Séminaire Équations aux dérivées partielles - École Polytechnique

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Séminaire Équations aux dérivées partielles (Polytechnique) (1986-1987), exp. nº 20, p. 1-6
<http://www.numdam.org/item?id=SEDP_1986-1987____A19_0>

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SEMINAIRE EQUATIONS AUX DERIVEES PARTIELLES 1986 - 1987

SOME MATHEMATICAL PROBLEMS IN INVERSE POTENTIAL SCATTERING.

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Exposé n° XX

26 Mai 1987

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We shall consider the Schrödinger operator $H_v = -\Delta + v(x)$ in $\mathbf{R}^n$ when $n = 3, 5, \ldots$. We shall work with real-valued potentials in the class $\mathcal{V}$. This is defined by the inequalities

$$(1) \qquad \int (1+|x|)^{|\alpha|-(n-2)} |D^\alpha v(x)|\, dx < \infty.$$

The notation $\|v\|$ will be used for different semi-norms in $\mathcal{V}$.

Remark. By making changes of scales we find that H_v is unitarily equivalent to $t^{-2} H_{v_t}$, where $v_t(x) = t^2 v(tx)$. The semi-norms with $|\alpha| = n-2$ are not changed then, and we let $\|\nabla^{n-2} v\|_1$ denote the sum of such semi-norms. We shall say that v is small if this number is small. If v is small, then H_v is conjugated to H_0 by an operator which is an isomorphism in L^p when $1 \le p \le \infty$, and there are no bound states then. This result will follow from our constructions of intertwining operators.

Intertwining operators.

Most of the ideas behind the construction of intertwining operators can be found in Faddeev's papers [**2–3**], where he introduced direction-dependent Green's functions. Since we are able to give explicit formulas for these functions, it will be possible for us to obtain precise information about the intertwining operators. This will allow us to generalize to higher dimensions the approach to inverse scattering that we developed in Melin [**4**] .

A continuous linear transformation $A : C_0^\infty(\mathbf{R}^n) \to \mathcal{D}'(\mathbf{R}^n)$ is called an intertwining operator (between H_v and H_0) if $H_v A = A H_0$. We shall always identify such operators with their distribution kernels , and the operator equation above can then be written as a differential equation

$$(2) \qquad (\Delta_x - \Delta_y - v(x)) A(x,y) = 0.$$

In order to describe the properties of these operators we introduce the space $\mathcal{M}$ of locally integrable functions $U(x,y)$ on $\mathbf{R}^n \times \mathbf{R}^n$ such that

$$\|U\|_{\mathcal{M}} = \max\{\sup_x \int |U(x,y|dy, \sup_y \int |U(x,y)|\, dx\} < \infty .$$

Then $\mathcal{M}$ is a Banach algebra under the composition

$$(U_1 \circ U_2)(x,y) = \int U_1(x,z) U_2(z,y)\, dz .$$

Every $U \in \mathcal{M}$ defines a linear operator $U : L^p \to L^p$, when $1 \le p \le \infty$, and

$$(3) \qquad \|U\|_{L^p \to L^p} \le \|U\|_{\mathcal{M}} .$$

THEOREM 1. *Assume that v is small (not necessarily real-valued). Then there exists an intertwining operator $A = I + U$ with $\|U\|_{\mathcal{M}} < 1$. H_v is then conjugated to H_0 by an operator which is an isomorphism in any L^p with $1 \le p \le \infty$.*

The distribution $A = A(x, y)$ is constructed by successive approximations,i.e.

$$A(x,y) = \sum_{0}^{\infty} U_N(x,y) \; , \tag{4}$$

where

$$U_0(x,y) = \delta(x-y) \; , \qquad (\Delta_x - \Delta_y)U_N(x,y) = v(x)U_{N-1}(x,y) \; .$$

In order to solve the equation to the right one makes use of special fundamental solutions E_θ for the ultra-hyperbolic operator. One defines U_N as $E_\theta * (vU_{N-1})$, where $(vU_N)(x,y) = v(x)U_N(x,y)$.

THEOREM 2. *There is a unique fundamental solution E_θ for $\Delta_x - \Delta_y$ when $\theta \in S^{n-1}$ such that*

(i) *$\langle y - x, \theta \rangle \ge 0$ in the support of E_θ;*
(ii) *$E_\theta(x+z, y+z) \to 0$ in $\mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n)$ as $|z| \to \infty$, $z/|z| \to \pm\theta$;*
(iii) *$E_\theta = \sum c_{\alpha\beta} \partial_x^\alpha \partial_y^\beta h_{\alpha\beta}$, where $\sup_x \int_{|x-y|<C} |h(x,y)|\, dy < \infty$ for every C.*

We shall now give some estimates for the $U_N = U_{N,\theta}$ from which it follows in particular that the series (4) is always convergent in the distribution sense at least.

THEOREM 3. *There is a continuous function $c(\lambda)$ on $\overline{\mathbf{R}}_+$ such that $c(\lambda) \to 0$ as $\lambda \to \infty$ and*

$$\sup_\theta \|e^{-\lambda\langle y-x,\theta\rangle} U_{N,\theta}\|_{\mathcal{M}} \le \left(c(\lambda)\|\nabla^{n-2}v\|_1\right)^N \tag{5}$$

when $\lambda \ge 0$, $N > 0$.

