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0. INTRODUCTION

In this paper, which is a condensed version of a paper which is to appear in Arkiv för Matematik, we study the propagation of singularities for a class of pseudo-differential operators having characteristics of variable multiplicity. We do not assume the characteristics to be in involution, in the sense that their Hamilton fields satisfy the Frobenius integrability condition. Instead, we assume that the characteristic set is a union of hypersurfaces tangent of exactly order $k_0 \geq 1$ along an involutive submanifold of codimension $d_0 \geq 2$. This means that the Hamilton fields are parallel at the intersection, and their Poisson brackets vanish of at least order $k_0 + 1$ there. We also assume a version of the generalized Levi condition. One example, with $k_0 = 1$, is the wave operator for uniaxial crystals, i.e. trigonal, tetragonal and hexagonal crystals. The main result is stated in Theorem 1.3, and it shows that the wave front set of the solution is propagated along the union of the Hamilton fields of the characteristic surfaces.

There have been many studies of singularities of solutions of symmetrizable hyperbolic systems, see [**15**] and references there. Nosmas [**12**] has studied the involutive case. Kumano-go and Taniguchi [**8**] have constructed parametrices for diagonalizable systems, but since they consider classical symbols, their results are not directly applicable here. The results on the propagation of singularities for the system in Proposition 2.3 may be obtained by the method of energy estimates of Ivrii [**6**] (see also [**16**]). For scalar operators, the case when the characteristics have transversal involutive self-intersection has been analyzed in [**1**], [**9**], [**13**] and [**14**]. Melrose and Uhlmann [**10**] considered the case of conical involutive singularity of the characteristic set. Morimoto [**11**] studied operators on the form (2.10) below, but with involutive characteristics. Ivrii [**7**] considered operators with L^∞ bounds on the Poisson brackets at double characteristic points.

In this paper, we shall consider classical, or polyhomogeneous, pseudo-differential operators. These have symbols which are asymptotic sums of homogeneous terms. But we shall also use the more general symbol classes of the Weyl calculus. Since all our metrics are split, we may use the standard calculus of pseudo-differential operators with these symbol classes. For notation and calculus results, see [**5**, Chapter 18].

1. Statement of result

We are going to study the pseudodifferential operator $P \in \Psi^m_{phg}(X)$ on a C^∞ manifold X. Let $p = \sigma(P)$ be the principal symbol and $\Sigma = p^{-1}(0)$ the characteristic set. Assume, microlocally near $(x_0, \xi_0) \in \Sigma$,

$$\Sigma = \bigcup_{j=1}^{r_0} S_j, \quad r_0 \geq 2, \quad \text{where } S_j \text{ are non-radial hypersurfaces} \\ \text{tangent at } \Sigma_2 = \bigcap_{j=1}^{r_0} S_j \text{ of exactly order } k_0 \geq 1. \tag{1.1}$$

This means that the Hamilton field of S_j does not have the radial direction $\langle \xi, \partial_\xi \rangle$. Also, the k_0:th jets of S_j coincide on Σ_2, but no $k_0{+}1$:th jet does, and the surfaces only intersect at Σ_2 in a neighborhood of (x_0, ξ_0). Observe that the surfaces need not be in involution, in the sense that their Hamilton fields satisfy the Frobenius integrability condition. Since p is homogeneous in ξ, Σ_i and S_j are conical. Next we assume, microlocally near (x_0, ξ_0),

$$\Sigma_2 \text{ is an involutive manifold of codimension } d_0 \geq 2, \\ \text{and } \Pi(\Sigma_2) = X, \text{ where } \Pi \text{ is the projection: } T^*(X) \to X. \tag{1.2}$$

Clearly the codimension cannot be equal to 1, and by non-degeneracy Σ_2 is a manifold near (x_0, ξ_0). In order to obtain conditions on lower order terms of P on the multiple characteristic set we assume the following version of the Levi condition. For $j = 1, \ldots, r_0$ there exist $m_j \in \mathbf{N}$, with the property that, if $\varphi_j \in C^\infty$, $(x, d_x\varphi_j) \in S_j$ near x_0, and $d_x\varphi_j(x_0) = \xi_0$, then

$$\left| e^{-i\varrho\varphi_j} P\left(e^{i\varrho\varphi_j} a\right) \right| \leq C(1 + \varrho\, d^{k_0+1}(x, d_x\varphi_j))^{m_0 - m_j}(1+\varrho)^{m-m_0}, \quad \varrho \to \infty, \tag{1.3}$$

$\forall\, a \in C_0^\infty$ supported near x_0. Here $m_0 = \sum_{j=1}^{r_0} m_j$, and $d(x, \xi)$ is the homogeneous distance to Σ_2, i.e. the distance with respect to the metric $|dx|^2 + |d\xi|^2/(1+|\xi|^2)$. This means that p vanishes of order m_j at $S_j \setminus \Sigma_2$, of order m_0 at Σ_2, and P satisfies the Levi condition on S_j and Σ_2 (see [**2**]). We also have uniform conditions on lower order terms on $\Sigma_1 = \Sigma \setminus \Sigma_2$ when approaching Σ_2. In order to avoid extra zeroes of the principal symbol at Σ_2, we assume

$$d^{m_0} p \neq 0 \text{ at } \Sigma_2, \quad m_0 = \sum_{j=1}^{r_0} m_j, \tag{1.4}$$

microlocally near (x_0, ξ_0), where $d^k p$ is the k:th differential of p.

Clearly, (1.1), (1.2) and (1.4) are invariant under multiplication with elliptic pseudodifferential operators and conjugation by elliptic Fourier integral operators corresponding to canonical transformations preserving the projection condition: $\Pi(\Sigma_2) = X$.

LEMMA 1.1. *Condition (1.3) is invariant under multiplication of P with elliptic pseudo-differential operators and conjugation of P by elliptic Fourier integral operators corresponding to canonical transformations preserving the projection condition.*

The proof follows by using the stationary phase (see [**5**, Th. 7.7.1 and 7.7.6]) when conjugating with elliptic Fourier integral operators, since $k_0 \geq 1$.

We shall now state the result for propagation of singularities for P. Since the surfaces are tangent at Σ_2, their Hamilton fields are parallel. Because Σ_2 is involutive and $\Sigma_2 = \bigcap S_j$, the Hamilton fields of S_j are tangent to Σ_2, and they define the same flow there.

DEFINITION 1.2. *The Hamilton flow on Σ is the union of the Hamilton flow on S_j, $j = 1, \ldots, r_0$.*

The following is the main result.

