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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

INVERSE SPECTRAL PROBLEM FOR THE SCHRÖDINGER OPERATOR WITH PERIODIC MAGNETIC AND ELECTRIC POTENTIALS.

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Consider the Schrödinger operator

$$H = \left(i\frac{\partial}{\partial x_1} + A_1(x)\right)^2 + (i\frac{\partial}{\partial x_2} + A_2(x))^2 + V(x), \tag{1}$$

where $x = (x_1, x_2) \in \mathbf{R}^2, A_1, A_2$ and V are periodic in x with respect to some lattice L. $\vec{A}(x) = (A_1(x), A_2(x))$ is called the magnetic (vector) potential, $V(x)$ is the electric (scalar) potential and

$$B(x) = \mathrm{curl}\vec{A} = \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} \tag{2}$$

is called the magnetic field.

The Schrödinger equation

$$H\psi = \lambda\psi \tag{3}$$

describes the spectrum of the electron in the periodic electromagnetic field (see [1]). Denote by $\mathrm{Spec}_0 H$ the periodic spectrum of H i.e. when the eigenfunctions $\psi(x)$ are periodic :

$$\psi(x + d) = \psi(x), \forall d \in L \ .$$

We shall study the following problem : **Recover** $B(x)$ **and** $V(x)$ **from** $\mathrm{Spec}_0 H$.

We shall assume that

$$\mathrm{div}\vec{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} = 0 \tag{4}$$

It follows from (2) that

$$\int\int_{T^2} B(x)dx_1dx_2 = 0 \ , \tag{5}$$

where $T^2 = \mathbf{R}^2/L$. Using (5) and the Fourier series expansions one can easily show that there are unique A_1, A_2 satisfying (2) and (4) and such that

$$\int\int_{T^2} A_k(x)dx_1dx_2 = 0, k = 1, 2 \ . \tag{6}$$

Therefore the problem of recovering $B(x)$ and $V(x)$ from Spec_0H is equivalent to the recovering of A_1, A_2 and $V(x)$ where A_1, A_2 satisfy (4) and (6). This work is a continuation of Eskin-Ralston-Trubowitz (see [2] and [3]) where the case $\vec{A} = 0$ was studied. We shall use some results and constructions from [2] and [3]. However the case when $\vec{A} \neq 0$ requires new methods.

Denote by $\mathrm{Spec}_k H$ the Floquet spectrum of H, i.e.

$$H\varphi_n(x) = \lambda_n(k)\varphi_n(x), \ k \in \mathbf{R}^2/L' \ , \tag{7}$$

where $\varphi_n(x + d) = e^{2\pi i k.d}\varphi_n(x), \forall d \in L$, L' is the dual lattice.

Repeating the proof of Theorem 6.2 in [2] we obtain :

Theorem 1.— *Assume that* $\vec{A}(x)$ *and* $V(x)$ *are real analytic and the lattice* L *has the following property :*

$$(8) \qquad |d| = |d'| \quad \textit{implies} \quad d = \pm d' \quad \textit{for any} \quad d, d' \in L\ .$$

Then $\mathrm{Spec}_0 H$ *determines* $\mathrm{Spec}_k H$ *for any* $k \in \mathbf{R}^2/L'$.

As in [2] denote by S the set of all "directions" in L' i.e. for any $\delta \in L'$ there is $\delta_0 \in S$ such that $\delta = m\delta_0$, $m \in \mathbf{Z}$ and $k\delta_0 \notin S$ for any $k \neq 1$.

Any periodic function $\vec{A}(x)$ has the following decomposition

$$(9) \qquad \vec{A}(x) = \sum_{\delta \in S} \vec{A}_\delta\left(\frac{x.\delta}{|\delta|}\right),$$

where

$$\vec{A}_\delta(s) = \sum_{n=-\infty}^{\infty} \vec{a}_{\delta n} e^{2\pi i n|\delta| s},$$

$$\vec{a}_{\delta n} = \frac{1}{|T^2|} \int\int_{T^2} \vec{A}(x) e^{-2\pi i n(\delta . x)} dx,$$

$|T^2|$ is the area of $T^2 = \mathbf{R}^2/L$.

