLYTECHNIQUE
F-91128 PALAISEAU Cedex (France)
Tél. (1) 69.41.82.00
Télex ECOLEX 601.596 F

Séminaire 1988-1989

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

PERIODIC SOLUTIONS OF SOME PROBLEMS OF 3-BODY TYPE

A. BAHRI and P.H. RABINOWITZ

Exposé n°XXI

21 Mars 1989

Periodic Solutions of Some Problems of 3-Body Type

by Abbas Bahri and Paul H. Rabinowitz

§1. Introduction

The study of time periodic solutions of the n-body problem is a classical one. See e.g [1]. The purpose of this paper is to sketch some of our recent research on the existence of time periodic solutions of Hamiltonian systems of 3-body type [2]. This work presents a new direct variational approach to the problem.

To describe it more fully, consider the Hamiltonian system of ordinary differential equations

$$(HS). \qquad m_i\ddot{q}_i + V_{q_i}(t,q) = 0, \quad 1 \le i \le 3$$

In (HS), $q_i \in \mathbf{R}^\ell$, $1 \le i \le 3$, $\ell \ge 3$, $m_i > 0$, where $F_3(\mathbf{R}^\ell)$ is the configuration space

$$(1.1) \qquad F_3(\mathbf{R}^\ell) = \{(q_1,q_2.q_3) \in (\mathbf{R}^\ell)^3 \mid q_i \neq q_j \quad \text{if} \quad i \neq j\}.$$

Furthermore V is T-periodic in t.

We are interested in T-periodic solutions of (HS). It is assumed that V is an interaction potential:

$$(1.2) \qquad V = \sum_{\substack{i,j=1 \\ i\neq j}}^{3} V_{ij}(t,q_i-q_j)$$

Each function V_{ij}, $1 \le i \neq j \le 3$, satisfies

(V_1) $V_{ij} \in C^2(\mathbf{R} \times (\mathbf{R}^\ell\backslash\{0\}),\mathbf{R})$ and is T-periodic in t,

(V_2) $V_{ij}(t,q) < 0$ for all $t \in [0,T]$, $q \in \mathbf{R}^\ell\backslash\{0\}$,

(V_3) $V_{ij}(t,q)$, $\frac{\partial V_{ij}}{\partial q_k}(t,q) \to 0$ as $|q| \to \infty$ uniformly in t, $1 \le k \le 3$,

(V_4) $V_{ij}(t,q) \to -\infty$ as $q \to 0$, uniformly in t,

(V_5) for all $M > 0$, there is an $R > 0$ such that

$$\frac{\partial V_{ij}}{\partial q} \cdot q > M\left|\frac{\partial V_{ij}}{\partial q}\right|$$

whenever $|q| > R$,

(V_6) there is a neighborhood, W, of 0 in $\mathbf{R}^\ell$ and $U_{ij} \in C^1(W\backslash\{0\}, \mathbf{R})$ such that $U_{ij}(q) \to \infty$ as $q \to 0$ and $-V_{ij}(q) \geq |U'_{ij}(q)|^2$ for $q \in W\backslash\{0\}$.

Condition (V_1) – (V_5) are satisfied in particular by potentials of the form

$$(1.3) \qquad V(q) = -\sum_{\substack{i,j=1 \\ i\neq j}}^{3} \frac{\alpha_{ij}}{|q_i - q_j|^{\beta_{ij}}}$$

where α_{ij} and β_{ij} are positive constants. Hypothesis (V_6) is also satisfied by V in (1.3) if $\beta_{ij} \geq 2$ for each i, j. The classical 3-body problem corresponds to the case in which $\beta_{ij} = 1$, for all $i \neq j$ and $\alpha_{ij} = \alpha_{ji}$. The significance of ($V_6$) will be discussed below.