THEOREM 1.3. *Assume that $P \in \Psi^m_{phg}(X)$ satisfies (1.1)–(1.4) microlocally near $w \in \Sigma$. If $u \in \mathcal{D}'(X)$, then $WFu \setminus WFPu$ is invariant under the Hamilton flow on $\Sigma = p^{-1}(0)$ near w.*

On Σ_1 this follows from the fact that the characteristics have constant multiplicity, see [**2**, Th. 1.1]. Theorem 1.3 will be proved in section 5.

2. REDUCTION TO A FIRST ORDER SYSTEM

We assume $P \in \Psi^m_{phg}(X)$ satisfies (1.1)–(1.4) microlocally near $w \in \Sigma_2$. Since the result is local and the conditions are invariant, we may assume $X = \mathbf{R}^n$. Because Σ_2 is involutive and $\Pi(\Sigma_2) = X$, we may choose symplectic, homogeneous coordinates $(x, \xi) \in T^*\mathbf{R}^n$ near $w \in \Sigma_2$, so that $w = (0; (0, \ldots, 1))$ and

$$\Sigma_2 = \{(x,\xi) \in T^*\mathbf{R}^n : \ \xi' = 0\}, \tag{2.1}$$

where $\xi = (\xi', \xi'') \in \mathbf{R}^{d_0} \times \mathbf{R}^{n-d_0}$. We may also assume

$$S_1 = \{(x,\xi) \in T^*\mathbf{R}^n : \ \xi_1 = 0\}, \tag{2.2}$$

near w. We rename $x_1 = t$, $(x_2, \ldots, x_{d_0}) = x'$ and $(x_{d_0+1}, \ldots, x_n) = x''$. Since S_j is tangent to S_1 at Σ_2, we obtain

$$S_j = \{(t, x; \tau, \xi) \in T^*(\mathbf{R} \times \mathbf{R}^n) : \ \tau + \beta_j(t, x, \xi) = 0\}, \tag{2.3}$$

with β_j real and homogeneous of degree 1 in ξ, $\beta_1 \equiv 0$, and

$$c|\xi'|^{k_0+1}/|\xi|^{k_0} \leq |\beta_j - \beta_k| \leq C|\xi'|^{k_0+1}/|\xi|^{k_0}, \quad j \neq k, \quad C, c > 0, \tag{2.4}$$

in a conical neighborhood of w. By taking $k = 1$, we obtain that β_j vanishes of exactly order $k_0 + 1$ at $\{\xi' = 0\}$.

Next, we prepare $P \in \Psi^m_{phg}(X)$. Assume P to be given by the expansion $p + p_{m-1} + p_{m-2} + \cdots$, where $p = \sigma(P)$ and $p_j \in S^j$. Conditions (1.3) (with $\varphi_1 = t$) and (1.4) give

$\partial_\tau^j p = 0$ at Σ_2 when $j < m_0$, and $\partial_\tau^{m_0} p \neq 0$, near $w \in \Sigma_2$. Thus Malgrange's preparation theorem gives, by homogeneity (see [**5**, Th. 7.5.5]),

$$p = c \sum_{j=0}^{m_0} a_{m_0 - j} \tau^j \qquad \text{near } \; w \in \Sigma_2,$$

where $0 \neq c \in S^{m-m_0}$, $a_j \in C^\infty(\mathbf{R}, S^j)$ are homogeneous in ξ, $a_0 \equiv 1$ and $a_j = 0$ at Σ_2, $j > 0$. By multiplication with an elliptic pseudo-differential operator, we may assume $m = m_0$ and $c \equiv 1$. By using Malgrange's preparation theorem repeatedly, we get

$$P \cong \sum_{j=0}^{m_0} A_{m_0-j} D_t^j \quad \text{mod } C^\infty, \quad \text{microlocally near } w, \tag{2.5}$$

where $A_j \in C^\infty(\mathbf{R}, \Psi^j_{phg})$ and $A_0 \equiv 1$. Now (1.3) gives more information about A_j, but we first have to introduce some symbol classes corresponding to the β_j's.

Let

$$m(\xi) = 1 + |\xi'|^{k_0+1} \langle \xi \rangle^{-k_0}, \tag{2.6}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, thus $m \approx 1 + \beta_j$. Put

$$g(dx, d\xi) = |dx|^2 + |d\xi'|^2 / (\langle \xi \rangle^\mu + |\xi'|)^2 + |d\xi''|^2 / \langle \xi \rangle^2 \quad \text{at } (x, \xi), \tag{2.7}$$

where $\mu = k_0/(k_0+1)$, which gives $h^2 = \sup g/g^\sigma = (\langle \xi \rangle^\mu + |\xi'|)^{-2} \le 1$. It is easy to see that g is σ temperate, and $m \approx \langle \xi \rangle^{-k_0} h^{-k_0-1}$ is a weight for g. We shall denote by $S(mh^j, g)$ the symbol classes in (x, ξ) of weight mh^j, $j \in \mathbf{Z}$, depending C^∞ on t, and $Op\, S(mh^j, g)$ the corresponding (classical) pseudo-differential operators. (Thus we shall suppress the t dependence.) The reason for using these classes is that $\beta_j \in S(m, g)$. Also, if $a(t, x, \xi)$ is homogeneous of degree j in ξ and $|a| \le c\, m^k$, then $a \in S(m^j, g)$. In fact, if $k < j$, then $a \equiv 0$, otherwise a vanishes of order $\ge j(k_0+1)$ at Σ_2.

LEMMA 2.1. *Assume that P is given by (2.5) and satisfies (1.1)–(1.4) with $m = m_0$ and $S_1 = \{\tau = 0\}$, near $w \in \Sigma_2$. Then $A_i \in Op\, S(m^i, g)$ and*

$$b_j = e^{-i\varphi_j} P(e^{i\varphi_j} a) \in S(m^{m_0 - m_j + r}, g) \quad \text{near } (t_0, x_0, \xi_0), \tag{2.8}$$

for all $a \in S(m^r, g)$, if $\varphi_j(t, x, \xi)$ is homogeneous of degree 1 in ξ, $(t, x, d_{t,x}\varphi_j) \in S_j$ near (t_0, x_0, ξ_0), $(t_0, x_0, d_{t,x}\varphi_j(t_0, x_0, \xi_0)) = w$, and $(t, x, d_{t,x}\varphi_j) \in \Sigma_2$ when $\xi' = 0$.