Talk arbitrary $\delta_0 \in S$. There is a basis $(d_0, d^{(0)})$ in L such that $d_0.\delta_0 = 0, d^{(0)}.\delta_0 = 1$. Denote

$$(10) \qquad A_{\delta_0}(s) = \vec{A}_{\delta_0}(s).\frac{d_0}{|d_0|}.$$

The following theorem holds :

Theorem 2.— *Knowing the Floquet spectrum* $\mathrm{Spec}_k H$ *for all* $k \in \mathbf{R}^2/L$ *one can recover the following integrals*

$$(11) \qquad H_{\delta_0}(\mu) = \frac{1}{\sqrt{2}} \int_0^{|\delta_0|^{-1}} \frac{ds}{\sqrt{\mu + 4A_{\delta_0}(s)}}$$

for all $\delta_0 \in S$ *and* $\mu > -4\min_s A_{\delta_0}(s)$.

The proof of Theorem 2 is based on the study of the asymptotics of the Green function for the nonstationary Schrödinger equation

$$(12) \qquad i\frac{\partial G}{\partial x_0} = HG(x, y, x_0), x_0 > 0\ ,$$

$$(12') \qquad G(x, y, 0) = \delta(x - y), x \in \mathbf{R}^2, y \in \mathbf{R}^2\ .$$

As in [2] it is easy to find out that the Floquet spectrum of H determines the integrals $\int\int_{T^2} G(x+d,x,x_0)dx, dx_2$ for any $d \in L$. Indeed the trace formula gives

$$\sum_{n=1}^{\infty} e^{-i\lambda_n(k)x_0} = \int\int_{T^2} G_k(x,x,x_0)dx, \tag{13}$$

where $G_k(x,y,x_0)$ satisfies (12), (12') for $x \in T^2, y \in T^2$ and the Floquet boundary conditions

$$G_k(x+d,y,x_0) = e^{2\pi i d.k} G_k(x,y,x_0), \forall d \in L\ . \tag{14}$$

The Green function $G_k(x,y,x_0)$ can be represented in the form

$$G_k(x,y,x_0) = \sum_{d\in L} e^{-2\pi i d.k} G(x+d,y,x_0)\ . \tag{15}$$

Substituting (15) into (13) we get that $\int_{T^2}\int G(x+d,x,x_0)dx$ are the Fourier coefficients of $\sum_{n=1}^{\infty} \exp(-i\lambda_n(k)x_0)$. The main result of the work is the following theorem :

Theorem 3.— *The following asymptotics holds as $N \to \infty$:*

$$\int\int_{T^2} G(x+Nd_0+md^{(0)},x,\frac{\tau_0}{N_1})dx = \tag{16}$$

$$= -\frac{iN_1|d_0|}{\pi(2\tau_0^3)^{1/4}} \exp(-\frac{iN_1^3}{4\tau_0} - i\frac{N_1}{\sqrt{2\tau_0}} S_0(\sqrt{\frac{\tau_0}{2}}))(a_0(\tau_0) + 0(\frac{1}{N_1})),$$

where $N_1 = N|d_0| + m(\frac{d_0}{|d_0|}.d^{(0)})$, $m > 0$ is fixed, $N \to \infty$,

$$\frac{d}{d\tau} S_0(\tau_m) = E(\tau_m), \tag{17}$$

$$\int_0^{|\delta_0|^{-1}} \frac{ds}{\sqrt{E(\tau_m) + 4A_{\delta_0}(s)}} = \frac{\sqrt{2}\tau_m}{m}, \tau_m = \sqrt{\frac{\tau_0}{2}}, \tag{18}$$

τ_0 is arbitrary and sufficiently small. There is an explicit expression for $a_0(\tau_0)$ and the further terms in the asymptotic expansion (16) can be found.

The proof of the Theorem 3 consists of the following three steps :

1) Make change of variables

$$s = x.\frac{\delta_0}{|\delta_0|}, t = \frac{d_0}{|d_0|}.x \tag{19}$$

and substitute in (12)

$$G = e^{i\gamma(s,t)}g \tag{20}$$

with appropriate choice of $\gamma(s,t)$ such that $g(s,t,s',t',x_0)$ will satisfy

$$i\frac{\partial}{\partial x_0}g = -\frac{\partial^2 g}{\partial s^2} - \frac{\partial^2 g}{\partial t^2} + 2iA_{\delta_0}(s)\frac{\partial g}{\partial t} + 2iA_3(s,t)\frac{\partial g}{\partial s} + C(s,t)g, \tag{21}$$

where $A_{\delta_0}(s)$ is the same as in (10).

