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SEMINAIRE EQUATIONS AUX DERIVEES PARTIELLES 1986 - 1987

# SUPERLINEAR ELLIPTIC EQUATIONS

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# I. INTRODUCTION.

We point out, in the following, a way to look at superlinear elliptic equations from an algebraic geometry and topology point of view. The proofs which are provided are sketchy, though enough precise to allow an interested reader to complete them

A superlinear elliptic equation is a variational-type equation where a self-adjoint unbounded operator "competes" with a non-linearity.

The model equation is :

$$(1)\qquad \begin{cases} \Delta u + g(x,u) = 0 \\ \Omega \subset \mathbb{R}^n \text{ bounded, regular} \\ u = 0_{|\partial\Omega} \end{cases}$$

$g$ is, for sake of simplicity, assumed to satisfy :

(H1) $\quad g : \Omega \times \mathbb{R} \to \mathbb{R}$ is $C^\infty$

(H2) $\quad \lim\limits_{|x|\to+\infty} \dfrac{g(x,s)}{s} = +\infty$

(H3) $\quad \exists C,\ \exists\ q < \dfrac{n+2}{n-2}$ such that $|g(x,s)| \leqslant C(|s|^q+1) \quad \forall (x,s) \in \Omega \times \mathbb{R}$

(H4) $\quad \exists\ \theta \in [0, \frac{1}{2}[\ , \exists\ C$ such that :

$$\theta g(x,s)s \geqslant \int_0^s g(x,\tau)d\tau - C$$

In the most general context, (H1) is replaced by Caratheodory conditions on $g$ .

(H2) remains unchanged and justifies the expression "superlinear".

(H3) is replaced, in dimension 1 and 2, by similar conditions.

Finally (H4) may be replaced, again for sake of simplicity, by the following more constraining equation :

(H5) $\quad \begin{matrix} \exists\ \mu \in [0,1) \text{ s.t} \\ \exists\ C \end{matrix} \qquad \mu g'_s(x,s)s^2 \geqslant g(x,s)s - C$ .

Observe that (H2) and (H5) imply (H4).

We will denote $X$ the set of hypotheses (H1)-(H2)-(H3)-(H5).

Under these hypotheses the operator :

(2)
$$H^1_0(\Omega) \longrightarrow H^1_0(\Omega)$$
$$u \longrightarrow \Delta_\Omega^{-1} g(x,u)$$

is a compact operator. ($\Delta_\Omega^{-1}$ is the inverse of the Laplace operator under Dirichlet boundary conditions).
Several different results lead to formulate the following conjecture :

Conjecture : Any equation satisfying $X$ has infinitely many solutions.

As we pointed out, $X$ may be weakened in the statement of the conjecture. Nevertheless,features as (H2) and (2) are essential.

The results which lead to formulate this conjecture are of three types :

1. The conjecture holds for n=1. In this case, when one is seeking for periodic solutions of such equations on a given interval, say [0,1], H. Jacobowitz [1] has proved the conjecture under hypotheses close to $X$ ; while P. Rabinowitz [2] has proved this conjecture, under the same type of hypotheses, for the Dirichlet boundary conditions.

2. In case g is odd in the second variable s , the conjecture holds in any dimension. Again, this follows from A. Ambrosetti - P. Rabinowitz [3] results; and in fact a careful analysis allows to trace it back to the Lusternik and Schnirelman celebrated theorem of existence of infinitely many critical points for an even functional defined on the unit sphere of a Hilbert space and satisfying some compacity condition as the Palais-Smale condition (see M. Krasnosls'kii [4] for further precisions).

3. In case g has the special form $g(x,u) = |u|^{p-1}u-f(x)$, with $1 < p < \frac{n+2}{n-2}$ , it follows from the results in [5] and [6] (A. Bahri - H. Berestycki and A. Bahri - P.L. Lions) that (1) has infinitely many solutions for $1 < p < \frac{n}{n-2}$ , while it follows from [7] that (1) has infinitely many solutions for a residual set of functions f(x) in $H^{-1}(\Omega)$.

Although 1., 2. and 3. would, by themselves, naturally lead to such a conjecture, there are some other results in [7] which show the unity in behaviour of superlinear type functionals and support therefore even more such a conjecture.
These results are the following :
Let

(3) $-\Delta e_i = \mu_i e_i$ ; $(\mu_i, e_i)$ i-th eigenvalue and eigenfunction of the Laplacian under Dirichlet boundary conditions, ordered with increasing $\mu_i$'s .

(4) $E_k = \bigoplus_{i \leqslant k} \mathbb{R} e_i$ ; $p_k : H_0^1 \to E_k$ the orthogonal projection.

(5) $G(x,s) = \int_0^s g(x,\tau)d\tau$

(6) $I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega G(x,u)dx$ ; $u \in H_0^1(\Omega)$

(7) For $u \in H_0^1(\Omega)$, let $|u| = (\int_0^1 |\nabla u|^2 dx)^{1/2}$ ; $S = \{u \in H_0^1(\Omega) \text{ s.t } |u| = 1\}$

(8) For $v \in S$ , $J(v) = \operatorname{Sup}_{\lambda \geqslant 0} I(\lambda v)$

It is proven, in [7], that critical points of $J$ of strictly positive energy are in one to one correspondence with critical points of I of strictly positive energy. It is also proven, in [7], that the functionals J and I satisfy a "Galerkin-type" (C) condition of Palais and Smale; namely :

<u>Proposition</u> A : Let $(u_k) \in E_k$ a sequence such that $p_k(I'(u_k)) \to 0$ while $(I(u_k))$ is bounded; then $(u_k)$ has a convergent subsequence.
Similarly, let $(v_k) \in E_k \cap S$ a qequence such that $p_k(J'(v_k)) \to 0$ while $(J(v_k))$ is bounded; then $(v_k)$ has a convergent subsequence.

Superlinear-type functionals $J$ have furthemore a key-feature, which is stated in the following proposition :

<u>Proposition</u> B : Let $(v_i)$ be a sequence on S . Then $(J(v_i))$ tends to $+\infty$ if and only if $v_i$ tends weakly to zero.

As a side-remark, one can see, considering the functional $\frac{1}{J}$ , the relationship between superlinear-type problems and the weakly continuous functionals studied by M. Krasnosels'kii [4] (see [5] and [7] for further precisions).

Propositions A and B lead to the following theorem, proven in [7] and which shows the remarkable unity in behaviour of superlinear-type problems :

<u>Theorem</u> 1 [7] : For any $a \in \mathbb{R}$ , there exists a $\mu(a) \geqslant a$ and $n_o(a) \in \mathbb{N}$ such

that if I has no critical value in $[a,\mu(a)]$, then the level set $I^a \cap E_k = \{u \in E_k \text{ s.t } I(u) \geqslant a\}$ is diffeomorphic to a disk for $k \geqslant n_o(a)$.

Theorem 1 is a negative-type result. It rests upon the hypothesis that a superlinear-type functional has <u>no</u> critical value in a large enough interval. As shown in [7], contradiction arguments allow sometimes to transform this negatively formulated result in existence results.

Some thought also shows that theorem 1. provides a tool in order to study the conjecture we introduced, or at least some strong form of it : Namely, assume that we wish to prove the existence of infinitely many solutions to (1), which are <u>stable under perturbation</u>. Some thought, together with the remarks ending [7] (see in particular Remark 18 in [7]), show that these solutions, considered as critical points, are <u>limits of critical points of I restricted to the finite dimensional subspaces</u> $E_k$ (for k large enough), which, because of the stability under perturbation, <u>have to induce a difference of topology in the level sets of the functional</u> $I|_{E_k}$. Therefore, this stronger statement implies that the level sets have to change at the crossing of the related critical values and differ from disks one one side or the other of these critical values.

Having pointed out the direction in which we think Theorem 1 may be useful, we show in this paper that this theorem has natural implications on a complexified version of the problem and leads to interesting questions in algebraic geometry and and algebraic topology.