PROOF: We obtain φ_j satisfying the conditions in the lemma by solving (3.3), according to Lemma 3.1. To compute (2.8) for homogeneous a, we may use the formal expansion in Lemma A.1, and homogeneity, to find

$$b_j \cong \sum_{k \ge 0} L_k(P, \varphi_j) a \qquad \text{mod } S^{-\infty},$$

since $h \leq \langle\xi\rangle^{-\mu}$. Here $L_k(P, s\varphi_j) = s^{m_0-k} L_k(P, \varphi_j)$ is differential operator of order k in (t, x), with principal symbol

$$\sigma(L_k(P, \varphi_j))(\varrho, \eta) = \sum_{|\alpha|=k} (\partial^\alpha_{\tau,\xi} p)(t, x, d_{t,x}\varphi_j)(\varrho, \eta)^\alpha / k!\,.$$

Applying this to $a \in S(1, g)$, homogeneous of degree 0 in ξ, (1.3) gives that $L_k(P, \varphi_j) \equiv 0$ when $k < m_j$, and that all coefficients of $L_k(P, \varphi_j)$ are bounded by $c\, m^{m_0 - m_j}$ when $k \geq m_j$ (since $m = m_0$). By homogeneity, all coefficients of $L_k(P, \varphi_j)$ are in $S(m^{m_0-k}, g)$ when $k \geq m_j$. Observe that this implies that p vanishes of order m_j at S_j, and $\partial^\alpha_{\tau,\xi} p\big|_{S_j} \in S(m^{m_0-|\alpha|}, g)$, $\quad |\alpha| \geq m_j$, near w.

By induction we obtain that p_{m_0-i} vanishes of order $(m_j - i)_+ = \max(m_j - i, 0)$ at S_j, and

$$\partial^\alpha_{\tau,\xi} p_{m_0-i}\Big|_{S_j} \in S(m^{m_0-i-|\alpha|}, g), \quad |\alpha| \geq m_j - i. \tag{2.9}$$

By using the expansion (A.4) for general a, we get (2.8). We obtain $A_i \in Op\, S(m^i, g)$, by using (2.9) for $j = 1$ (i.e. $\tau = 0$).

LEMMA 2.2. *Assume that P satisfies the conditions in Lemma 2.1. Then we can find A, $A_I \in Op\, S(1, g)$, $I = (i_1, \dots, i_{r_0}) \in \mathbf{N}^{r_0}$, so that $\sigma(A) \equiv 1$ and*

$$P = A \prod_{j=1}^{r_0} Q_j^{m_j} + \sum_{\substack{|I|<m_0 \\ i_j \leq m_j}} A_I \prod_{j=1}^{r_0} Q_j^{i_j}\,, \tag{2.10}$$

microlocally near $w \in \Sigma_2$. Here $Q_j = D_t + B_j$, $B_j \in Op\, S(m, g)$ and $\sigma(B_j) = \beta_j$.

PROOF: Observe that the products in (2.10) are commutative modulo lower order terms. We find that $\sigma(P) = p = \prod q_j^{m_j}$, where $q_j = \sigma(Q_j)$, since it is a monic polynomial of degree m_0 in τ, vanishing of order m_j at $\tau = -\beta_j$. We shall consider the cases $|\xi'| \gtrless c\langle\xi\rangle^\mu$, by using a partition of unity in $S(1, g)$. When $|\xi'| \leq c\langle\xi\rangle^\mu$, we find $S(m^k, g) \subset S(1, g)$, $\forall k$. Replacing D_t^k by $\prod Q_j^{k_j}$, where $\sum k_j = k$ and $k_j \leq m_j$, only changes terms of lower order in D_t. Thus we only have to consider $|\xi'| \geq c\langle\xi\rangle^\mu$. The result will follow if we can write p_{m_0-k}, $k > 0$, on the form

$$p_{m_0-k} = \sum_{\substack{0 \leq i_j \leq m_j \\ |I|<m_0}} a_I^k \prod_j q_j^{i_j}\,, \qquad a_I^k \in S(1, g), \tag{2.11}$$

when $|\xi'| \geq c\langle\xi\rangle^\mu$.

The proof of Lemma 2.1 implies that p_{m_0-k} vanishes of order $(m_j - k)_+$ at $\{\tau = -\beta_j\}$. Since p_{m_0-k}/p is rational in τ, residue calculus gives

$$p_{m_0-k}/p = \sum_{\substack{1 \leq i \leq \min(m_j, k) \\ j}} a^j_{ki} (q_j)^{-i}.$$

By (2.9) and the fact that $q_i^{-1}\big|_{\tau=-\beta_j} = (\beta_i - \beta_j)^{-1} \in S(m^{-1}, g)$ when $|\xi'| \geq c\langle\xi\rangle^\mu$, we find $a^j_{ki} \in S(1, g)$ when $|\xi'| \geq c\langle\xi\rangle^\mu$. This proves (2.11) and the lemma.

Now it is simple to reduce (2.10) to a first order diagonalizable system. See Morimoto [11] for the details. Summing up, we obtain the following result.

Proposition 2.3. *Assume that $P \in \Psi^m_{phg}$ satisfies (1.1)–(1.4). Then, by conjugation with elliptic Fourier integral operators and multiplication by elliptic pseudo-differential operators, the equation $Pu = f$, $u \in \mathcal{D}'(X)$, can be reduced to the $N_0 \times N_0$ system*

$$D_t U + K(t, x, D_x)U = F, \tag{2.12}$$

microlocally near $w \in \Sigma_2$. Here $WF F = WF f$, $WF U = WF u$, $N_0 = \sum_{j=1}^{m_0} m_0!/j!$, and $K \in Op\, S(m, g)$ with principal symbol

$$k_1(t, x, \xi) = (\delta_{jk}\beta_{i_k})_{j,k=1,\dots,N_0} \tag{2.13}$$

being diagonal matrix, with real eigenvalues $\beta_i \in S(m, g)$ homogeneous of degree 1 in ξ, satisfying (2.4), and $\beta_1 \equiv 0$.

3. The Cauchy Problem

We shall study the Cauchy problem for the $N_0 \times N_0$ system

$$P = D_t Id_{N_0} + K(t, x, D_x), \tag{3.1}$$

having the properties in Proposition 2.3. Let π_j be the projection on the eigenvectors corresponding to the eigenvalue β_j, along the others, thus $k = \sum_{j=1}^{r_0} \beta_j \pi_j$. We are going to solve

$$\begin{cases} PE \cong 0 \\ E\big|_{t=0} \cong Id_{N_0} \end{cases} \tag{3.2}$$

microlocally near $(0, (0, \xi_0), (0, \xi_0))$, $\xi_0' = 0$, with E: $\mathcal{E}'(\mathbf{R}^{n-1}) \to \mathcal{D}'(\mathbf{R}^n)$. We shall use Lax' method of oscillatory solutions. In order to do that, we must solve the eiconal equations

$$\begin{cases} \partial_t \phi_j + \beta_j(t, x, d_x\phi_j) = 0 \\ \phi_j(0, x, \eta) = \langle x, \eta\rangle \end{cases} \quad \text{for } j = 1, \dots, r_0. \tag{3.3}$$

By Hamilton-Jacobi, this has a unique local solution, homogeneous of degree 1 in η.