2) Construct $g(s,t,s',t',x_0)$ as a kernel of a Fourier integral operator with nonhomogeneous phase function :

$$g = \frac{1}{(2\pi)^2}\int\int a(x_0\wedge, s, t, \xi, \eta)e^{-i\wedge L(x_0\wedge, s, t, s', t', \xi, \eta)}d\xi d\eta, \tag{22}$$

where

$$\wedge = (\xi^2 + \varepsilon_0^4\eta^4 + \varepsilon_0^{-8})^{\frac{1}{4}}, \tag{23}$$

ε_0 is small and fixed, L satisfies the eiconal equation

$$L_\tau - L_s^2 - L_t^2 - 2\wedge^{-1}A_{\delta_0}(s)L_t - 2\wedge^{-1}A_3(s,t)L_s = 0, \tag{24}$$

$$L(0,s,t,s',t',\xi,\eta) = (s-s')\eta\wedge^{-1} + (t-t')\xi\wedge^{-1}, \tag{24'}$$

$\tau = x_0\wedge$, $a = a_0 + a_1 + \cdots + a_N$ where a_k satisfy corresponding transport equations.

3) Compute the trace of the Fourier integral operator by the stationary phase method. The stationary points form a curve corresponding to the whirling motion of the pendulum

$$\frac{d^2 s}{d\tau^2} - 4\frac{dA_{\delta_0}(s)}{ds} = 0\ . \tag{25}$$

Such curves are defined by their period τ_m and they satisfy the following conditions

$$s(\tau_m) = y + m|\delta_0|^{-1}, p(\tau_m) = \eta, \tag{26}$$

where $s(0) = y$, $p(0) = \eta$ are the initial conditions and $p(\tau) = \frac{1}{2}\frac{ds(\tau)}{d\tau}$. For each sufficiently small τ_m there is such a curve and $\tau_m \to 0$ as $E(\tau_m) \to \infty$ and vice versa. Here

$$E(\tau_m) = \frac{1}{2}(\frac{ds}{d\tau})^2 - 4A_{\delta_0}(s) = 2\eta^2 - 4A_{\delta_0}(y) \tag{27}$$

is the energy. It is enough to consider the case $m = 1$ since $\tau_m = m\tau_1$ and $E(\tau_m) = E(\tau_1)$. The Theorem 2 follows from (18) if we take $\mu = E(\tau_1)$ and denote by $H_{\delta_0}(\mu)$ the function inverse to $E(\tau_1)$.

Remark 1 The asymptotics of the integral $\int\int_{T^2} G(x+Nd_0+md^{(0)}, \frac{\tau_0}{N_1})dx$ is more difficult in the case $m = 0$. It requires a Maslov's type global construction of the Fourier integral operators. The stationary points in this case form curves corresponding to the periodic trajectories of the pendulum (25). We didn't consider the case $m = 0$ since the spectral invariants obtained by the asymptotics for $m = 0$ can be easily obtained from (11) by the analytic continuation in μ.

Now we shall apply Theorem 2 and 3 to the inverse spectral problem.

Note that $H_{\delta_0}(\mu)$ can be extended analytically to the whole complex plane μ with the cut along the real axis from $-\infty$ to $-4\min_s A_{\delta_0}(s)$. Using this analytic continuation we can find the following functions

(28)
$$H^{(1)}_{\delta_0}(\mu) = \int_{\mu+YA_{\delta_0}(s)<0} (-\mu - 4A_{\delta_0}(s))^{-\frac{1}{2}} ds \ ,$$

(29)
$$H^{(2)}_{\delta_0}(\mu) = \int_{\mu+4A_{\delta_0}(s)>0} (\mu + 4A_{\delta_0}(s))^{\frac{1}{2}} ds \ .$$

The spectral invariant $H^{(2)}_{\delta_0}(\mu)$ can be used to prove the following theorem :