## II. MILNOR'S FIBRATION AND SUPERLINEAR FUNCTIONALS :

We need to decide whether the set $I^a \cap E_k$ is a disk or not. We first consider a simple case, where n=2 and $g(x,s) = s^3$. Writing down a vector u in $E_k$ as $u = \sum_{i=1}^{k} x_i e_i$, the set $I^a \cap E_k$ is as well :

(9) $$I^a \cap E_k \simeq \{(x_1,\dots,x_k) \in \mathbb{R}^k \text{ s.t } \frac{1}{2} \sum_{i=1}^{k} \mu_i x_i^2 - \frac{1}{4} \int_\Omega (\sum_{i=1}^{k} x_i e_i)^4 dx \geqslant a\}$$

that is $I^a \cap E_k$ is a real algebraic set.

Introducing the polynomials :

(10) $$p(x_1,\dots,x_k) = \frac{1}{4} \int_\Omega (\sum_{i=1}^{k} x_i e_i)^4$$

and

(11) $$q(x_1,\dots,x_k) = \frac{1}{2} \sum_{i=1}^{k} \mu_i x_i^2 - p(x_1,\dots,x_k) - a$$

we consider then the homogeneized polynomial of q :

(11bis) $$Q(x_1,\dots,x_k,t) = \frac{1}{2} t^2 \left(\sum_{i=1}^{k} \mu_i x_i^2\right) - p(x_1,\dots,x_k) - at^4$$

We now consider $Q$ as a real coefficient polynomial in the complex variables $(z_1,\dots,z_{k+1})$ and we introduce :

(12) $$X = \{(z_1,\dots,z_{k+1}) \in \mathbb{C}^{k+1} \text{ s.t } Q(z_1,\dots,z_{k+1}) = 1\}$$

$X$ is equipped with a natural involution :

(13) $$T : X \to X$$
$$(z_1,\dots,z_{k+1}) \to (\bar{z}_1,\dots,\bar{z}_{k+1})$$

and, denoting Fix T the fixed point set of $T$ , we have :

(14) $$\text{Fix } T = \{(x_1,\dots,x_k,t) \in \mathbb{R}^{k+1} \text{ s.t } Q(x_1,\dots,x_k,t) = 1\}$$

Due to the fact that $t$ cannot be zero on Fix T, we have the following easy proposition :

Let

(15) $$\mathring{I}^a \cap E_k = \{(x_1,\dots,x_k) \in \mathbb{R}^k \text{ s.t } \frac{1}{2} \sum_{i=1}^{k} \mu_i x_i^2 - \frac{1}{4} \int_\Omega \left(\sum_{i=1}^{k} x_i e_i\right)^4 dx > a\}$$

We have :

<u>Proposition</u> 1 : Fix T is diffeomorphic to two disjoint copies of $\mathring{I}^a \cap E_k$ .

The proof of Proposition 1 is straightforward and implies the following corollary :

<u>Corollary</u> 1 : $I^a \cap E_k$ is a (closed) disk if and only if Fix T is diffeomorphic to two disjoint copies of a disk.

Therefore, we may view properties of $I^a \cap E_k$ as properties of Fix T.
On another hand, we have the following remarkable result on the topology of sets as $X$ :

<u>Theorem</u> 2. ([8]; Milnor) Let $P(z_1,\dots,z_{k+1})$ be a homogeneous polynomial, of order m, which is non singular outside the origin (0,...,0).

Then, for any $c \in \mathbb{C}^*$, the set $F_c = \{(z_1,\dots,z_{k+1}) \in \mathbb{C}^{k+1}$ s.t $P(z_1,\dots,z_{k+1})=c\}$ has the homotopy type of a wedge of $(m-1)^{k+1}$ -spheres $S^k$ .

Therefore, if $Q(z_1,\dots,z_{k+1})$ is non singular outside the origin, then X has the homotopy type of a wedge of $3^{k+1}$-spheres $S^k$ . Observe also that, under such a condition, X is a 2k-dimensional riemannian manifold (with the induced standard metric) and that T defines an isometry of X ; the problem being to decide when Fix T is made up of two disks. Observe lastly that T is an involution of X . In order to be able to apply Theorem 2, we need Q to be non singular outside the origin. The critical points of Q are given in the following :

<u>Proposition</u> 2 : Consider Q as a polynomial on $\mathbb{C}^{k+1}$ .
The critical points of Q are then either critical points of $p(z_1,\dots,z_k)$ with $z_{k+1} = 0$ and $\frac{\partial Q}{\partial z_{k+1}}(z_1,\dots,z_k,0) = 0$ or critical points $(z_1,\dots,z_{k+1})$ such that $(z_1/z_{k+1},\dots,z_k/z_{k+1})$ is critical for q , with $z_{k+1}$ arbitrar $\neq 0$ and $q(z_1/z_{k+1},\dots,z_k/z_{k+1}) = a$ .
<u>Proof of Proposition</u> 2 : Either a critical point $(z_1,\dots,z_{k+1})$ has $z_{k+1} = 0$ . Then $(z_1,\dots,z_k)$ is critical for p and $\frac{\partial Q}{\partial z_{k+1}}(z_1,\dots,z_k,0) = 0$. Or

$z_{k+1}$ is non zero. Then, $(z_1/z_{k+1},\dots,z_k/z_{k+1})$ is critical for $z_{k+1}^4(q(z_1/z_{k+1},\dots,z_k/z_{k+1})-a)$ in the variables $(z_1/z_{k+1},\dots,z_k/z_{k+1})$ while the vanishing of the derivative along $z_{k+1}$ provides with $q(z_1/z_{k+1},\dots,z_k/z_{k+1}) = a$ .

It is clear that we may assume, after Proposition 2, that the only critical points are (0,...,0). Indeed, we may perturb a, p, Q and even $\Omega$ slightly in order to be in this situation. That this situation is generic requires some technicalities which we leave aside here.Nevertheless the interested reader will convince himself of this after some thought.

The example we developped here generalizes of course to other polynomials p of degree 2m ; $m \geq 1$. These polynomials need not to be homogeneous (Observe however that, in Proposition 2, p will be replaced by its 2m-homogeneous part). Again, perturbation arguments on a, p and Q (even $\Omega$ which changes the $\mu_i$'s) imply a Proposition similar to Proposition 2 and therefore allows to apply Theorem 2.
We are thus left with non-linearities g which are not polynomial; and these

are the most frequent ones.
Although it is not possible then to replace $g$ globally by a polynomial, it is still possible, in these superlinear problems, for given $a$ and $k$ to do so in order to study $I^a \cap E_k$.
Let us consider the simple case $g(x,s) = g(s)$ with a growth of $g$ controlled by $|s|^\alpha, \alpha > 1$, at infinity. $G(s)$ is $\int_0^s g(\tau)d\tau$ and grows faster than $s^2$ ; $E_k$ is finite dimensional. Hence $I^{a-1} \cap E_k$ is bounded and there is a uniform $L^\infty$-bound, say $b$, on functions in $I^{a-1} \cap E_k$.
Let $p$ be a polynomial such that :

(16) $$|p-G|_{C^2([-3b,3b])} < \varepsilon$$

Set :

(17) $$q_m(s) = p(s) + \left(\frac{s}{2b}\right)^{2m}$$

Clearly, $q_m$ is lower bounded by $p$ ; $|q_m - p|_{C^1([-b,b])}$ tends to zero when $m$ tends to $+\infty$ ; and $q_m(s) \geqslant G(s) - \varepsilon$ for $m$ large enough. Observe also that $q_m$ behave as $cs^{2m}$ at infinity in $s$.
Therefore, for $m$ large enough, the set :

(18) $$\tilde{I}^a \cap E_k = \{u \in E_k \text{ s.t } \frac{1}{2} \sum_{i=1}^k \mu_i x_i^2 - \int_\Omega q_m\left(\sum_{i=1}^k x_i e_i\right)dx \geqslant a\}$$

is contained in $I^{a-\varepsilon} \cap E_k$. If $\varepsilon$ is less than 1, we thus have $|u|_{L^\infty} < b$, $\forall u \in \tilde{I}^a \cap E_k$ and thus :

(19) $$|q_m(u) - G(u)| < \varepsilon \ \forall u \in (\tilde{I}^a \cap E_k) \cup (I^a \cap E_k) \ ; \ \forall x \in \Omega .$$

When $\varepsilon$ is small enough, the two sets $\tilde{I}^a \cap E_k$ and $I^a \cap E_k$ have then the same homotopy type and we are brought back to our polynomial situation.
Similar arguments, of one kind or another, allow to bring back <u>all</u> superlinear type problems to this polynomial type situation.
We stress, at the end of this section, that all these approximation arguments are suitable for the general conjecture and not only for a generic form of it :
Indeed, in the superlinear-type problems, having bounds on critical values for these perturbed problems allow to pass to the limit. Again, this is a technical point which can be checked directly.