To formulate (HS) as a variational problem, let $E = W_T^{1,2}(\mathbf{R}, (\mathbf{R}^\ell)^3)$, the Hilbert space of T-periodic functions from $\mathbf{R}$ into $(\mathbf{R}^\ell)^3$ with norm:

$$(1.4) \qquad \|q\| = \left(\int_0^T |\dot{q}|^2 dt + |q|^2\right)^{1/2}$$

where

$$(1.5) \qquad [q] = \frac{1}{T}\int_0^T q(s)ds.$$

The functional associated with (HS) is

$$(1.6) \qquad I(q) = \int_0^T \left(\frac{1}{2}\sum_{i=1}^{3} m_i|\dot{q}_i|^2 - V(t,q)\right) dt.$$

Set

$$(1.7) \qquad \Lambda = \{q \in E \mid q(t) \in F_3(\mathbf{R}^\ell) \text{ for all } t \in [0,T]\}.$$

It is not difficult to prove that

Lemma 1.8: If V satisfies (V_1), (V_2), (V_4), and (V_6), then for each $c > 0$, there is a $\delta(c) > 0$ such that if $I(q) \leq c$, then

$$\inf_{\substack{t\in[0,T] \\ i\neq j}} |q_i(t) - q_j(t)| \geq \delta(c).$$

See e.g. [2] or [3]. An immediate consequence of Lemma 1.8 is the variational problem can be posed on Λ rather than E. Moreover it is easy to verify that if $q \in \Lambda$ and $I'(q) = 0$, then q is a classical T-periodic solution of (HS).

Our main result is:

Theorem 1.9. If V satisfies $(V_1) - (V_6)$, then I possesses an unbounded sequence of critical values.

If condition (V_6) is dropped, it is possible that $q \in E$ with $I(q) < \infty$ but $q_i(t) = q_j(t)$ for some $i \neq j$ and $t \in [0,T]$, i.e. a *collision* occurs at time t. Thus without (V_6) it is possible that collisions can occur for periodic solutions of (HS). Since a collision orbit cannot be a classical solution of (HS), following [4], we say $q \in E$ is a *generalized T-periodic solution of*

$$(1.10) \qquad \begin{cases} (i) & \mathcal{D} = \{t \in [0,T] \mid q(t) \notin F_3(\mathbf{R}^\ell)\} \textit{ has measure } 0. \\ (ii) & q \in C^2 \textit{ and satisfies (HS) in } [0,T]\backslash\mathcal{D}. \\ (ii) & -\int_0^T V(t,q(t))dt < \infty. \\ (iv) & \textit{If } V \textit{ is independent of } t, \frac{1}{2}\sum_{i=1}^3 |\dot{q}_i(t)|^2 + V(q(t)) \equiv \\ & \textit{constant for } t \in [0,T]\backslash\mathcal{D}, \textit{ i.e. energy is conserved} \\ & \textit{on the set on which it is defined.} \end{cases}$$

Remark 1.11. *Conditions (1.10) (i)-(iv) are not mutually exclusive but we prefer to define generalized T-periodic solution in this way since it is these conditions that one verifies in applications.*

Given Theorem 1.9, using an approximation argument from [4], it is not difficult to show:

Theorem 1.12. *If V satisfies $(V_1) - (V_5)$, then (HS) possesses a generalized T-periodic solution.*

Corollary 1.13. *If in addition, V is independent of t and $V'(q) \neq 0$ for all $q \in (\mathbf{R}^\ell)^3$, (HS) has infinitely many distinct T-periodic solutions.*

In the next two sections, we will discuss some of the preliminaries that go into the proof of Theorem 1.9. Then in §4, the proof of Theorem 1.9 itself will be sketched. Finally in §5, a few remarks will be made about the proofs of Theorem 1.12 and Corollary 1.13.

§2. The breakdown of (PS) and a Morse Lemma for neighborhoods of infinity.

A standard condition used in the study of variational problems is the Palais-Smale condition or (PS) for short. It says any sequence (q^k) satisfying

$$(2.1) \qquad I(q^k) \textit{ is bounded and } I'(q^k) \to 0$$

is precompact. Unfortunately (PS) does not hold for (1.6). However the behavior of (PS) sequences can be characterized precisely.