Lemma 3.1. *Let ϕ_j solve (3.3) with β_j satisfying the conditions in (2.4), and $\beta_1 \equiv 0$. Then we find that $\varphi_j(t, x, \eta) = \phi_j(t, x, \eta) - \langle x, \eta\rangle$ satisfies*

$$\partial^\gamma_{\eta'}\varphi_j \equiv 0 \qquad \text{when } \eta' = 0, \quad |\gamma| \le k_0, \quad \forall j. \tag{3.4}$$

Idea of proof: Clearly (3.3) gives $\varphi_j \equiv 0$ when $\eta' = 0$. Successively differentiating the equation, we find that $\partial_t \partial^\gamma_{\eta'}\varphi_j \equiv 0$ when $\eta' = 0$ and $\gamma \le k_0$.

Now we define E_j: $\mathcal{E}'(\mathbf{R}^{n-1}) \to \mathcal{D}'(\mathbf{R}^n)$, $j = 1, \dots, r_0$, as oscillatory integrals

$$E_j u(t, x) = (2\pi)^{1-n} \iint e^{i(\phi_j(t,x,\eta) - \langle y, \eta\rangle)} a_j(t, x, \eta) u(y)\, dy d\eta, \tag{3.5}$$

with $a_j \in S(1,g)$. Assume that a_j is supported in a conical neighborhood of $\{\eta' = 0\}$. By Lemma A.1 in the appendix, we get

$$PE_j u(t,x) = (2\pi)^{1-n} \iint e^{i(\phi_j(t,x,\eta)-\langle y,\eta\rangle)} b_j(t,x,\eta) u(y)\, dy d\eta, \tag{3.6}$$

where

$$b_j(t,x,\eta) = (\partial_t \phi_j Id_{N_0} + k(t,x,d_x\phi_j))a_j + L_j a_j + R_j a_j, \tag{3.7}$$

R_j is continuous $S(m^i h^l, g) \to S(m^i h^{l+1}, g)$, $\forall i,j,l$, and

$$L_j a_j = D_t a_j + \sum_i (\partial_{\xi_i} k)(t,x,d_x\phi_j) D_{x_i} a_j + M_j a_j,$$

with $M_j \in S(1,g)$. In general, we cannot find homogeneous a_j making $b_j \in S^{-\infty}$. However, we have the following result.

LEMMA 3.2. *Assuming (2.13), we can find $a_j \in S(1,g)$ such that $b_j \in S(m^{-N},g)$, $\forall N$, in (3.7), $j = 1,\dots,r_0$, and*

$$\sum_j a_j\big|_{t=0} \equiv Id_{N_0}. \tag{3.8}$$

PROOF: Let $a_j \sim a_j^0 + a_j^{-1} + \cdots$, where $a_j^{-k} \in S(m^{-k},g)$. The term in $S(m^{-r},g)$, $r \geq 0$, in the expansion (3.7), is given by

$$\sum_i \phi_j^*(\beta_i - \beta_j)\pi_i a_j^{-1-r} + L_j a_j^{-r} + R_j a_j^{1-r},$$

where $\phi_j^* f = f(t,x,d_x\phi_j)$, since $h \leq m^{-1}$ $(a_j^1 \equiv 0)$. For $r = -1$, we obtain $a_j^0 \in \operatorname{Im} \pi_j = \bigcap_{i\neq j} \operatorname{Ker} \pi_i$. If we take $a_j^0 = \pi_j$ at $t = 0$, we obtain $\sum a_j^0\big|_{t=0} = Id_{N_0}$. Now $\phi_j^*(\beta_i - \beta_j) \in S(m,g)$ is invertible modulo $S(m^{-1},g)$ according to (2.4), when $j \neq i$, since $d_{x'}\phi_j = \mathcal{O}(|\eta'|)$ by (3.4). Thus, it suffices to solve successively, with suitable initial data,

$$\pi_j(L_j a_j^{-r} + \widetilde{R}_j a_j^{1-r}) = 0, \quad r \geq 0, \tag{3.9}$$

where $a_j^1 \equiv 0$, and $(Id_{N_0} - \pi_j)a_j^{-r}$ has been determined in the previous step. Here $\widetilde{R}_j$ is continuous $S(m^i,g) \to S(m^{i-1},g)$, $\forall i$.

Now fix j, let $\{v_j^i\}_i$ be a base for Im π_j, and consider $\sum_i \alpha_i v_j^i$, $\alpha_i \in S(m^{-r},g)$. We obtain $\pi_j L_j \sum_i \alpha_i v_j^i = \sum_i \gamma_i v_j^i$, where

$$\gamma_i = D_t \alpha_i + \sum_l \phi_j^*(\partial_{\xi_l}\beta_j) D_{x_l} \alpha_i + \sum_l \mu_i^l \alpha_l \in S(1,g), \tag{3.10}$$

with $\mu_i^l \in S(1,g)$. If we introduce local g orthogonal coordinates, then $\sum_l \phi_j^*(\partial_{\xi_l}\beta_j) D_{x_l}$ transforms into a uniformly bounded C^∞ vector field. Thus, by adding a suitable linear combination of v_j^i to each column of a_j^{-r} we may solve (3.9) for all $1 \leq j \leq r_0$, with initial data making (3.8) hold modulo $S(m^{-r-1},g)$.

Now the symbols in $\bigcap_N S(m^{-N},g)$ are integrable in η'. We obtain new symbol classes after integrating (3.6), according to the following lemma.

LEMMA 3.3. *If $a(t,x,\eta) \in \bigcap_N S(m^{-N}, g)$ has support where $|\eta'| \le c\,|\eta''|$, and $\varphi(t,x,\eta)$ is homogeneous of degree 1 satisfying (3.4), then*

$$\tilde{a}(t,x,y',\eta'') = \int e^{i(\varphi(t,x,\eta)+\langle x'-y',\eta'\rangle)} a(t,x,\eta)\,d\eta' \in S^{\nu}_{1,\mu,0}\,, \tag{3.11}$$

where $\nu = \mu(d_0-1)$, $\mu = k_0/(k_0+1)$, $d_0 = $ codim Σ_2. Here $S^{\nu}_{1,\mu,0}$ is defined by

$$\left|D_t^k D_{x'}^{\alpha'} D_{x''}^{\alpha''} D_{y'}^{\beta'} D_{\eta''}^{\gamma''} b(t,x,y',\eta'')\right| \le C_{\alpha\beta\gamma k} \langle\eta''\rangle^{\nu+\mu|\alpha'+\beta'|-|\gamma''|}. \tag{3.12}$$