III. INVOLUTIONS ON A WEDGE OF SPHERES $S^k$.

We provide here a theorem relating the topology of the fixed point set of an involution $T$ acting on a wedge of spheres $S^k$ to an algebraic invariant of $T$.

Namely, let :

(20) $X : S^k \vee \dots \vee S^k$

(21) $T : X \to X \;;\; T^2 = Id$

(22) $T_*: H_k(X;\mathbb{Z}) \longrightarrow H_k(X,\mathbb{Z})$.

We have

(23) $T_*^2 = Id$

On the other hand, any homomorphism of a free $\mathbb{Z}$-module may be reduced over $\mathbb{Z}$ to the following form :

(24) $$T_* = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & 0 & \\ & & -1 & & & & & \\ & & & -1 & & & & \\ & & & & 0 & 1 & & \\ & & & & 1 & 0 & & \\ & 0 & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{pmatrix}$$

i.e $T_*$ may be read as a matrix built up with 1 or -1 terms on the diagonal or also blocks of type $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Definition : We define $i(T_*)$ to be the number of 1 and -1 on the diagonal of such a reduction of $T_*$.

We then have :

Theorem 3[(1)]: Let X be a finistic space which is topologically a wedge of spheres $S^k$ ; T an involution acting on X with fixed point set made up of two connected components.

Then Fix T is acyclic over $\mathbb{Z}_2$ if and only if $i(T_*) = 1$.

Remark : If furthermore X is riemannian and T is an isometry, then Fix T is made up of two disks implies the diagonal term is $(-1)^k$ by an Euler-Poincaré characteristic argument. Indeed, we then have : $\chi(\mathrm{Fix}\,T) = 2 = 1+(-1)^k \mathrm{tr}T_*$. Hence the conclusion. Conversely, if $i(T_*)=1$ and the diagonal term is $(-1)^k$ for T an involution which is an isometry, then Fix T is $\mathbb{Z}_2$- acyclic and each component has its Euler-Poincaré characteristic equal to 1.

Proof of Theorem 3 : Let $E_{\mathbb{Z}_2} \to B_{\mathbb{Z}_2}$ be the classifying bundle for the $\mathbb{Z}_2$-action .

Let :

$$X \times_{\mathbb{Z}_2} E_{\mathbb{Z}_2} \to B_{\mathbb{Z}_2} \tag{25}$$

be the related Borel construction, which is a fibration with fiber X . Consider the spectral sequence for this fibration as in G.E. Bredon [9]. The $E_2$-term for such a fibration is :

$$E_2^{p,q} = \check{H}^p(B_{\mathbb{Z}_2} ; \check{H}^q(X;\mathbb{Z}_2)) \tag{26}$$

where $\check{H}$ is Cech-cohomology.

As X is a wedge of spheres $S^k$ , $E_2^{p,q}$ is zero for $q \neq 0$ and $q \neq k$ :

$$E_2^{p,q} = 0 \quad \text{for} \quad q \neq 0 ; q \neq k . \tag{27}$$

and we also have :

$$\text{rank } E_2^{p,0} = 1 \quad ; \quad \text{rank } E_2^{p,k} = i(T_*) \tag{28}$$

Indeed, as $E_2^{p,0} = \check{H}^p(B_{\mathbb{Z}_2} ; \check{H}^o(X;\mathbb{Z}_2)) = \check{H}^p(B_{\mathbb{Z}_2} ; \mathbb{Z}_2)$ , rank $E_2^{p,0} = 1$

On the other hand $E_2^{p,k} = \check{H}^p(B_{\mathbb{Z}_2} ;\check{H}^k(X;\mathbb{Z}_2)) = \ker(Id+T_*)/\mathrm{Im}(Id+T_*)$ (see G.E. Bredon [9]); where $T_*$ acts on $\check{H}^k(X;\mathbb{Z}_2)$. Using then the matrix of $T_*$ acting on $\check{H}^k(X;\mathbb{Z})$ provided by (24), which still holds on $\check{H}^k(X;\mathbb{Z}_2) = \check{H}^k(X;\mathbb{Z}) \otimes \mathbb{Z}_2$ , we derive :

$$\text{rank}(\ker(Id+T_*)/\mathrm{Im}(Id+T_*)) = i(T_*) . \tag{29}$$

hence (28).

We thus have only two non zero terms in the $E_2$-sequence.

Observe now that Fix T is non empty.

Therefore, (25) admits a global section. This implies that $E_\infty^{p,0} \simeq E_2^{p,0}$ ; which implies in turn that $E_\infty^{p,k} \simeq E_2^{p,k}$ .

We thus have :

(30) $E_\infty^{p,q} = 0$ for $q \neq 0$ ; $q \neq k$ ; rank $E_\infty^{p,0} = 1$ ; rank $E_\infty^{p,k} = i(T_*)$.

Hence :

(31) $$\text{rank } \check{H}^p(X \underset{\mathbb{Z}_2}{\times} E_{\mathbb{Z}_2} ; \mathbb{Z}_2) = \sum_{q=0}^{+\infty} \text{rank } E_\infty^{p,q} = 1 + i(T_*) .$$

Using again results of G.E. Bredon [9], we have :

(32) $$\text{rank } \check{H}^p(X \underset{\mathbb{Z}_2}{\times} E_{\mathbb{Z}_2} ; \mathbb{Z}_2) = \text{rank } \check{H}^p(\text{Fix}\, T \times B_{\mathbb{Z}_2} ; \mathbb{Z}_2) \text{ for } p > k .$$

Therefore :

(33) $$1 + i(T_*) = \text{rank}(\check{H}^*(\text{Fix}\, T; \mathbb{Z}_2))$$

Fix T is assumed to have two connected components. Clearly, (33) implies that these two connected components are $\mathbb{Z}_2$- acyclic if and only if $i(T_*) = 1$. Hence the proof of theorem 3.

## IV. PROPERTIES OF $X-X_{\mathbb{R}} = X-\text{Fix}\, T$ .

Let

(34) $$X_{\mathbb{R}} = \text{Fix } T$$

From now on, X is defined to be the Milnor's fiber at 1 for a homogeneous non singular polynomial $Q(z_1,\dots,z_{k+1})$. T is complex conjugation. We have a natural map, which is equivariant with respect to the $\mathbb{Z}_2$-action on $X-X_{\mathbb{R}}$ via T :

(35) $$h : X-X_{\mathbb{R}} \to S^k$$
$$(z_1,\dots,z_{k+1}) \to \frac{(\text{Im}z_1,\dots,\text{Im}z_{k+1})}{\sqrt{\sum_{i=1}^{k+1}(\text{Im}z_i)^2}}$$

We want in this section to derive equivalent forms of the conjecture translated as properties of $X-X_{\mathbb{R}}$ , on which h is defined. Observe that, choosing a point $x_1$ in $X_{\mathbb{R}}$ and considering a sphere $S^{k-1}$ in the normal bundle at $x_1$ to $X_{\mathbb{R}}$ in X , we have a well-defined map :