Proposition 2.2. *Suppose V satisfies $(V_1)-(V_4)$ and (V_6). Let (q^k) satisfy (2.1). Then the following alternative holds: Either*

(i) there exists a subsequence, still denoted by (q^k), and a sequence $(v_k) \subset \mathbf{R}^\ell$ such that $(q_i^k - v_k)$ converges in $W_T^{1,2}(\mathbf{R}, \mathbf{R}^\ell)$ for $i = 1, 2, 3$, or

(ii) there exists a subsequence, still denoted by (q^k), a sequence $(v_k) \subset \mathbf{R}^\ell$, and $i \in \{1, 2, 3\}$ such that

a. $\|[q_i^k - v_k]\| \to \infty$ and $\|\dot{q}_i^k\|_{L^2} \to 0$ as $k \to \infty$, and

b. if $j \neq r \in \{1,2,3\}\backslash\{i\}$, $(q_j^k - v_k, q_r^k - v_k)$ converges in $W_T^{1,2}(\mathbf{R}, \mathbf{R}^\ell)^2)$ to a classical solution of the two-body problem associated with the potential $V_{ji} + V_{ij}$.

Moreover if

$$I_{jr}(q_j, q_r) \equiv \int_0^T \left(\frac{1}{2}(m_j|\dot{q}_j|^2 + m_r|\dot{q}_r|^2) - V_{jr}(t, q_j - q_r) - V_{rj}(t, q_r - q_j)\right) dt, \tag{2.3}$$

then $I_{jr}(q_j^k, q_r^k) \to c$.

Proposition 2.2 tells us that if a (PS) sequence is not precompact in the usual sense, it has a subsequence which converges to a "two-body solution at infinity". We further note that one can take $v^k = \frac{1}{2}[q_j^k + q_r^k]$.

One major new idea in our work is to use a "Morse Lemma" in a neighborhood of a sequence of type (ii) in Proposition 2.2.

Proposition 2.4. *Let V satisfy $(V_1)-(V_5)$. Then*

(1) for all $C > 0$, there exists an $\alpha(C) > 0$ such that if $q = (q_1, q_2, q_3) \in \Lambda$ satisfies

$$(i) \quad \sum_{i=1}^{2} \|q_i - v(q)\|_{L^\infty} \leq C,$$

and

$$(ii) \quad \frac{1}{2}m_3\|\dot{q}_3\|_{L^2}^2 + \frac{1}{1 + \|[q_3 - v(q)]\|^2} \leq \alpha(C)$$

for some $v = v(q) \in \mathbf{R}^\ell$, then there is a unique $\lambda(q) > 0$, continuousl y differentiable in q, and satisfying

$$I(q) = I_{12}(q_1, q_2) + \frac{1}{2}\int_0^T m_3|\dot{Q}_3|^2 dt + \frac{1}{1 + \|[Q_3 - \frac{1}{2}(q_1 + q_2)]\|^2}$$

where

$$Q_3 = \frac{1}{2}[q_1 + q_2] + \frac{1}{\lambda(q)}(q_3 - [q_3]) + \lambda(q)[q_3 - \frac{1}{2}(q_1 + q_2)].$$

(2) Conversely for all $C > 0$, there exists $\overline{\alpha}(C) > 0$ such that if $(q_1, q_2, Q_3) \in \Lambda$ satisfies

$$(iii) \quad \sum_{i=1}^{2} \|q_i - v(q)\|_{L^\infty} \leq C,$$

and

$$(iv) \quad \frac{1}{2}m_3\|\dot{Q}_3\|_{L^2}^2 + \frac{1}{1 + |[Q_3 - \frac{1}{2}(q_1 + q_2)]|^2} \leq \overline{\alpha}(C)$$

for some $v = v(q) \in \mathbf{R}^\ell$, then there is a unique $\mu(q_1, q_2, Q_3) > 0$, continuously differentiable in its argument, and satisfying

$$I(q_1, q_2, q_3) = I_{12}(q_1, q_2) + \frac{1}{2}\int_0^T m_3|\dot{Q}_3|^2 dt + \frac{1}{1 + |[Q_3 - \frac{1}{2}(q_1 + q_2)]|^2}$$

where

$$q_3 = \frac{1}{2}[q_1 + q_2] + \mu(q_1, q_2, Q_3)(Q_3 - [Q_3]) + \frac{1}{\mu(q_1, q_2, Q_3)}[Q_3 - \frac{1}{2}(q_1 + q_2)].$$

(3) If $\alpha(C) = \overline{\alpha}(C)$ is sufficiently small, then $\lambda(q_1, q_2, q_3)\mu(q_1, q_2, Q_3) = 1$ and the transformations defined in 1 and 2 are inverse diffeomorphisms.