PROOF: If $N(k_0+1) \ge d_0 + |\alpha|$, we obtain

$$\begin{aligned} \int_{|\eta'|\le c|\eta''|} & \eta'^{\alpha} \left(1+|\eta'|^{k_0+1}\langle\eta\rangle^{-k_0}\right)^{-N} d\eta' \\ & \le \langle\eta''\rangle^{(|\alpha|+d_0-1)\mu} \int \xi'^{\alpha}\left(1+|\xi'|^{k_0+1}\right)^{-N} d\xi' \le C_\alpha \langle\eta''\rangle^{(|\alpha|+d_0-1)\mu}, \end{aligned} \tag{3.13}$$

by putting $\xi' = \eta'/\langle\eta''\rangle^{\mu}$. This gives $|\tilde{a}| \le C\langle\eta''\rangle^{\nu}$. When differentiating (3.11), the derivatives falling on a gives the right factors. The derivatives falling on the exponent gives either η' factors, or factors

$$\left|\partial_t^k \partial_x^{\alpha} \partial_{\eta''}^{\gamma''} \varphi(t,x,\eta)\right| \le C_{k\alpha\gamma''} \langle\eta\rangle^{-|\gamma''|} m,$$

by (3.4) and homogeneity. The η' factors gives only $\langle\eta''\rangle^{\mu}$ factors by (3.13), and the m factors are harmless since $a \in S(m^{-N}, g)$, $\forall N$. This completes the proof.

The lemma gives

$$PE_j u = (2\pi)^{d_0-n} \iint e^{i\langle x''-y'',\eta''\rangle} r_j(t,x,y',\eta'') u(y)\,dy d\eta'' \tag{3.14}$$

where $r_j \in S^{\nu}_{1,\mu,0}$, $j = 1,\ldots,r_0$. We shall compensate for these terms by adding a similar term E_0: $\mathcal{E}'(\mathbf{R}^{n-1}) \to \mathcal{D}'(\mathbf{R}^n)$ with symbol $a_0 \in S^{\nu}_{1,\mu,0}$. By lemma A.2 in the appendix, we obtain that PE_0 has symbol $b_0 \in S^{\nu}_{1,\mu,0}$ given by

$$b_0 = D_t a_0 + e^{i\langle D_y, D_\xi\rangle} \tilde{k}(t,x,\xi) a_0(t,y,z',\eta'')\Big|_{\substack{y=x \\ \xi=(0,\eta'')}} \tag{3.15}$$

if $\tilde{k}$ is the full symbol of K. By using Proposition 4.1, we may solve $b_0 \cong -\sum r_j$, $0 < t < c$, $a_0|_{t=0} \cong 0$, modulo $S^{-\infty}$. Since we can do this with t replaced by $t-s$, for small s, we obtain

PROPOSITION 3.4. *Let $K(t,x,D_x) \in Op\,S(m,g)$ be an $N_0 \times N_0$ system with principal symbol $k(t,x,\xi)$ satisfying (2.13). Then the Cauchy problem for $|s| < \varepsilon$*

$$\begin{cases} D_t E^{(s)} + K(t,x,D_x)E^{(s)} \cong 0, & t > s, \\ E^{(s)}\big|_{t=s} \cong Id_{N_0}\,, \end{cases} \tag{3.16}$$

microlocally near $(0,(0,\xi_0),(0,\xi_0))$, $\xi_0'=0$, *has a solution* $E^{(s)}\colon \mathcal{E}'(\mathbf{R}^{n-1})\to\mathcal{D}'(\mathbf{R}^n)$ *on the form*

$$E^{(s)}=\sum_{j=0}^{r_0}E_j^{(s)}.$$

Here

$$E_j^{(s)}u(t,x)=(2\pi)^{1-n}\iint e^{i(\phi_j(t,x,\eta)-\langle y,\eta\rangle)}a_j(t,x,\eta)u(y)\,dyd\eta,\quad j\ge 1,$$

ϕ_j *solves (3.3),* $a_j\in S(1,g)$*; and*

$$E_0^{(s)}u(t,x)=(2\pi)^{d_0-n}\iint e^{i\langle x''-y'',\eta''\rangle}a_0(t,x,y',\eta'')u(y)\,dyd\eta'',$$

where $a_0\in S^{\nu}_{1,\mu,0}$, $\nu=\mu(d_0-1)$, $\mu=k_0/(k_0+1)$, $d_0=$ *codim* Σ_2.

4. The micro-local pseudo-differential operator

We are going to study the system

$$\text{(4.1)}\qquad \begin{cases} D_tf+e^{i\langle D_y,D_\xi\rangle}k(t,x,\xi)f(t,y,z',\eta'')\Big|_{\substack{y=x\\ \xi=(0,\eta'')}}\cong r(t,x,z',\eta''), & t>0,\\ f(0,x,z',\eta'')\cong f_0(x,z',\eta''), \end{cases}$$

modulo $S^{-\infty}$, where f_0, $r\in S^{\nu}_{1,\mu,0}$ have values in $\mathbf{C}^{N_0}$, and $k\in S(m,g)$ is $N_0\times N_0$ system (see section 3). By lemma A.2 in the appendix, we have $r\in S^{\nu}_{1,\mu,0}$ if $f\in S^{\nu}_{1,\mu,0}$. We shall also assume that k is symmetrizable, i.e. $\exists$ symmetric $N_0\times N_0$ system $M(t,x,\xi)\in S(1,g)$ such that $0<c\le M$ and $Mk-(Mk)^*\in S(1,g)$.

Proposition 4.1. *Assume that* $k(t,x,\xi)\in S(m,g)$ *is a symmetrizable* $N_0\times N_0$ *system. Then, for every* f_0, $r\in S^{\nu}_{1,\mu,0}$, *the equation (4.1) has a solution* $f\in S^{\nu}_{1,\mu,0}$ *in a conical neighborhood of* $(0,0,(0,\eta_0''))\in\mathbf{R}\times\mathbf{R}^{2d_0-2}\times T^*\mathbf{R}^{n-d_0}$.

