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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

CR MAPPINGS BETWEEN REAL HYPERSURFACES IN COMPLEX SPACE.

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Let M be (a germ) of a smooth real hypersurface in $\mathbf{C}^{n+1}$ containing the origin defined by $\rho(Z,\bar{Z})=0$, where ρ is a smooth real valued function satisfying $\rho(0,0)=0,\ d\rho(0)\neq 0$. We may assume $\frac{\partial\rho}{\partial Z_{n+1}}(0)\neq 0$. By a CR function h defined on M we mean a germ of a smooth function h at 0 satisfying $L_j h=0,\ j=1,\ldots,n$, with

$$L_j=\frac{\partial}{\partial\bar{Z}_j}-\frac{\rho_{\bar{Z}_j}(Z,\bar{Z})}{\rho_{\bar{Z}_{n+1}}(Z,\bar{Z})}\frac{\partial}{\partial\bar{Z}_{n+1}}.$$

If M' is another hypersurface of $\mathbf{C}^{n+1}$, a smooth mapping $H:M\to M'$, $H(0)=0$, is called CR if $H=(h_1,\ldots,h_{n+1})$, where the h_j's are CR functions defined on M. We shall give some local geometric and analytic properties of such mappings. We refer to [4] and [5] for complete details.

If M is real analytic, after a holomorphic change of coordinates we can assume that

$$\rho(Z,0)=\alpha(Z)Z_{n+1}\ ,\ \alpha(0)\neq 0. \tag{1}$$

If M is only smooth such a change of variables can be done formally (i.e. in formal power series of Z). If (1) is satisfied we say that Z_{n+1} is a **transversal (holomorphic or formal) coordinate for** M. If $Z'=(Z_1',\ldots,Z_{n+1}')$ are coordinates in $\mathbf{C}^{n+1}$ such that Z_{n+1}' is transversal to M', and if $H=(h_1,\ldots,h_{n+1})$ is a CR map from M to M' given by $Z_j'=h_j(Z)$, for $Z\in M$, we say that h_{n+1} is a **transversal component** of H.

If j is a CR function defined on M, we associate to j a formal power series $J(Z)$, $Z=(Z_1,\ldots,Z_{n+1})$, such that the Taylor series of j at 0 coincides with $J(Z)_{|M}$. If Z_{n+1} is transversal to M, we write (z,w) instead of Z (i.e. $w=Z_{n+1}$), $z\in\mathbf{C}^n$, $w\in\mathbf{C}$. Similarly if h_{n+1} is a transversal component of H, we write $H=(f,g)$, $f=(f_1,\ldots,f_n)$ (i.e. $h_j=f_j, 1\leq j\leq n$, $h_{n+1}=g$), and $F_j(z,w)$, $G(z,w)$ the associated formal power series. It follows from (1) that

$$G(z,w)=w\ G_1(z,w), \tag{2}$$

where $G_1(z,w)$ is another power series.

If H is a CR mapping as above then it is said to be of finite multiplicity if

$$\dim_{\mathbf{C}}\mathcal{O}[[z]]/(F(z,0))<\infty, \tag{3}$$

where $\mathcal{O}[[z]]$ is the ring of formal power series in n indeterminates $z_1,\ldots,z_n$ and $(F(z,0))$ is the ideal generated by $(F_1(z,0),\ldots,F_n(z,0))$. The number given by the left hand side of (3) is called the **multiplicity** of H at 0.

As in Baouendi-Jacobowitz-Treves [3] in the real analytic case, and D'Angelo [1] in the smooth case, we say that M is **essentially finite** at 0 if

$$\dim_{\mathbf{C}}\mathcal{O}[[z]]/(a_\alpha(z))<\infty, \tag{4}$$

with $\rho(z,0,\zeta,0)=\sum_\alpha a_\alpha(z).\zeta^\alpha$, and $(a_\alpha(z))$ the ideal generated by the power series $a_\alpha(z)$ for all $\alpha\in\mathbf{Z}_+^n$. Note that it follows from (1) that $a_\alpha(0)=0$. The number given by the left hand side of (4) is called the **essential type** of M at 0 and is denoted by ess. type ${}_0M$.

We are now ready to state our main results.

Theorem 1.— *Let $H : M \to M'$ be a smooth CR mapping defined near 0, with M and M' C^∞ hypersurfaces in $\mathbf{C}^{n+1}$. Let w be any (formal) transversal coordinate for M and G any (formal) transversal coordinate of H. Assume that M is essentially finite at 0.*

(i) If $G \equiv 0$ then either H is not of finite multiplicity at 0, or M' is not essentially finite at 0.