In Proposition 2.4, we could replace $\frac{1}{2}[q_1 + q_2]$ by any convex combination of $[q_1]$ and $[q_2]$. In particular we could have taken the center of mass $\frac{m_1[q_1]+m_2[q_2]}{m_1+m_2}$. If we do so, the representation provided by Proposition 2.4 has the physical interpretation that the interaction of the motion of the body q_3 with the two other bodies can be replaced by the motion of a new body Q_3 which interacts (at the level of mean values) only with the center of mass of the other bodies. Proposition 2.4 allows us to represent I in a simple fashion in a neighborhood of a sequence violating the (PS) condition, i.e. near a "critical point at infinity". In this sense we have a Morse Lemma for neighborhoods of critical points at infinity.

One final technical result will be given in this section. Let

$$I^c = \{q \in \Lambda \mid I(q) \leq c\}. \tag{2.5}$$

Proposition 2.6. *Let V satisfy $(V_1) - (V_5)$. Then there is an $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, I^ϵ retractby deformation on an ENR $X \subset \mathbb{R}^\ell$. In particular the singular homology of I^ϵ (with rational coefficients) vanishes in all dimensions $\geq \ell$.*

§3. An abstract theorem in Morse Theory.

The proof of Theorem 1.9 involves in part the construction of a certain deformation retraction. In this section, a finite dimensional version of this result will be stated. In the next section, the extensions needed for the proof of Theorem 1.9 will be discussed. More details can be found in [2] and [5].

Let $\mathcal{M}$ be a compact n-dimensional Riemannian manifold and let $f \in C^2(\mathcal{M}, \mathbf{R})$. Assume all of the critical points of f are nondegenerate. Let Z denote a pseudogradient vector field for f, i.e. Z is a locally Lipschitz continuous vector field on $\mathcal{M}$ and

$$(3.1) \qquad \langle f'(x), Z(x) \rangle \geq |f'(x)|^2$$

and

$$(3.2) \qquad |Z(x)| \leq \gamma |f'(x)|$$

for all $x \in \mathcal{M}$ where $\gamma > 0$ is a constant. Further assume the critical points of f are nondegenerate zeroes of Z. Consider the ordinary differential equation

$$(3.3) \qquad \frac{dx}{ds} = -Z(x), \qquad x(0, y) = y.$$

Let $\varphi(s, y)$ denote the solution of (3.3). Suppose $Z(x_0) = 0$. Set

$$(3.4) \qquad W_u(x_0) = \{x \in \mathcal{M} \mid \varphi(s, x) \to x_0 \quad as \quad s \to -\infty\},$$

i.e. $W_u(x_0)$ is the unstable manifold for the flow (3.3) which emanates from x_0. Let $a < b$ be noncritical values of f and

$$f^c = \{x \in M \mid f(x) \leq c\}.$$

Let

$$C_a^b = \{x \in \mathcal{M} \mid f'(x) = 0 \quad and \quad a \leq f(x) \leq b\}.$$

Then we have

Theorem 3.5. *Let Z_1 be a peudogradient vector field for f. Then in any C^1 neighborhood of Z_1, there exists a pseudogradient vector field Z for f such that f^b retracts by deformation onto*

$$f^a \cup \left(\bigcup_{x \in C_a^b} W_u(x) \right).$$

If C_a^b is a single point, then Theorem 3.5 is a classical deformation result. See e.g. [6, p.156-160]. There are many extensions of Theorem 3.5, especially in an infinite dimensional space. The theorem can also be formulated in different ways. We refer to Bahri [5] for such extensions, alternate formulations, and applications. We note that Theorem 3.5 can not be generalized as such to situations where (PS) fails. In the next section we will discuss how to extend Theorem 3.5 so as to apply to the functional (1.6).

§4. The construction of a pseudogradient vector field for I and a sketch of the proof of Theorem 1.9.