Proof: We shall solve (4.1) by iteration, modulo $S^{\nu-\mu}_{1,\mu,0}$, $\mu=k_0/(k_0+1)<1$. By Lemma A.2, we have

$$\text{(4.2)}\qquad e^{i\langle D_y,D_\xi\rangle}k(t,x,\xi)f(t,y,z',\eta'')\Big|_{\substack{y=x\\ \xi=(0,\eta'')}}\cong e^{i\langle D_{y'},D_{\xi'}\rangle}kf\Big|_{\substack{y'=x'\\ \xi'=0}}=k(t,x,D_{x'},\eta'')f,$$

modulo terms in $S^{\nu-1}_{1,\mu,0}$. Also, we may assume k supported where $|\xi-(0,\eta'')|<\varepsilon\langle\eta''\rangle$ and $|t|<c$. By cutting off, we may assume $\nu=0$, k, f supported where $\langle\eta''\rangle\approx\langle\eta_0''\rangle$, and f_0, r having compact support. Put $\lambda=\langle\eta_0''\rangle^{-\mu}\le 1$, and choose $w=(x'',\lambda^{-1}z',\lambda^{1/\mu}\eta'')$ as new coordinates. If we make the symplectic dilation $(y,\eta)=(\lambda^{-1}x',\lambda\xi')$, then it suffices to solve the system

$$\text{(4.3)}\qquad \begin{cases} D_tf(t,y,w)+k_\lambda(t,y,w,D_y)f(t,y,w)\cong r(t,y,w), & t>0,\\ f(0,y,w)\cong f_0(y,w), \end{cases}$$

modulo $S(\lambda, |dw|^2 + |dy|^2)$, where $k_\lambda(t,y,w,\eta) \in S(\langle\eta\rangle^{k_0+1}, g_\lambda)$, $g_\lambda = \lambda^2|dy|^2 + |dw|^2 + |d\eta|^2/\langle\eta\rangle^2$ and f_0, $r \in B^\infty$, uniformly in λ. Here B^∞ is the set of C^∞ functions having L^∞ bounds on all derivatives. By assumption, there exists a symmetric $N_0 \times N_0$ system $0 < c \le M_\lambda(t,y,w,\eta) \in S(1,g_\lambda)$, such that $M_\lambda k_\lambda$ is symmetric modulo $S(1,g_\lambda)$. To complete the proof we need to solve (4.3) with $f \in B^\infty$, uniformly in λ. Going back, we obtain a solution in $S^\nu_{1,\mu,0}$ to (4.1) modulo $S^{\nu-\mu}_{1,\mu,0}$.

Choose a partition of unity $\{\chi_j(y)\} \in S(1,|dy|^2)$, such that there is a fixed bound of the diameter of the supports of χ_j, and on the number of overlapping supports. Replacing f_0, r with $\chi_j f_0$, $\chi_j r$, and translating in y, it suffices to solve (4.3) with $f \in \mathcal{S}$ uniformly, when f_0, $r \in C_0^\infty$ uniformly with fixed support. Since

$$\lambda^{-1}\left(k_\lambda(t,y,w,\eta) - k_\lambda(t,0,w,\eta)\right) \in S\left(\langle y\rangle\langle\eta\rangle^{k_0+1}, g_1\right), \quad \lambda \le 1,$$

uniformly, we can replace $k_\lambda(t,y,w,D_y)$ by $k_\lambda(t,w,D_y) = k_\lambda(t,0,w,D_y)$ in the system (4.3). By taking $M_\lambda(t,w,\eta) = M_\lambda(t,0,w,\eta)$ we obtain that $M_\lambda k_\lambda$ is symmetric, mod $S(1,g_\lambda)$.

Now taking the Fourier transform with respect to y in (4.3), we want to solve

$$\text{(4.4)} \qquad \begin{cases} D_t\hat{f}(t,\eta,w) + k_\lambda(t,w,\eta)\hat{f}(t,\eta,w) = \hat{r}(t,\eta,w), & t > 0, \\ \hat{f}(0,\eta,w) = \hat{f}_0(\eta,w). \end{cases}$$

The unique temperate solution to (4.4) is given by

$$\text{(4.5)} \qquad \hat{f}(t,\eta,w) = F_\lambda(t,\eta,w)\left(\hat{f}_0(\eta,w) + i\int_0^t F_\lambda^{-1}(s,\eta,w)\hat{r}(s,\eta,w)\,ds\right),$$

if $F_\lambda(t,\eta,w)$ is temperate solution to

$$\text{(4.6)} \qquad \begin{cases} D_tF_\lambda(t,\eta,w) + k_\lambda(t,w,\eta)F_\lambda(t,\eta,w) = 0, & t > 0, \\ F_\lambda(0,\eta,w) = Id_{N_0}. \end{cases}$$

Thus the proof is completed by showing that $f \in \mathcal{S}$ uniformly, which is done in the following

LEMMA 4.2. *F_λ is temperate, and the mapping $\mathcal{S} \times \mathcal{S} \ni (f_0, r) \to f \in \mathcal{S}$ defined by (4.5) is continuous, uniformly with respect to λ.*

PROOF: Since Fourier transformation and integration are continuous in $\mathcal{S}$, it remains only to prove that multiplication with $F_\lambda^{\pm1}$ is uniformly continuous. This will follow from

$$\text{(4.7)} \qquad 0 < c \le |F_\lambda(t,\eta,w)| \le C$$

$$\text{(4.8)} \qquad \left|D_t^j D_\eta^\alpha D_w^\beta F_\lambda(t,\eta,w)\right| \le C_{j\alpha\beta}\langle\eta\rangle^{(j+|\beta|)(k_0+1)+|\alpha|k_0}.$$

To prove (4.7), we let $\|v\|_\lambda = \langle M_\lambda v, \bar{v}\rangle$, $v \in \mathbf{C}^{N_0}$, then $c \le \|v\|_\lambda^2 / |v|^2 \le C$ uniformly. We obtain by (4.6) that $\left|\partial_t \|F_\lambda v\|_\lambda^2\right| \le C\|F_\lambda v\|_\lambda^2$, so Grönwall's lemma gives (4.7). By differentiating (4.6), we get (4.8) by induction . This completes the proof.

REMARK 4.3. *The unique $f \in \mathcal{S}$ solving (4.4) with f_0, $r \in \mathcal{S}$, gives a continuous map $B^\infty \times B^\infty \to B^\infty$, uniformly in λ.*

This follows easily by writing (4.5) as an oscillatory integral and integrating by parts, using (4.8).

We shall construct a microlocal parametrix for the $N_0 \times N_0$ system P in Proposition 3.4. As before, it suffices to consider $w = (0, 0, \eta_0'') \in \Sigma_2$. Let ϱ_s be the restriction to $\{t = s\}$, and $\varphi \in S^0_{1,0}$ have support in a conical neighborhood of w, such that $w \notin \mathrm{WF}(\varphi - 1)$ and $N^*\{t = s\} \bigcap \mathrm{WF}\,\varphi = \emptyset$, $\forall s$, where N^* is the conormal bundle. Then the composition $\varrho_s \circ \varphi$ is well defined, so for sufficiently small $\varepsilon > 0$ we may define

$$Ef = \int_{-\varepsilon}^{t} E^{(s)} \circ \varrho_s \circ \varphi f \, ds \qquad f \in \mathcal{D}'(\mathbf{R}^n), \tag{5.1}$$

$t \in \,]-\varepsilon, \varepsilon[$, where $E^{(s)}$ is the solution to (3.16). Then E is a microlocal parametrix near w, and we shall study the singularities of this parametrix. Recall that $\Sigma = \bigcup_{j=1}^{r_0} S_j$, where S_j are non-radial hypersurfaces. Let $C_j \subset S_j \times S_j$ be the forward (in t) Hamilton flow on S_j, $j = 1, \dots, r_0$, and Δ^* the diagonal in $T^*\mathbf{R}^n \times T^*\mathbf{R}^n$.