(36) $$\tau : S^{k-1} \to X \quad \text{such that} \quad \tau(-x) = T\tau(x).$$

Let now $\Sigma$ be a tubular neighbourhood of $X_{\mathbb{R}}$ in X ; $\partial\Sigma$ being thus the (k-1)-dimensional sphere bundle over $X_{\mathbb{R}}$ ; $(\Sigma,\partial\Sigma)$ is then a relative $(D^k,S^{k-1})$

bundle over $X_{\mathbb{R}}$ and $(X,\Sigma,\partial\Sigma)$ is an excisive triad, which we may assume to be stable under T. $\Sigma$ retracts by deformation on $X_{\mathbb{R}}$ , while $X-X_{\mathbb{R}}$ retracts by deformation on $X-\Sigma$ and we therefore have the following Mayer-Victoris exact sequence, where we are using $\mathbb{Z}$-coefficients :

(37) $$\cdots \to H_{i+1}(X) \to H_i(\partial\Sigma) \to H_i(X_{\mathbb{R}}) \oplus H_i(X-X_{\mathbb{R}}) \to H_i(X) \to \cdots$$

All sets are invariant under the action of T and we, in fact, have a commutative diagram :

(38) $$\begin{array}{ccccccccc} \cdots \to & H_{i+1}(X) & \to & H_i(\partial\Sigma) & \to & H_i(X_{\mathbb{R}}) \oplus H_i(X-X_{\mathbb{R}}) & \to & H_i(X) & \to \cdots \\ & \downarrow T_* & & \downarrow T_* & & \downarrow T_* & & \downarrow T_* & \\ \cdots \to & H_{i+1}(X) & \to & H_i(\partial\Sigma) & \to & H_i(X_{\mathbb{R}}) \oplus H_i(X-X_{\mathbb{R}}) & \to & H_i(X) & \to \cdots \end{array}$$

We assume in the sequel that :

(39) $$\pi_1(X_{\mathbb{R}}) = 0$$

This assumption is justified by the following :

Lemma 2 : Given a superlinear type functional I and $a \in \mathbb{R}$, there exists $k_o(a) \in \mathbb{N}$ such that $\pi_1(I^a \cap E_k) = 0$ for $k \geqslant k_o(a)$.

Proof of lemma 2 : The critical points of I under the energy level a are of finite generalized Morse index upperbounded by an integer $k_1(a)$. (The definition and main properties of this generalized Morse index are stated in [10]; [11]).Indeed, under a given energy level a, the critical points are bounded uniformly in $L^\infty$ and this implies the uniform boundedness of the dimension of the negative or null eigenspace of the linearized operator. Using Proposition A , we derive a uniform bound on the generalized Morse indices of the critical points of energy level under a, not only for I, but also for $I_{|E_k}$ ; and this bound $k_2(a)$ does not depend on k . Considering then k such that $k-k_2(a)$ is larger than 2, standard homotopy type perturbation arguments as in [10] (see also [11]; theorem 6.4) imply lemma 2.

In order to exploit (38), we need the following :

Lemma 3 : Assume $H_{k-1}(X_{\mathbb{R}}) = 0$ . Then, $H_k(\partial\Sigma) = 0$ ; $H_{k-1}(\partial\Sigma) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ where $e_1$ and $e_2$ are spheres of the normal bundle to $X_{\mathbb{R}}$ at two points $x_1$ and $x_2$ belonging to the two distinct components of $X_{\mathbb{R}}$ .

Proof : $(\Sigma,\partial\Sigma)$ is a $(D^k,S^{k-1})$ bundle over $X_{\mathbb{R}}$ , which is simply connected. Therefore, by Thom isomorphism theorem, we have, with $\mathbb{Z}$-coefficients :

(40) $$H_{k+1}(\Sigma,\partial\Sigma) \simeq H_1(X_{\mathbb{R}}) = 0$$

(41) $$H_k(\Sigma,\partial\Sigma) \simeq H_o(X_{\mathbb{R}}) = \mathbb{Z}\oplus\mathbb{Z}.$$

Writing down the exact sequence of the pair $(\Sigma,\partial\Sigma)$ we derive :

(42) $$\ldots \to H_{k+1}(\Sigma,\partial\Sigma) \to H_k(\partial\Sigma) \to H_k(\Sigma) \to H_k(\Sigma,\partial\Sigma) \to H_{k-1}(\partial\Sigma) \to H_{k-1}(\Sigma) \to \ldots$$

As $H_k(\Sigma) = H_k(X_{\mathbb{R}}) = 0$ ($X_{\mathbb{R}}$ is a k-dimensional non compact manifold), we derive

(43) $$H_k(\partial\Sigma) = 0$$

On the other hand, we also have $H_{k-1}(\Sigma) = H_{k-1}(X_{\mathbb{R}}) = 0$ by assumption. Therefore :

(41) $$H_{k-1}(\partial\Sigma) \simeq H_k(\Sigma,\partial\Sigma) \simeq \mathbb{Z}\oplus\mathbb{Z}$$

The generators are clearly $e_1$ and $e_2$ ; hence the proof of lemma 3.

We show later on why the assumption $H_{k-1}(X_{\mathbb{R}}) = 0$ fits superlinear-type problems.

Lemma 3 and (38) imply that we have the following short exact sequence :

(45) $$0 \to H_k(X-X_{\mathbb{R}}) \xrightarrow{f^*} H_k(X) \xrightarrow{g^*} \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \to H_{k-1}(X-X_{\mathbb{R}}) \to 0$$

This sequence is stable by $T_*$ .

Let

(46) $$C = \mathrm{Im}g_* .$$

We thus have :

(47) $$0 \to H_k(X-X_{\mathbb{R}}) \to H_k(X) \to C \to 0 .$$

This sequence is again stable by $T_*$ as $T_*$ commute to $g_*$ .
We recall now that, if K is a group on which $\mathbb{Z}_2$ acts (the action is denoted $T_*$), we have : (see Bredon [9 ])

$$\check{H}^p(B_{\mathbb{Z}_2};K) = \begin{cases} \ker(Id-T_*) \text{ for } p=0 \\ \ker(Id-T_*)/Im(Id+T_*) \text{ for } p>0 \text{ even} \\ \ker(Id+T_*)/Im(Id-T_*) \text{ for } p>0 \text{ odd.} \end{cases} \tag{48}$$

We thus have :

Lemma 4 :The following sequence is exact

$$\ldots \to \check{H}^p(B_{\mathbb{Z}_2}, H_k(X-X_{\mathbb{R}})) \to \check{H}^p(B_{\mathbb{Z}_2}, H_k(X)) \to \check{H}^p(B_{\mathbb{Z}_2}, C) \to \check{H}^{p+1}(B_{\mathbb{Z}_2}, H_k(X-X_{\mathbb{R}})) \to \ldots$$

Proof : Let $K_1$, $K_2$, $K_3$ denote respectively $H_k(X-X_{\mathbb{R}})$ ,$H_k(X)$ and C .

As $T_*^2 = Id$, we have, for i = 1,2,3

$$\ldots K_i \xrightarrow{Id-T_*} K_i \xrightarrow{Id+T_*} K_i \xrightarrow{Id-T_*} K_i \quad \ldots \tag{49}$$

and we are dealing with chain complexes.
(48) allows to identify their homology as $\check{H}^p(B_{\mathbb{Z}_2};K_i)$ and lemma 4 follows then immediately from (47).

Corollary 2 : rank $\check{H}^p(B_{\mathbb{Z}_2}, H_k(X-X_{\mathbb{R}})) \leq 3$ $\forall p$ if $X_{\mathbb{R}}$ is $\mathbb{Z}_2$-acyclic.

Proof : If $X_{\mathbb{R}}$ is $\mathbb{Z}_2$-acyclic, then rank $\check{H}^p(B_{\mathbb{Z}_2}, H_k(X)) \leq 1$ by theorem 3.
C has on the other hand at most rank 2 ; hence the result.