Let C_1 be a constant and $\beta \in C(\mathbf{R}^+, \mathbf{R}^+)$. Let $\mathcal{V}_3$ denote the set of $(q_1, q_2, q_3) \in \Lambda$ satisfying

and

$$(ii) \quad \frac{1}{2} m_3 \|\dot{q}_3\|_{L^2}^2 + \frac{1}{1 + \|[q_3 - \frac{1}{2}(q_1 + q_2)]\|^2} \leq \beta(C_1).$$

The sets $\mathcal{V}_1, \mathcal{V}_2$ are defined in a similar way via a permutation of indices. If β is sufficiently small, Proposition 2.4 is valid on $\mathcal{V}_3$ (resp. $\mathcal{V}_1, \mathcal{V}_2$) and one can therefore use, as in the classical situation in Morse Theory [6], the new coordinates (q_1, q_2, Q_3) to define a pseudogradient vector field $\tilde{Z}$ for I on $\mathcal{V}_3$. Then $\tilde{Z}$, similarly defined on $\mathcal{V}_1, \mathcal{V}_2$ can be extended to Λ by taking convex linear combinations of I' and $\tilde{Z}$.

We will give an idea for the construction of $\tilde{Z}$ on $\mathcal{V}_3$. A detailed proof can be found in [2]. Suppose we have a pseudogradient vector field Z_{12} for I_{12}. To simplify our presentation, assume that the critical points of I_{12} and I are nondegenerate. (This, of course, is not the case due to the translational symmetry possessed by I_{ij} and I.) Let $(\overline{q}_1, \overline{q}_2)$ be a critical point of I_{12} and therefore a zero of Z_{12}. Let $W_u(\overline{q}_1, \overline{q}_2)$ be the unstable manifold associated with $(\overline{q}_1, \overline{q}_2)$ for the differential equation

$$\frac{d}{ds}(q_1, q_2) = -Z_{12}(q_1, q_2). \tag{4.2}$$

Now we define $\tilde{Z}$ in the coordinates $(q_1, q_2, Q_3 - [Q_3], [Q_3 - \frac{1}{2}(q_1 + q_2])$ via

$$\frac{dq}{ds} = -\tilde{Z}(q) \tag{4.3}$$

if and only if

$$(4.4)\qquad \begin{cases} \frac{d}{ds}(q_1,q_2) = -Z_{12}(q_1,q_2) \\ \frac{d}{ds}(Q_3 - [Q_3]) = -(Q_3 - [Q_3]) \\ \frac{d}{ds}[Q_3 - \frac{1}{2}(q_1+q_2)] = 0. \end{cases}$$

Using Proposition 2.4. it is easy to verify that $\tilde{Z}$ is a pseudogradient vector field for I on $\mathcal{V}_3$. The solution $\varphi(s,q)$ of (4.3) compactifies the "critical points at infinity" in the sense of [7], i.e. the decreasing (with respect to I as $s \to +\infty$) orbits of the gradient flow that are not compact. In doing so, we introduce new equilibrium points for $\tilde{Z}$ which are distinct from the critical points. This prevents Theorem 3.5 from being extended directly to (4.3). It is necessary to take into account the "unstable manifolds of critical points at infinity". By doing so, one can prove a version of Theorem 3.5 for the current situation. Let

$$\mathcal{C}_a^b = \{q \in \Lambda \mid I'(q) = 0 \quad \text{and} \quad a \le I(q) \le b\}, \text{ cr}$$

$$\mathcal{C}_a^b(i,j) = \{(\overline{q}_i,\overline{q}_j) \mid I_{ij}(\overline{q}_i,\overline{q}_j) = 0 \quad \text{and} \quad a \le I_{ij}(\overline{q}_i,\overline{q}_j) \le b\},$$

$$\mathcal{D}_a^b = \bigcup_{q\in\mathcal{C}_a^b} W_u(q). \quad \text{and} \quad \mathcal{D}_a^b(\infty) = \bigcup_{i\neq j}\ \bigcup_{(\overline{q}_i,\overline{q}_j)\in\mathcal{C}_a^b(i,j)} W_u^\infty(\overline{q}_i,\overline{q}_j).$$

Here $W_u^\infty(\overline{q}_i,\overline{q}_j)$ denotes the unstable manifolds of critical points at infinity.