PROPOSITION 5.1. *Let $P = D_t + K(t, x, D_x)$ be an $N_0 \times N_0$ system, with $K \in Op\,S(m, g)$ having principal symbol k satisfying (2.13). If E is the parametrix for P defined by (5.1), then $WF'\,E \subset (\bigcup_{j=1}^{r_0} C_j) \bigcup \Delta^*$, microlocally near $(w, w) \in \Sigma_2 \times \Sigma_2$.*

PROOF: We have $\mathrm{WF}(\varrho_s \varphi f) = \pi(\mathrm{WF}(\varphi f))\,|_{t=s}$, where π: $(t, x; \tau, \xi) \to (t, x, \xi)$ is the projection. Thus, it suffices to show

$$\mathrm{WF}(E^{(s)} f_0)\big|_{t>s} \subset \bigcup_{j=1}^{r_0} C_j \circ \iota_s^{*-1}(\mathrm{WF}\, f_0), \quad f_0 \in \mathcal{D}'(\mathbf{R}^{n-1}), \tag{5.2}$$

where ι_s^*: $T^*_{t=s}\mathbf{R}^n \to T^*\mathbf{R}^{n-1}$ is the dual to the inclusion of $\mathbf{R}^{n-1}$ as the surface $\{t = s\}$ in $\mathbf{R}^n$. Now, (5.2) holds for $E_j^{(s)} f_0$, $j > 0$, since φ_j solves (3.3). It is clear that

$$\mathrm{WF}(E_0^{(s)} f_0)\,|_{t>s} \subset C_0 \circ \iota_s^{*-1}(\mathrm{WF}\, f_0), \quad f_0 \in \mathcal{D}'(\mathbf{R}^{n-1}),$$

where $C_0 \subset \Sigma_2 \times \Sigma_2$ is the set of (w_1, w_2) such that w_1 and w_2 are in the same leaf of Σ_2 and $t(w_1) > t(w_2)$. Thus it suffices to prove that $E_0^{(s)} \in C^\infty$ microlocally near $(t, x, (0, \eta_0''), z, (0, \eta_0''))$ when $x' \neq z'$. By translation we may assume $s = 0$.

Now applying P to $E_0^{(0)}$, we obtain by (3.15) and Lemma A.2 in the appendix

$$\begin{cases} D_t a_0 + e^{i\langle D_{y'}, D_{\xi'}\rangle} \tilde{k}(t, x, z', \xi', \eta'') a_0(t, y', x'', z', \eta'')\Big|_{\substack{\xi'=0 \\ y'=x'}} \cong R_0 a_0, \quad t > 0, \\ a_0(0, x, z', \eta'') \cong 0, \end{cases} \tag{5.3}$$

mod $S^{-\infty}$, microlocally when $|x' - z'| \geq \varepsilon > 0$. Here R_0: $S^\nu_{1,\mu,0} \to S^{\nu-1}_{1,\mu,0}$, $\forall \nu$, and $\tilde{k}$ is the full symbol of K. (This follows since (5.2) holds for $E_j^{(0)}$, $j > 0$.) Also, (5.3) is determined mod $S^{-\infty}$ by the restriction of a_0 to $\{|y' - z'| > \varepsilon/2\}$, and $\tilde{k}$ to $\{|\xi'| \leq C\langle \eta'' \rangle\}$. We shall prove $a_0 \in S^{-\infty}$ in $\{x' \neq z'\}$, by showing that $a_0 \in S^\nu_{1,\mu,0} \Rightarrow a_0 \in S^{\nu-\mu/2}_{1,\mu,0}$, $\forall \nu$, there.

Thus assume $a_0 \in S^\nu_{1,\mu,0}$ near $(t_0, x_0, z_0', \eta_0'')$, $|x_0' - z_0'| \ge \varrho > 0$. By translation and localization, we may assume $x_0' = 0$, $a_0 \in S^\nu_{1,\mu,0}$ supported where $\langle \eta'' \rangle \cong \langle \eta_0'' \rangle$, and $\tilde{k}$ supported where $|\xi'| \le C\langle \eta'' \rangle \cong C\langle \eta_0'' \rangle$. Let $\lambda = \langle \eta_0'' \rangle^{-\mu}$, and change variables as in section 4. Then $a_0(t, y, w) \in S(\lambda^{-\nu/\mu}, e)$, $\tilde{k}(t, y, w, \eta) \in S(\langle \eta \rangle^{k_0+1}, g_\lambda)$ uniformly, where e is equal to the Euclidean metric and we may assume $\nu = 0$. Clearly $|w| > \varrho\lambda^{-1}$, and (5.3) holds mod $S(\lambda^N, e)$, $\forall N$, when $|y| = |\lambda^{-1}x'| < \varrho\lambda^{-1}/2$. Choose $\phi(s) \in C_0^\infty(\mathbf{R})$, such that $\phi(s) = 1$ when $|s| \le 1/2$, $\phi(s) = 0$ when $|s| > 1$, and put $\chi(y, w) = \phi(4\lambda|y|^2/\varrho^2 + |w|^2) \in S(1, \lambda|dy|^2 + |dw|^2)$. Then $b_0 = \lambda^{-1/2}\chi a_0$ satisfies

$$\text{(5.4)} \qquad \begin{cases} D_t b_0 + \tilde{k}_0(t, w, D_y) b_0 = r_1, & 0 < t < \varepsilon, \\ b_0 \,|_{t=0} = r_0, \end{cases}$$

where $\tilde{k}_0(t, w, \eta) = \tilde{k}(t, 0, w, \eta)$, and $r_j \in C_0^\infty$ are uniformly bounded in B^∞. In fact, $\chi a_0 \in S(\lambda^N, e)$, $\forall N$, at $t = 0$. Also, the calculus gives

$$\lambda^{-1/2}[\tilde{k}_0(t, w, D_y), \chi] \in Op\, S(\langle \eta \rangle^{k_0}, \tilde{g}_\lambda),$$

and

$$\lambda^{-1/2}\chi(\tilde{k}(t, y, w, D_y) - \tilde{k}_0(t, w, D_y)) \in Op\, S(\langle \eta \rangle^{k_0+1}, \tilde{g}_\lambda),$$

where $\tilde{g}_\lambda = \lambda|dy|^2 + |dw|^2 + |d\eta|^2/\langle \eta \rangle^2$. Then Remark 4.3 gives that b_0 is uniformly in B^∞, $0 \le t < \varepsilon$. Thus $\chi a_0 \in S(\lambda^{1/2}, e)$, and since this is uniform in λ when $|x' - z'| \ge \varrho > 0$, we obtain the proposition.