(ii) If $G \not\equiv 0$ then $\frac{\partial G}{\partial w}(0) \neq 0$, H is of finite multiplicity and M' is essentially finite.

In addition, if M and M' are real analytic and H is holomorphic, then $G \not\equiv 0$ if and only if H maps any neighborhood of 0 in M onto a neighborhood of 0 in M'.

Theorem 2.— *Let $H : M \to M'$ be a smooth CR map if, either M is essentially finite and $G \not\equiv 0$, or M' is essentially finite and H of finite multiplicity then*

$$\text{(5)} \qquad \textit{ess. type}_0 M = (\text{mult}_0 H) \times (\textit{ess. type}_0 M'),$$

with all three integers in (5) being finite.

The proofs of Theorems 1 and 2 could be found in [4] and [5]. Several tools of commutative algebra such as the Nullstellensatz and Nakayama's lemma are used in these proofs.

If $M, M' \subset \mathbf{C}^{n+1}$ are real analytic and $H : M \to M'$ is a smooth CR map, we are interested in the following question : when is H the restriction of a (local) holomorphic mapping in $\mathbf{C}^{n+1}$? Several results could be found in the literature starting with Lewy [9] and Pincuk [10] when M and M' are strictly pseudoconvex and H is a diffeomorphism. Recent results closely related to ours (Theorem 3) have been independently proved by Diederich and Fornaess [8].

Before stating our extension results we need to introduce another definition. If $H = (f_1, \dots, f_n, g)$ is a CR map as above, with g a transversal imponent, and if (z, w) are coordinates for M such that w is transversal to M, we say that H is **totally degenerate** if

$$\text{(6)} \qquad \det\left(\frac{\partial F_j}{\partial z_k}(z,0)\right) = 0 \ ,$$

i.e. the formal power series defined by the left hand side of (6) is 0. We have the following result.

Theorem 3.— *Let $H : M \to M'$ be a smooth CR map, $H(0) = 0$, where M and M' are real analytic hypersurfaces in $\mathbf{C}^{n+1}$, and g a transversal CR component. Then H extends holomorphically to a neighborhood of 0 in $\mathbf{C}^{n+1}$ if any of the following conditions holds :*

(i) M is essentially finite and g is not flat at 0.

(ii) M' is essentially finite and H is of finite multiplicity at 0.

(iii) M' is essentially finite and H is not totally degenerate at 0.

Note that it follows from Theorems 1 and 2 that (i) $\Leftrightarrow$ (ii). We can also show that (i) and (ii) imply (iii). However condition (iii) is weaker than (i) and (ii) as is shown by the following example. Let M and M' be embedded in $\mathbf{C}^3$ given by $M = \{(z,w) : \operatorname{Im} w = |z_1|^2 + |z_1 z_2|^2\}$, and $M' = \{(z',w') : \operatorname{Im} w' = |z'_1|^2 + |z'_2|^2\}$, and $H = (f_1, f_2, g)$ with $f_1(z,w) = z_1$, $f_2(z,w) = z_1 z_2$ and $g = w$. Here M' is essentially finite, M is of finite type (but not essentially finite), H is not totally degenerate but not of finite multiplicity at 0.

When $n = 1$, (i.e. $M, M' \subset \mathbf{C}^2$), then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) ; in this case Theorem 3 was proved by the authors jointly with S. Bell [2]. Theorem 3 generalizes the result of [3] which deals with the diffeomorphic case. A complete proof could be found in [4] and [5].

We give some corollaries of Theorem 3.

Corollary 1.— *Let $\mathcal{H} : D \to D'$ be a proper holomorphic mapping between two bounded domains in $\mathbf{C}^{n+1}$ with real analytic boundaries. If $\mathcal{H} \in C^\infty(\bar{D})$, and if at every $p \in \partial D$ a transversal component of $\mathcal{H}$ at p is not flat at p, then $\mathcal{H}$ extends as a proper holomorphic mapping from a neighborhood of $\bar{D}$ into a neighborhood of $\bar{D}'$.*

Using the result of Bell-Catlin [6] and Diederich-Fornaess [7] Corollary 1 yields :

Corollary 2.— *If $\mathcal{H} : D \to D'$ is a proper holomorphic mapping between two bounded pseudoconvex domains in $\mathbf{C}^{n+1}$ with real analytic boundaries, then the conclusion of Corollary 1 holds.*

Several other corollaries of Theorems 1, 2 and 3 could be found in [4] and [5].

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