Theorem 4.5. *Let $a < b$ be noncritical values of I. Then I^b retracts by deformation on*

$$I^a \cup \mathcal{D}_a^b \cup \mathcal{D}_a^b(\infty)$$

and $W_u^\infty(\overline{q}_i,\overline{q}_j)$ is a trivializable fiber over $W_u(\overline{q}_i,\overline{q}_j)$, the fiber having the homotopy type of a sphere $S^{\ell-1}$.

With the aid of Theorem 4.5, the proof of Theorem 1.9 can now be sketched. To simplify matters, assume that I has no critical points. Then, for any $b > a = \epsilon > 0$, $\mathcal{C}_\epsilon^b = \phi$ so by Theorem 4.5,

$$(4.6)\qquad I^b \simeq I^\epsilon \cup \mathcal{D}_\epsilon^b(\infty)$$

where $\simeq$ denotes retraction by deformation. The proof continues via three steps.

Step 1. *Since $\Lambda = \bigcup_{b\in\mathbf{R}^+} I^b$ and (4.6) holds for all $b > \epsilon$, it can be shown that*

$$(4.7)\qquad \Lambda \sim I^\epsilon \cup \mathcal{D}_\epsilon^\infty(\infty)$$

where $\sim$ *denotes homotopy equivalence.*

Step 2. *Let*

$$\mathcal{B}_{ij} = \bigcup_{(\overline{q}_i,\overline{q}_j)\in\mathcal{C}_\epsilon^\infty(i,j)} W_u(\overline{q}_i,\overline{q}_j);$$

$$\mathcal{B}_{ij}^\infty = \bigcup_{(\overline{q}_i,\overline{q}_j)\in\mathcal{C}_\epsilon^\infty(i,j)} W_u^\infty(\overline{q}_i,\overline{q}_j)$$

and

$$\mathcal{B}^\infty = \bigcup_{i\neq j} \mathcal{B}_{ij}^\infty.$$

An improved version of Theorem 4.5 [2] says that $\mathcal{B}_{ij}^\infty$ *fibers over* $\mathcal{B}_{ij}$*, the fiber being trivializable and having the homotopy type of a sphere* $S^{\ell-1}$*. Set*

$$\Lambda_{ij} = \{(q_i, q_j) \in W_T^{1,2}(\mathbf{R},(\mathbf{R}^\ell)^2) \mid q_i(t) \neq q_j(t) \quad \textit{for all} \quad t\in[0,T]\}. \tag{4.8}$$

It is not difficult to check that I_{ij} *satisfies (PS) on* Λ_{ij} *up to translations, i.e. if* I_{ij} *is bounded and* $I_{ij}' \to 0$ *along the sequence* (q_i^m, q_j^m)*, then there is a sequence* $(v_m) \subset \mathbf{R}^\ell$ *such that* $(q_i^m - v_m, q_j^m - v_m)$ *possesses a convergent subsequence. Therefore an infinite dimensional version of Theorem 3.5 [4], and an argument related to (4.7), yields*

$$\Lambda_{ij} \sim I_{ij}^\epsilon \cup \mathcal{B}_{ij}. \tag{4.9}$$

Step 3. *By Proposition 2.6, the rational homology of* I^ϵ *vanishes in dimension* $\geq \ell$*. Applying the Mayer-Vietoris sequence to the excisive triad* $(\Lambda, I^\epsilon, \mathcal{B}^\infty)$ *shows that*

$$H_k(\Lambda) = H_k(\mathcal{B}^\infty) \quad \textit{for} \quad k \geq \ell. \tag{4.10}$$

Similarly

$$H_k(\Lambda_{ij}) = H_k(\mathcal{B}_{ij}) \quad \textit{for} \quad k \geq \ell. \tag{4.11}$$

Moreover, from the fibration of $\mathcal{B}_{ij}^\infty$ *over* $\mathcal{B}_{ij}$*, one deduces that*

$$H_k(\mathcal{B}_{ij}^\infty) = H_k(\mathcal{B}_{ij}) \oplus H_{k-\ell+1}(\mathcal{B}_{ij}) \quad \textit{for} \quad k \geq \ell. \tag{4.12}$$

Combining (4.10)-(4.12) yields:

$$H_k(\Lambda) = \bigoplus_{i\neq j} H_k(\Lambda_{ij}) \oplus H_{k-\ell+1}(\Lambda_{ij}) \quad \textit{for} \quad k \geq \ell. \tag{4.13}$$