Theorem 4.— *Assume that $A_{\delta_0}(s)$ is even and real analytic. Assume that $A_{\delta_0}(s)$ has $2m$ local maxima and minima : $0, s_0^{\pm}, \ldots, s_{m-1}^{\pm}$ where $s_{m-1}^{+} = \frac{1}{2}|\delta_0|^{-1}$ and $s_k^{-} = -s_k^{+}$. Assume that $A''_{\delta_0}(0) \neq 0$ and $A_{\delta_0}(0) \neq A_{\delta_0}(s_k^{\pm})$ for $0 \leq K \leq m-1$. Then there is at most $2m$ even real analytic functions having the same spectral invariant $H^{(2)}_{\delta_0}(\mu)$ as $A_{\delta_0}(s)$.*

Computing the first term in the asymptotic expansion (16) that depends on $V(x)$ and using Theorem 4 one can prove the following theorem on the rigidity of isospectral deformations :

Theorem 5.— *Let $\vec{A}^{(t)}(x) = (A_1^{(t)}(x), A_2^{(t)}(x))$ and $V^{(t)}(x)$ be continuous family of even real analytic magnetic and electric potentials, $0 \leq t \leq 1$. Assume that the lattice L satisfies the condition (8) and $A^{(0)}_{\delta_0}(s)$ for all $\delta_0 \in S$ satisfies the same conditions as in Theorem 4. Assume that the periodic spectrum of $H^{(t)}$ is independent of $t, 0 \leq t \leq 1$ where $H^{(t)}$ is the Schrödinger operator corresponding to $\vec{A}^{(t)}(x), V^{(t)}(x)$. Then $\vec{A}^{(t)}(x) = \vec{A}^{(0)}(x), V^{(t)}(x) = V^{(0)}(x)$ for all $t \in (0,1]$.*

The asymptotic formula (16) shows that some kind of quantum mechanical semiclassical asymptotics appears in the direction δ_0. The same semiclassical nature of the problem appears when one considers the asymptotics of eigenvalues for the operator H with periodic boundary conditions.

Let n be an integer, $n \to \infty$. Denote

$$\xi_n = \frac{2\pi}{|d_0|} n, h_n = \frac{1}{\sqrt{|\xi_n|}} = \sqrt{\frac{|d_0|}{2\pi n}}, h_n \to 0 \ . \tag{30}$$

Let $\mu_{m,n}$ be the approximative eigenvalues for the semiclassical eigenvalue problem

$$-h_n^2 \frac{d^2\varphi(s)}{ds^2} + (\mu_{m,n} - 2A_{\delta_0}(s) + h_n^2 C_{\delta_0}(s))\varphi(s) = 0(h_n^N), \tag{31}$$

where $h_n \to 0$ and $C_{\delta_0}(s)$ has the same relation to $C(s,t)$ as $\vec{A}_\delta(s)$ to $\vec{A}(x)$ (see (9) and (21)).

It is known (see [4]) that

$$\mu_{m,n} = \mu_{m,n,0} + h_n^2 \mu_{m,n,1} + \dots , \tag{32}$$

where μ_{mn0} satisfies the Bohr-Sommerfeld quantization condition

$$\int_{2A_{\delta_0}(s) > \mu_{m,n,0}} (2A_{\delta_0}(s) - \mu_{m,n,0})^{\frac{1}{2}} ds = \pi h_n (m + \frac{1}{2}), m \in \mathbf{Z} \ . \tag{33}$$

Theorem 6.— *Let $\lambda_{m,n} = \xi_n^2 - \xi_n \mu_{m,n}$. There exists a subsequence $\lambda_{m,n}^*$ in* $\mathrm{Spec}_0 H$ *such that*

$$|\lambda_m^* - \lambda_m| \leq C \ h_n \tag{34}$$

for all n sufficiently large and m such that

$$0 < C_1 \leq (m + \frac{1}{2}) h_n \leq C_2 \ . \tag{35}$$

The approximative eigenfunctions (quasimodes) have the following form

$$(H - \lambda_{m,n})[e^{i\gamma(s,t) + i\xi_n t}(w_{m,n}(s) + h_n w_{m,n,1}(s,t) + h_n^2 w_{m,n,2}(s,t))] = 0(h_n), \tag{36}$$

where $w_{m,n}(0) = 1$, $w_{m,n,k}(s,t) = 0(1), k = 1, 2, \gamma(s,t)$ is the same as in (20).

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