Proof of Theorem 1.3: As mentioned before, we only have to consider $w \in \Sigma_2$. By Proposition 2.3 it suffices to prove the propagation of singularities for the system $P = D_t Id_{N_0} + K(t, x, D_x)$ with principal symbol satisfying (2.13). The adjoint P^* satisfies the same conditions, so by Proposition 5.1 we can construct a parametrix E for P^* such that $\mathrm{WF}'\, E \subset (\bigcup C_j) \bigcup \Delta^*$, microlocally near $(w, w) \in \Sigma_2 \times \Sigma_2$. Cutting off, we may assume $u \in \mathcal{E}'$ and $w \in \Sigma_2 \setminus \mathrm{WF}\, Pu$. Then $u \cong E^* Pu$ modulo C^∞, and since we may change t to $-t$, this gives the result.

Appendix. Some calculus lemmas

We are going to study the composition of conormal distributions having non-standard symbols. Let $a_\varphi(x, D) \in \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n)$ be given by

$$\text{(A.1)} \qquad a_\varphi(x, D)u(x) = (2\pi)^{-n} \int e^{i(\langle x - y, \eta \rangle + \varphi(x, \eta))} a(x, \eta) u(y)\, dy d\eta,$$

$u \in C_0^\infty(\mathbf{R}^n)$, where $a \in S(m^k, g)$, and $\varphi(x, \eta) \in C^\infty(T^*\mathbf{R}^n \setminus 0)$ is homogeneous of degree 1 in the η variables, satisfying (3.4). Here g, m are defined by (2.6)–(2.7). The composition with $p(x, D)$ is given by $p(x, D)a_\varphi(x, D)u(x) = b_\varphi(x, D)u(x)$, if p, $a \in \mathcal{S}$, where

$$\text{(A.2)} \qquad b(x, \eta) = e^{i\langle D_y, D_\theta \rangle} f(x, \theta; y, \eta) \Big|_{\substack{\theta = \eta \\ y = x}}$$

if we put

$$\theta = \xi - \int_0^1 \partial_x \varphi(x + s(y - x), \eta)\, ds, \tag{A.3}$$

and $f(x, \theta; y, \eta) = p(x, \xi)a(y, \eta)$, since $\left|\frac{d(y,\xi)}{d(y,\theta)}\right| \equiv 1$.

LEMMA A.1. *Assume $\varphi(x, \eta) \in C^\infty(T^*\mathbf{R}^n \backslash 0)$ is homogeneous of degree 1 in the η variables and satisfies (3.4). If $a \in S(m^k, g)$, $k \in \mathbf{Z}$, has support in a sufficiently small conical neighborhood of $\{\eta' = 0\}$ and $p \in S(m, g)$, then the composition of $p(x, D)$ and $a_\varphi(x, D)$ is equal to $b_\varphi(x, D)$, where $b \in S(m^{k+1}, g)$ satisfies (A.2), and has the expansion*

$$b(x, \eta) \cong \sum_{j=0}^{N-1} (i\langle D_\xi, D_y - (\partial\theta/\partial y)D_\xi\rangle)^j p(x, \xi)a(y, \eta)/j!\,\Big|_{\substack{y=x \\ \xi=\eta+d_x\varphi(x,\eta)}} \tag{A.4}$$

modulo $S(m^{k+1}h^N, g)$, with θ given by (A.3).

IDEA OF PROOF: If $\varphi \equiv 0$ then (A.4) follows from the Weyl calculus, since $g(t, -\tau) = g(t, \tau)$ (see Th. 18.5.4 and 18.5.10 in [5]). Now $p(x, \xi)a(y, \eta) \in S(M, G)$ where $M(\xi, \eta) = m(\xi)m^k(\eta)$ is a weight for $G = g_{x,\xi}(dx, d\xi) + g_{y,\eta}(dy, d\eta)$. Since $\partial_\xi\chi = (0, Id; 0, 0)$ and $\partial_y\chi = (0, \partial\theta/\partial y; Id, 0)$, the result follows by proving $\chi^*S(M, G) = S(M, G)$, where χ: $(x, \xi; y, \eta) \to (x, \theta; y, \eta)$ is a diffeomorphism, using Lemma 8.2 in [4].

Next, let $S^\nu_{1,\mu,0}$ be the symbol classes defined by (3.12), $\mu = k_0/(k_0 + 1)$. For $a \in S^\nu_{1,\mu,0}$, we define $a(x, D'') \in \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n)$ by

$$a(x, D'')u(x) = (2\pi)^{d_0 - n} \iint e^{i\langle x'' - y'', \eta''\rangle} a(x, y', \eta'')u(y)\, dy d\eta'', \tag{A.5}$$

$u \in C_0^\infty(\mathbf{R}^n)$. If p, $a \in \mathcal{S}$, then the composition is given by $p(x, D)a(x, D'')u(x) = b(x, D'')u(x)$, where

$$b(x, z', \eta'') = e^{i\langle D_y, D_\xi\rangle} p(x, \xi)a(y, z', \eta'')\Big|_{\substack{y=x \\ \xi=(0,\eta'')}} . \tag{A.6}$$

LEMMA A.2. *If $p \in S(m, g)$ and $a \in S^\nu_{1,\mu,0}$, then the composition of $p(x, D)$ and $a(x, D'')$ is equal to $b(x, D'')$, where $b \in S^\nu_{1,\mu,0}$ satisfies (A.6) and*

$$b(x, z', \eta'') = e^{i\langle D_{y'}, D_{\xi'}\rangle} p(x, \xi', \eta'')a(y', x'', z', \eta'')\Big|_{\substack{\xi'=0 \\ y'=x'}} + Ra, \tag{A.7}$$

where R: $S^\nu_{1,\mu,0} \to S^{\nu-1}_{1,\mu,0}$ is continuous. Also, b and Ra are determined modulo $S^{-\infty}$ by the restriction of a to $\{|y - x| < \varepsilon\}$, and p to $\{|\xi - (0, \eta'')| < \varepsilon\langle\eta''\rangle\}$, $\forall\varepsilon > 0$.

IDEA OF PROOF: The composition is well defined since the metrics and weights are A temperate with respect to the diagonal, if $A(x, \xi, y, z', \eta'') = \langle y, \xi\rangle$. By using $e^{i\langle D_y, D_\xi\rangle} = e^{i\langle D_{y'}, D_{\xi'}\rangle} \circ e^{i\langle D_{y''}, D_{\xi''}\rangle}$, we obtain (A.7) since the restrictions of the metrics and weights to $\{x'' = y'' \wedge \xi'' = \eta''\}$ also are temperate.

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