Let α_k be the dimension of $H_k(\Lambda)$ and β_k the dimension of $H_k(\Lambda_{ij})$. Then by (4.13)

$$\alpha_k = 3(\beta_k + \beta_{k-\ell+1}) \quad \textit{for} \quad k \geq \ell. \tag{4.14}$$

However Λ_{ij} has the homotopy type of the free loop space on $S^{\ell-1}$ — see [2] — and therefore β_k is bounded independently of k [8]. On the other hand, by a Theorem of Sullivan and Vigué-Poirrier [8]. the sequence (α_k) is unbounded. This contradiction shows that I has at least one positive critical value.

A more complicated variant of this argument given in [2] which takes $\mathcal{D}_a^\infty$ into account proves that I, in fact, has an unbounded sequence of critical values.

§5. The proof of Theorem 1.12 and Corollary 1.13.

We will give a brief sketch of the ideas involved in getting Theorem 1.12 from Theorem 1.9. First for all $\delta > 0$, the potentials V_{ij} are approximated by V_{ij}^δ which satisfy $(V_1)-(V_6)$. $V_{ij}^\delta(t,x) = V_{ij}(t,x)$ if $|x| \geq \delta$, and $-V_{ij}^\delta(t,x) \geq -V_{ij}(t,x)$ if $|x| < \delta$. Then Theorem 1.9 applies to the functional

$$I_\delta(q) = \int_0^T \left(\frac{1}{2} \sum_{i=1}^3 m_i |\dot{q}_i|^2 - V^\delta(t,q) \right) dt. \tag{5.1}$$

Next it is shown that there are constants M and ϵ_1 which are independent of δ such that I_δ has a critical value c_δ in $I_\delta^M \backslash I_\delta^{\epsilon_1}$. Thus

$$\epsilon_1 \leq c_\delta \leq M \tag{5.2}$$

independently of δ. Let q^δ be a critical point of I_δ corresponding to c_δ. The bounds (5.2) and the properties of V_δ lead to upper bounds depending only on ϵ_1 and M for

$$\sum_{i=1}^3 \| q_i^\delta - \frac{1}{2}[q_1^\delta + q_2^\delta] \|_{W^{1,2}} \tag{5.3}$$

and for

$$-\int_0^T V_\delta(q_\delta) dt. \tag{5.4}$$

These bounds enable us to let $\delta \to 0$ and find a subsequence of (q^δ) converging to a generalized T-periodic solution of

To prove Corollary 1.13, we use a standard argument. By Theorem 1.12, we have a generalized T-periodic solution q^1. By the assumption that $V'(q) \neq 0$ for $q \in (\mathbf{R}^\ell)^3$, $q^1 \not\equiv$

const.. Let T/k_1 denote its minimal period. Applying Theorem 1.12 again with T replaced by $T/(1+k_1)$, there exists a $T/(1+k_1)$ periodic solution q^2 having a minimal period $\leq \frac{T}{1+k_1}$. Clearly q^2 is geometrically distinct from q^1. Repeating this argument generates a sequence of geometrically distinct generalized T-periodic solutions of

REFERENCES

[1] *Poincaré, H., Les méthodes nouvelles de la mécanique céleste, Lib. Albert Blanchard. Paris, 1987.*

[2] *Bahri, A. and P. H. Rabinowitz. Periodic solutions of Hamiltonian systems of 3-body type, in progress.*

[3] *Greco, C., Periodic solutions of a class of singular Hamiltonian systems. Nonlinear Analysis, T.M.A.,* **12**, *(1988), 259-270.*

[4] *Bahri, A. and P. H. Rabinowitz, A minimax method for a class of Hamiltonian systems with singular potentials, J. Functional Anal.,* **82**, *(1989), 412-428.*

[5] *Bahri. A., work in preparation.*

[6] *Hirsch, M. W., Differential Topology, Springer-Verlag 1975.*

[7] *Bahri, A., Critical points at infinity in some variational problems, to appear, Pitman Research Notes in Mathematics.*

[8] *Sullivan, D. and M. Vigué-Poirrier, The homology theory of the closed geodesic problem, J. Diff. Geom.* **11**, *(1976), 633-644.*