Observe now that, by (47), $H_k(X-X_{\mathbb{R}})$ is a subgroup of $H_k(X)$; therefore $H_k(X-X_{\mathbb{R}})$ is free abelian over $\mathbb{Z}$ and $T_*$ , as acting on $H_k(X-X_{\mathbb{R}})$ , has a reduction of type (24). The following proposition relates the number of 1 and -1 on the diagonal of such a reduction to the topology of $X_{\mathbb{R}}$ .

Proposition 3 : Let $i_1(T_*)$ be the number of 1 and -1 in the reduction of $T_*$ acting on $H_k(X-X_{\mathbb{R}})$ . If $\chi(X_{\mathbb{R}}) = 2$ and $X_{\mathbb{R}}$ has two connected components $\mathbb{Z}_2$-acyclic, and if $2e_1$ is a boundary in $X-X_{\mathbb{R}^n}$ . $i_1(T_*) = 1$ or $i_1(T_*) = 3$ .

If $i_1(T_*) = 1$, the diagonal term is $(-1)^{k-1}$ ; if $i_1(T_*) = 3$, one diagonal term is $(-1)^k$ and the two other ones are $(-1)^{k-1}$ .
In this latter case, if $2e_1$ is a boundary in $X-X_{\mathbb{R}}$ with $2e_1 = g_*([\delta])$, $T_*([\delta]) = (-1)^k[\delta]$ , and if $X_{\mathbb{R}}$ is made up of two disks, then $X-X_{\mathbb{R}}$ is a wedge of spheres $S^k$ .
Conversely, if $i_1(T_*) = 1$, the diagonal term being $(-1)^{k-1}$, or $i_1(T_*) = 3$ with one $(-1)^k$ and two $(-1)^{k-1}$, $X_{\mathbb{R}}$ is $\mathbb{Z}_2$- acyclic.

As a corollary to Proposition 3, we have :

Theorem 4 : Let $Q(z_1,\dots,z_{k+1})$ be associated to a superlinear-type problem at the value a as described in section 2. Assume $2e_1$ is a boundary in $X-X_{\mathbb{R}}$ . Then, if $I^a \cap E_k$ is a disk, $i_1(T_*)$ is equal to 1 or 3 ; with a diagonal term equal to $(-1)^{k-1}$ in the first case and diagonal terms equal to one $(-1)^k$ and two $(-1)^{k-1}$ in the second case. Conversely, if $i_1(T_*)$ is such, then $I^a \cap E_k$ is $\mathbb{Z}_2$-acyclic . Finally, assume that I has a non degenerate critical point of Morse index $k_o$ and critical level b . Let Q be associated to I at the value $b+\varepsilon, \varepsilon > 0$ small enough. Then, if $I^{b+\varepsilon} \cap E_k$ is a disk, for $k-k_o$ odd, $k-k_o \geqslant 3$ , $i_1(T_*) = 1$ or $i_1(T_*) = 3$ (one $(-1)^{k-1}$ or two $(-1)^{k-1}$ and one $(-1)^k$), and if $i_1(T_*)$ is equal to 3, $X-X_{\mathbb{R}}$ is a wedge of spheres $S^k$ .
Remark : in this latter case, i.e. when $X-X_{\mathbb{R}}$ is a wedge of spheres $S^k$ , the map $\tau$ of (36) is null-homotopic. The map h of (35) is essential and generates one $(-1)^{k-1}$ in $\check{H}^{k+1}(B_{\mathbb{Z}_2}, H^k(X-X_{\mathbb{R}}))$ .

Proof of Theorem 4 : The first part follows from Proposition 3. For the second part, we show in lemma 8 below that under the assumptions of the theorem on $k-k_o$ , $2e_1$ is a boundary in $X-X_{\mathbb{R}}$ , with $2e_1 = g_*([\delta])$, $T_*([\delta]) = (-1)^k[\delta]$ : Hence Proposition 3 applies again $X-X_{\mathbb{R}}$ is a wedge of spheres $S^k$ if $i_1(T_*)=3$. The proof of Theorem 4 is thereby complete.

Proof of Proposition 3 : $e_1$ and $e_2$ are (k-1)-dimensional spheres and T acts as antipodal involution on them. Therefore :

$$T_*e_1 = (-1)^k e_1 \ ; \ T_*e_2 = (-1)^k e_2 \ . \tag{50}$$

C is a subgroup of $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and contains, by assumption, $2\mathbb{Z}e_1 \oplus 2\mathbb{Z}e_2$ .
Thus :

$$C = \mathbb{Z} \oplus \mathbb{Z} \tag{51}$$

(50) implies.

(52) $$(T_* + (-1)^{k-1}\,\mathrm{Id})(C) = 0 \ .$$

(50) and (51) imply :

(53) $$\ker(T_* + (-1)^k \mathrm{Id})\ (C) = \ker(2\mathrm{Id})(C) = 0 \ .$$

Therefore, using (48), we have :

(54) $$\check{H}^p(B_{\mathbb{Z}_2}, C) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv k \quad (2) \\ 0 & \text{if } p \equiv k+1 \quad (2) \ . \end{cases}$$

On the other hand, by theorem 3 and the remark following it, $X_{\mathbb{R}}$ is $\mathbb{Z}_2$-acyclic, (made up of two connected components), with $\chi(X_{\mathbb{R}}) = 2$, if and only if there is only one diagonal term, equal to $(-1)^k$ on the diagonal of $T_*$ acting on $H_k(X;\mathbb{Z})$. Therefore, under such hypotheses, we have :

(55) $$\check{H}^p(B_{\mathbb{Z}_2}, H_k(X)) = \begin{cases} \mathbb{Z}_2 & \text{if } p \equiv k \quad (2) \\ 0 & \text{if } p \equiv k+1 \quad (2). \end{cases}$$

Lemma 4, (54) and (55) imply :

(56) $$0 \to \check{H}^k(B_{\mathbb{Z}_2}, H_k(X-X_{\mathbb{R}})) \to \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \check{H}^{k+1}(B_{\mathbb{Z}_2}, H_k(X-X_{\mathbb{R}})) \to 0$$

We thus have two possibilities :
either

(57) $$\check{H}^k(B_{\mathbb{Z}_2}, H_k(X-X_{\mathbb{R}})) = 0 \ ; \ \check{H}^{k+1}(B_{\mathbb{Z}_2}, H_k(X-X_{\mathbb{R}})) = \mathbb{Z}_2$$

or, covering the other case in Proposition 3 :

(58) $$H^k(B_{\mathbb{Z}_2}, H_k(X-X_{\mathbb{R}})) = \mathbb{Z}_2 \ ; \ \check{H}^{k+1}(B_{\mathbb{Z}_2}, H_k(X-X_{\mathbb{R}})) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

and the first part of Proposition 3 is proven.
For the latter part, the assumption that $2e_1 = g_*([\delta])$ where $T_*([\delta]) = (-1)^k[\delta]$ rules out immediately (58) if $2e_1$ may be taken as a generator of C . Indeed, if so, either $2e_1$ disappears in $\check{H}^k(B_{\mathbb{Z}_2}, C)$ or, if not, is image of an element in $\check{H}^k(B_{\mathbb{Z}_2}, H_k(X))$. By symmetry, this holds also for $2e_2$ .

We are thus left with the case $C = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ .

Then $H_{k-1}(X-X_{\mathbb{R}}) = 0$ ; $H_k(X-X_{\mathbb{R}})$ is free abelian.

In higher dimensions, we have

(59) $$0 \to H_r(\partial\Sigma) \to H_r(X-X_{\mathbb{R}}) \to 0 \; ; \; r > k \; .$$

and also :

(60) $$0 \to H_{r+1}(\Sigma,\partial\Sigma) \simeq H_{r+1-k}(X_{\mathbb{R}}) \to H_r(\partial\Sigma) \to 0$$

Therefore, if $X_{\mathbb{R}}$ is made up of two discs, $H_r(X-X_{\mathbb{R}}) = 0$ for $r > k$ .

Thus $X-X_{\mathbb{R}}$ is a wedge of spheres $S^k$ .

This ends the proof of Proposition 3.

We justify in the following four lemmas the assumptions in lemma 3, Proposition 3 for superlinear-type problems. Namely, we show, under some hypotheses on a and k that $H_{k-1}(I^a \cap E_k) = 0$ and that $2e_1$ is a boundary in $X-X_{\mathbb{R}}$ , $2e_1 = g_*([\delta])$, with $T_*([\delta]) = (-1)^k[\delta]$ .

Lemma 5 : There exists $a_1 > 0$ such that, for $a \geqslant a_1$ , $H_{k-1}(I^a \cap E_k) = 0$

Proof of lemma 5 : $J(v)$, $v \in S$ , has been defined in (8).

Let

(61) $$\alpha_k = \underset{v \in S \cap E_k}{\mathrm{Min}} \; J(v)$$

One can see easily that the $\alpha_k$'s are bounded and converge, when k tends to $+\infty$ to $a_o = \inf_{v \in S} J(v)$.

Let

(62) $$\bar{a}_1 = \underset{k \in \mathbb{N}}{\mathrm{Sup}} \; \alpha_k + 1 \; .$$

Clearly, we have, for any $a \geqslant \bar{a}_1$ :

(63) $$J^a \cap E_k = \{v \in S \cap E_k / J(v) \geqslant a\} \subsetneq S \cap E_k \; .$$

$S \cap E_k$ is a sphere of dimension k-1 in which $J^a \cap E_k$ is strictly contained.

Therefore :

(64) $$H_{k-1}(J^a \cap E_k) = 0 \; .$$

On the other hand, under (H1)-(H2)-(H3)-(H5), it is proven in [7] (see [7]; Theorem 3), that there exists a constant $\overline{a}_2 > 0$ such that if $J(v) \geqslant \overline{a}_2$ :

(65) $$J(v) = I(\lambda(v)v) \text{ for a unique } \lambda(v) > 0$$

(66) the function $\lambda \to I(\lambda v)$ is increasing for $0 \leqslant \lambda < \lambda(v)$ and decreasing for $\lambda > \lambda(v)$

Hence, for any $v$ in $J^a \cap E_k$ , there exists two values $0 < \lambda_1(v,a) \leqslant \lambda_2(v,a)$, such that :

(67) $$I(\lambda_1(v,a)v) = I(\lambda_2(v,a)v) = a$$

(68) $$I(\lambda v) \geqslant a \quad \text{for} \quad \lambda \in [\lambda_1(v,a),\lambda_2(v,a)]$$

$\lambda_1(v,a)$ and $\lambda_2(v,a)$ are continuous functions of $v$ and allow to identify $I^a \cap E_k$ :

(69) $$I^a \cap E_k = \{u \in E_k \text{ s.t } u = \lambda \frac{u}{|u|} ; \lambda \in [\lambda_1(\frac{u}{|u|},a), \lambda_2(\frac{u}{|u|}, a)]; \frac{u}{|u|} \in J^a \cap E_k\}$$

Hence, $I^a \cap E_k$ retracts by deformation onto the set :

(70) $$F^a \cap E_k = \{\lambda(v)v ; v \in J^a \cap E_k\}$$

This set is diffeomorphic to $J^a \cap E_k$ ; hence :

(71) $$H_{k-1}(I^a \cap E_k) = H_{k-1}(F^a \cap E_k) = H_{k-1}(J^a \cap E_k) = 0 .$$

Lemma 5 is there by proven.

We turn now to proving that, under the assumptions on a, I and k , $2e_1$ is a boundary in $X-X_{\mathbb{R}}$ , $2e_1 = g_*([\delta])$ with $T_*([\delta]) = (-1)^k[\delta]$ .

We first consider a polynomial $Q_1(z_1,\dots,z_{k+1})$ on $\mathbb{C}^{k+1}$ , with real coefficients, on which we assume :

(A1) $Q_1$ has a <u>real</u> critical point $(a_1,\dots,a_{k+1})$, non degenerate of index $k_o$ .

Let

(72) $$c = Q_1(a_1,\dots,a_{k+1}) .$$

As $(a_1,\dots,a_{k+1})$ is non degenerate, there is a Morse reduction of $Q_1$

around $(a_1,\ldots,a_{k+1})$. This Morse lemma may be completed in complex coordinates, while preserving real coordinates.
Hence, we have a local diffeomorphism of $\mathbb{C}^{k+1},\varphi$, preserving $\mathbb{R}^{k+1}$, such that :

$$(73)\quad \begin{cases} Q_1(a_1+\varphi_1(z_1,\ldots,z_{k+1}),\ldots,a_{k+1}+\varphi_{k+1}(z_1,\ldots,z_{k+1}))=c+\sum_{k_o+1}^{k+1} z_j^2 - \sum_1^{k_o} z_j^2 . \\ \sum_{j=1}^{k+1} |z_j|^2 \leqslant \alpha \end{cases}$$

Consider then $\varepsilon > 0$, small enough, and :

$$(74)\qquad F_\varepsilon = \{(z_1,\ldots,z_{k+1});\ \sum_{j=1}^{k+1} |z_j|^2 \leqslant \alpha\ ;\ \sum_{k_o+1}^{k+1} z_j^2 - \sum_1^{k_o} z_j^2 = \varepsilon\}$$

$$(75)\qquad \begin{aligned} \delta : S^k &\longrightarrow F_\varepsilon \\ (x_1,\ldots,x_{k+1}) &\longrightarrow (i\varepsilon x_1,\ldots,i\varepsilon x_{k_o}, \varepsilon x_{k_o+1},\ldots,\varepsilon x_{k+1}) \end{aligned}$$

where $S^k$ is the standard sphere in $\mathbb{R}^{k+1}$ .

$$(76)\qquad R_\varepsilon = F_\varepsilon \cap \mathbb{R}^{k+1}$$

$F_\varepsilon$ is a 2k-dimensional manifold with boundary $\partial F_\varepsilon$, having the homotopy type of a sphere $S^k$ . Assuming $k > 1$, $\pi_1(F_\varepsilon)$ is zero and $F_\varepsilon$ is therefore orientable. Then $R_\varepsilon$ is a k-dimensional manifold with boundary $\partial R_\varepsilon$ contained in $\partial F_\varepsilon$ ; therefore, $R_\varepsilon$ defines a cycle in $(F_\varepsilon,\partial F_\varepsilon)$. Let $[R_\varepsilon]$ be the homology class in $H_k(F_\varepsilon,\partial F_\varepsilon;\mathbb{Z})$ defined by $R_\varepsilon$ and $[\delta]$ be the one defined by $\delta$ .
We then have :

<u>Lemma 6</u> : The intersection number of $[\delta]$ and $[R_\varepsilon]$ is $\pm 2$ if $k-k_o$ is non zero, even. We then also have $T_*[\delta] = (-1)^k[\delta]$ in $H_k(F_\varepsilon;\mathbb{Z})$ , where $T$ is the complex conjugation acting on $F_\varepsilon$ .

<u>Proof</u> : Observe that $\delta(S^k)\cap F_\varepsilon = \delta(S^{k-k_o})$, where $S^{k-k_o}$ is the standard $k-k_o$-dimensional sphere. As $k-k_o$ is even and non zero, $S^{k-k_o}$ is connected and has an Euler-characteristic equal to 2. We may thus pick up a vector-field $v$ on $S^{k-k_o}$ having only two singularities each counting with an index 1. We use $v$ to set $\delta$ in general position with respect to $R_\varepsilon$ :
Let $\nu$ be the tangent bundle to $\delta(S^{k-k_o})$ and $\eta$ be its normal bundle in $\delta(S^k)$. Let $\eta_D$ be the associated disk-bundle which we may think of as being

a tubular neighbourhood of $\delta(S^{k-k_o})$ in $\delta(S^k)$ .

Let $\theta$ be the tangent bundle to $F_\varepsilon$ along $\delta(S^k)$.

We have :

(77) $$\theta = \nu \oplus \eta \oplus i\nu \oplus i\eta$$

(78) $\nu \oplus i\eta$ is the tangent bundle to $R_\varepsilon$ along $\delta(S^{k-k_o})$.

We call $\theta_D$ the disk-bundle of $\theta$ . This also may be thought as a tubular neighbourhood of $\delta(S^{k-k_o})$ in $F_\varepsilon$ .

Let :

(79) $\omega$ be a function equal to 1 on the zero section of $\theta_D$ and equal to zero outside $\theta_D$ .

We pick up an orientation of $S^{k-k_o}$ and of $S^k$ . This provides with orientations of $\nu$, $\eta$ , $i\nu$ , $i\eta$ .

We extend v to $\theta_D$ , in a trivial way, i.e. v is valued in $\nu$ and depends only on the base point in $S^{k-k_o}$ .

We consider the vector-field $i\omega v$ which is therefore defined on $\theta_D$ , or else on a tubular neighbourhood of $\delta(S^{k-k_o})$ in $F_\varepsilon$ . It generates a local one parameter group $\phi_s$ , which is the identity outside $\theta_D$ , as $\omega$ is zero there. We consider, for $s > 0$ , $\phi_s(\delta(S^k))$ and we study $\phi_s(\delta(S^k)) \cap R_\varepsilon$ . This corresponds to study, at an infinitesimal level, for which (x,y), $x \in S^{k-k_o}$, $y \in (\eta_D)_x$ , we have :

(80) $$y + is\omega(x,y)v(x,y) \in \nu_x \oplus i\eta_x \quad .$$

As v(x,y) belongs to $\nu_x$ , $is\omega(x,y)v(x,y)$ belongs to $i\nu_x$ and (80) imposes :

(81) $$y = 0 \; ; \; \omega(x,y) = 1 \; ; \; v(x,0) = 0 \; .$$

We are thus left with $(x_1,0)$ and $(x_2,0)$ where $x_1$ and $x_2$ are the singularities of v on $S^{k-k_o}$. At those points, the tangent space at $\phi_s(\delta(S^k))$ is

$(Id+is\, Dv_y(x_j,0)(\eta_{x_j}) \oplus (Id + is\, Dv_x(x_j,0))(\nu_{x_j})$ ,where Dv is the differential of v in a local chart of $F_\varepsilon$ around $x_j$ .

As v is y-independent, $Dv_y(x_j,0)(\eta_{x_j}) = 0$ and the tangent space is :

(82) $$\eta_{x_j} \oplus (Id + isDv_x(x_j, 0))(\nu_{x_j}) .$$

Therefore, the local index of the intersection with $\nu_{x_j} \oplus i\eta_{x_j}$ in $\theta_{x_j} = \nu_{x_j} \oplus \eta_{x_j} \oplus i\nu_{x_j} \oplus i\eta_{x_j}$ is given by the sign of the determinant of the endomorphism $Dv_x(x_j,0)$ of $\nu_{x_j}$. This is exactly the index of $v$ at $x_j$. We thus have :

(83) $\phi_s(\delta(S^k))$ intersects $R_\varepsilon$ with the same index at two points.

Thus :

(84) $$[\delta] \cdot [R_\varepsilon] = \pm 2 .$$ (. is the intersection number).

Observe now that the Lefschetz number of $T$ acting on $\delta(S^k)$ is equal to the Euler-characteristic of the fixed point set ; hence, this number is 2 and necessarly :

(85) $$T_*([\delta]) = (-1)^k[\delta] .$$

Lemma 6 is there by proven.

We apply now lemma 6 to superlinear-type problems :

Let $Q(z_1,\dots,z_{k+1})$ be a polynomial provided by a superlinear-type problem, with $X_{\mathbb{R}} = I^a \cap E_k$.

Consider :

(86) $Q_1(z_1,\dots,z_{k+1}) = Q(z_1,\dots,z_{k+1}) - \varepsilon z_{k+1}$ ; $\varepsilon > 0$ small ; $Q$ homogeneous of degree $2m$ ; $Q(z_1,\dots,z_{k+1}) = z_{k+1}^{2m}(q(z_1/z_{k+1},\dots,z_k/z_{k+1})-a)$

We have :

<u>Lemma</u> 7 : The critical points of $Q_1(z_1,\dots,z_{k+1})$ are such that $(z_1/z_{k+1},\dots,z_k/z_{k+1})$ is critical for $Q(z_1/z_{k+1},\dots,z_k/z_{k+1},1) = q(z_1/z_{k+1},\dots,z_k/z_{k+1}) - a$. $z_{k+1}$ satisfies then the equation : $2m\, z_{k+1}^{2m-1}(q(z_1/z_{k+1},\dots,z_k/z_{k+1})-a) = \varepsilon$.

Furthermore, after having possibly perturbed a, the following holds :

The real positive critical values of $Q_1$ are, for any $\varepsilon > 0$, equal to $\frac{2m-1}{2m} \frac{\varepsilon^{2m/2m-1}}{(2m(a-b))^{1/2m-1}}$, where $b$ is any real critical value of $q$ strictly less than $a$.

Lastly, if $(a_1,\dots,a_k)$ is a real non degenerate critical point of $q$, of Morse index $k_o$, with critical value $b < a$, the corresponding real critical point of $Q_1$, $(\theta a_1,\dots,\theta a_k,\theta)$, $\theta = (\frac{\varepsilon}{2m}\ \frac{1}{b-a})^{1/2m-1}$, has a Morse index equal to $k_o+1$.

Proof of lemma 7 : The first part of lemma 7, about the critical points of $Q_1$, is straightforward . The critical values of $Q_1$ are thus equal to $\frac{1-2m}{2m}\varepsilon z_{k+1}$ where $z_{k+1}$ satisfies :

(87) $$2m\, z_{k+1}^{2m-1}(b-a) = \varepsilon$$

and $b$ is any critical value of $q$. Clearly, given these critical values, which are in finite number (if not, perturb slightly q), we may perturb if necessary a, so that (88) has only solutions which are not real if $b$ is not real. The statement about the real positive critical values of $Q_1$ in lemma 7 follows.

We consider now a real non degenerate critical point of q $(a_1,\dots,a_k)$, of Morse index $k_o$, and critical value $b < a$. Around $(a_1,\dots,a_k)$, q may be read, in a local chart, as $b + \sum_{k_o+1}^{k} x_j^2 - \sum_1^{k_o} x_j^2$ ; hence $Q_1$ reads :

(89) $$z_{k+1}^{2m}(b-a+\sum_{k_o+1}^{k} x_j^2 - \sum_1^{k_o} x_j^2) - \varepsilon z_{k+1} = \theta^{2m}(b-a) - \varepsilon\theta +$$

$$+ \theta^{2m}(\sum_{k_o+1}^{k} x_j^2 - \sum_1^{k_o} x_j^2) + m(2m-1)(z_{k+1}-\theta)^2\theta^{2m-2}(b-a) + O((z_{k+1}-\theta)^3 + \sum_1^{k}|x_j|^3).$$

As $b < a$, the Morse index of $Q_1$ at $(\theta a_1,\dots,\theta a_k,\theta)$ is $k_o+1$ . The proof of lemma 7 is thereby complete.

We then have :

Lemma 8 : Let $b_o$ be the largest real critical value of $q$ less than a ($b_o < a$). Assume that $q$ has at the value $b_o$ a real critical point $(a_1,\dots,a_k)$ of Morse index $k_o$, with $k-k_o$ odd, $k-k_o \geq 3$ . Assume furthermore that we are in the situation when lemma 7 applies. Then $2e_1$ is a boundary in $X-X_{\mathbb{R}}$ ; and $2e_1 = g_*([\delta])$, with $[\delta]$ such that $T_*([\delta]) = (-1)^k[\delta]$ .

Proof of lemma 8 : By lemma 7, the real positive critical values of $Q_1(z_1,\dots,z_{k+1}) = Q(z_1,\dots,z_{k+1}) - \varepsilon z_{k+1}$ are the $\frac{2m-1}{2m} \frac{\varepsilon^{2m/2m-1}}{(2m(a-b))^{1/2m-1}}$ where $b$ is a real critical value of $Q$ less than a. The largest one is :

(90) $$c_\varepsilon = \frac{2m-1}{2m} \frac{\varepsilon^{2m/2m-1}}{(2m(a-b_o))^{1/2m-1}}$$

Clearly, for $\varepsilon$ small enough, we have :

(91) $$0 < c_\varepsilon < 1$$

By lemma 7, we also know that $Q_1$ has, at the value $c_\varepsilon$ , a real critical point of index $k_o+1$ .

The assumption of lemma 8 imply that $k-(k_o+1)$ is even, larger than 2. We may then apply lemma 6 :

We choose $\gamma > 0$ , small enough. Introducing :

(92) $$F_{c_\varepsilon+\gamma} = \{(z_1,\dots,z_{k+1}) \text{ s.t. } Q_1(z_1,\dots,z_{k+1}) = c_\varepsilon+\gamma\}$$

(93) $\hat{\Sigma}$ a tubular neighbourhood of $F_{c_\varepsilon+\gamma} \cap \mathbb{R}^{k+1}$ in $F_{c_\varepsilon+\gamma}$ invariant under the complex conjugation T . $\partial\hat{\Sigma}$ is its boundary.

We derive from lemma 6 the existence of a cycle $[\delta]$ in $F_{c_\varepsilon+\gamma}$ intersecting a connected component of the real part of $F_{c_\varepsilon+\gamma}$ with an intersection number equal to $\pm 2$ and such that $T_*([\delta]) = (-1)^k[\delta]$ , This means that $2\hat{e}_1 = \hat{g}_*([\delta])$, when $\hat{e}_1$ is a sphere in the normal bundle in $F_{c_\varepsilon+\gamma}$ to the real part of it at a point of the connected component we are considering; and where $\hat{g}_*$ is the connecting homomorphism :

(94) $$\hat{g}_*: H_k(F_{c_\varepsilon+\gamma}) \to H_{k-1}(\partial\hat{\Sigma})$$

Thus, lemma 8 is proven with $F_{c_\varepsilon+\gamma}$ instead of X .

In order to prove the same result on X , we prove that there exists a diffeomorphism :

(95) $$\psi : F_{c_\varepsilon+\gamma} \to X$$

such that

(96) $$\psi \circ T = T \circ \psi$$

The existence of such a diffeomorphism implies immediately the result on X . The existence of $\psi$ is proven in two steps.
First, we introduce :

(97) $$F_1 = \{(z_1,\dots,z_{k+1}) \in \mathbb{C}^{k+1} \text{ s.t } Q_1(z_1,\dots,z_{k+1}) = 1\}$$

Considering then the differential equation :

(98) $$\frac{\partial z_j}{\partial s} = \frac{\overline{\frac{\partial}{\partial z_j} Q_1}}{\sum_{j=1}^{k+1} \left|\frac{\partial}{\partial z_j} Q_1\right|^2} \qquad j = 1,\dots,k+1$$

one can prove local and global existence on $[0,+\infty[$ for initial data in $F_{c_\varepsilon+\gamma}$ . Local existence follows from the Cauchy-Lipschitz existence theorem for any $(z_1,\dots,z_{k+1})$ such that $\sum_{j=1}^{k+1} \left|\frac{\partial Q_1}{\partial z_j}\right|^2$ is non zero. This holds on a flow-line (for $s \geqslant 0$) starting at a point in $F_{c_\varepsilon+\gamma}$ as $Q_1$ remains real lower bounded by $c_\varepsilon+\gamma$ on such a flow-line $(z_1(s),\dots,z_{k+1}(s))$. In fact, we have :

(99) $$\frac{d}{ds} Q_1(z_1(s),\dots,z_{k+1}(s)) = 1 .$$

Global existence follows from the fact that $\sum_{j=1}^{k+1} \left|\frac{\partial Q_1}{\partial z_j}\right|^2$ tends to $+\infty$ with $\sum_{j=1}^{k+1} |z_j|^2$ , as can be easily checked, using also the fact that Q is non singular, homogeneous.

Calling $\eta(s,.)$ the flow generated this way, $\eta(1-c_\varepsilon-\gamma,.)$ is a diffeomorphism between $F_{c_\varepsilon+\gamma}$ and $F_1$ , which commutes to complex conjugation.

The second step consists in showing that $F_1$ and X an also diffeomorphic through a diffeomorphism commuting to complex conjugation.
For this, we introduce the differential equations

(101) $$\frac{\partial z_j}{\partial s} = - z_{k+1} \frac{\overline{\frac{\partial}{\partial z_j}(Q-(\varepsilon-s)z_{k+1})}}{\sum_{j=1}^{k+1} \left|\frac{\partial}{\partial z_j}(Q-(\varepsilon-s)z_{k+1})\right|^2} \quad j = 1,\dots,k+1 .$$

Observe that on a flow-line of (101), we have :

$$(102) \qquad \frac{d}{ds}\left(Q(z_1(s),\ldots,z_{k+1}(s)) - (\varepsilon-s)z_{k+1}(s)\right) = 0$$

Local existence for initial data in $F_1$ again follows from the Cauchy-Lipschitz theorem. As $Q(z_1(s),\ldots,z_{k+1}(s)) - (\varepsilon-s)z_{k+1}(s)$ remains then equal to 1 and as, by lemma 7, the real positive critical values of $Q(z_1,\ldots,z_{k+1}) - (\varepsilon-s)z_{k+1}$ are the $\frac{2m-1}{2m}\ \frac{(\varepsilon-s)^{2m/2m-1}}{(2m(a-b))^{1/2m-1}}$, we may rapply the Cauchy-Lipschitz theorem for $s\in[0,\varepsilon]$. Global existence for $s\in[0,\varepsilon]$ thus holds if the trajectories do not go to infinity during this time ; which again holds because $\sum_{j=1}^{k+1}\left|\frac{\partial}{\partial z_j}(Q-(\varepsilon-s)z_{k+1})\right|^2$ tends to $+\infty$ uniformly with s when $\sum_{j=1}^{k+1}|z_j|^2$ tends to $+\infty$. The flow of this differential equation $\phi(s,.)$, taken at time $s=\varepsilon$, $\phi(\varepsilon,.)$ is the desired diffeomorphism. The proof of lemma 8 is thereby complete.

(1) As may be checked through the proof of theorem 3, Theorem 3 is a particular case of a more general theorem which states that the total dimension of the $\mathbb{Z}_2$-homology of Fix T is equal to $i(T_*)+1$ (see in particular (33)).

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# ERRATA

EXPOSE N XXI          A. BAHRI

SUPERLINEAR ELLIPTIC EQUATIONS

page 9 line 25

au lieu de :

which implies in turn that $E_\infty^{p,k} \simeq E_2^{p,k}$ .

lire :

which implies in turn that $E_\infty^{p-k,k} \simeq E_2^{p-k,k}$

page 10 formule 30

au lieu de :

$E^{p,q} = 0$ for $q \neq 0$ ; $q \neq k$ ; rank $E_\infty^{p,0} = 1$ ; rank $E_\infty^{p,k} = i(T_*)$.

lire :

$E_\infty^{p,q} = 0$ for $q \neq 0$ ; $q \neq k$ ; rank $E_\infty^{p,0} = 1$ ; rank $E_\infty^{p-k,k} = i(T_*)$ for $p > k$ .

page 10 formule 31

au lieu de :

$$\text{rank } \check{H}^p(X \underset{\mathbb{Z}_2}{\times} E_{\mathbb{Z}_2}; \mathbb{Z}_2) = \sum_{q=o}^{+\infty} \text{rank } E_\infty^{p,q} = 1 + i(T_*) \text{ .}$$

lire :

$$\text{rank } \check{H}^p(X \underset{\mathbb{Z}_2}{\times} E_{\mathbb{Z}_2}; \mathbb{Z}_2) = \sum_{q=o}^{+\infty} \text{rank } E_\infty^{p-q,q} = 1 + i(T_*).$$