Remark . The set

$$\{U; e^{-\lambda\langle y-x,\theta\rangle} U(x,y) \in \mathcal{M}\}$$

is a Banach-algebra which operates on

$$L^p_{\lambda,\theta} = \{f; e^{\lambda\langle y,\theta\rangle} f(y) \in L^p\} \; .$$

COROLLARY 4. *H_v is conjugated to H_0 by an isomorphism in $L^p_{\lambda,\theta}$, when λ is large and $1 \le p \le \infty$.*

We can say much more about the U_N. It is convenient to introduce the following definition.

DEFINITION 5. *$\mathcal{M}_\theta$ is the set of all $U \in \mathcal{M}$ which satisfy the following conditions.*

(i) *$\langle y - x, \theta\rangle \ge 0$ in the support of U;*
(ii) *The repeated commutators of U with the ∂_i are in $\mathcal{M}$;*
(iii) *U satisfies the following conditions at ∞*

$$\int |U(x,y)|\, dy \to 0 \text{ as } |x| \to \infty \; , x/|x| \to \theta$$

and

$$\int |U(x,y)|\, dx \to 0 \text{ as } |y| \to \infty \; , y/|y| \to -\theta \; .$$

By carefully estimating the U_N we can prove the following result.

THEOREM 6. *Assume that $v \in \mathcal{V}$. Then there is a continuous family*

$$A_\theta = I + U_\theta \in I + \mathcal{M}_\theta \ , \theta \in S^{n-1}$$

and a continuous family $R_\theta(x,y)$ with all derivatives in $L^1(\mathbf{R}^n \times \mathbf{R}^n)$ so that

$$A_\theta^{-1} H_v A_\theta = H_0 + R_\theta \ . \tag{6}$$

Moreover, $A_\theta^{-1} \in I + \mathcal{M}_\theta$. The R_θ are of finite rank and vanish identically if v is small.

Remark. The A_θ have properties similar to elliptic pseudo-differential operators of order 0, and the kernel of U_θ^N becomes more and more smooth as N tends to ∞. The U_θ will have a small spectral radius in $\mathcal{M}$. Using the polar decomposition of A_θ it will then be possible for us to construct a unitary operator $T \in I + \mathcal{M}$ so that

$$T^* H_v T = H_0 + K \ , \tag{7}$$

where K is a finite sum of tensor products $f \otimes \bar{f}$, and all derivatives of f are in L^1.

The scattering matrix.

In this section we assume that $v \in \mathcal{V}$ is real-valued. The scattering matrix can be expressed in terms of the operators A_θ in Theorem 6. This is true at least in the case of a small potential.

THEOREM 7. *Assume that v is small. Then the scattering matrix can be identified with the function*

$$\{(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^n ; |\xi| = |\eta|\} \ni (\xi,\eta) \to (\widehat{vA_{-\eta/|\eta|}})(\xi,\eta) \ . \tag{8}$$

We shall give a heuristic motivation for this. Consider an initial state u with $\hat{u}$ supported outside the origin and near the ray defined by θ. Then the wave $u(t) = e^{-itH_v}u$ travels to infinity in the direction of $-\theta$ as $t \to -\infty$, and it follows from the definition of $\mathcal{M}_\theta$ that $A_\theta u(t)$ is close to $u(t)$, when t is close to $-\infty$. But $A_\theta u(t) = e^{-itH_v}u$ by the intertwining property, so we obtain an approximation for $W_- u$ by $A_\theta u$. A strict proof involves a partition of unity w.r.t. the frequency variables and gives us formulas for the wave operators

$$W_\pm = \lim_{t \to \pm\infty} e^{itH_v} e^{-itH_0} \ ,$$

and then also for the scattering operator $S = W_+^* W_-$.

In the case of a large potential the presence of bound states causes some problems. However , the construction of the intertwining operators shows that we can always define $A_\theta(1 - \chi(\varepsilon D))$ where χ is some cut-off function with compact support and ε is small. The conclusion of Theorem 7 is then still true if we just restrict the map (8) to the set where the energy $|\xi|^2 = |\eta|^2$ is large. Since we know that $A_\theta - I \in \mathcal{M}$, we can also easily conclude from this result that v can be recovered from S.

Trace formulas and the miracle.

We start with some general remarks. To a real-valued potential $v \in \mathcal{V}$ we have associated distributions such as $W_\pm$, A_θ and S, $Q_\theta = A_\theta^* A_\theta$. The objects in the first category are intertwining operators between H_v and H_0, while the objects in the second category commute with H_0. This means that they solve the ultra-hyperbolic equation when they are identified with their distribution kernels. The distributions we consider have their leading singularities concentrated near the diagonal in $\mathbf{R}^n \times \mathbf{R}^n$, and we shall see that all relevant information is sitting in these singularities. For the case of simplicity we assume here that the potential is small in order to have some objects globally defined.

It is natural to blow up the singularities around the diagonal. We therefore consider the map

$$\gamma : \mathbf{R}^n \times \overline{\mathbf{R}}_+ \times S^{n-1} \ni (x,r,\omega) \to (x, x+r\omega) \in \mathbf{R}^n \times \mathbf{R}^n \ . \tag{9}$$

This map induces a map between distributions

$$\gamma_* : \mathcal{D}'(\mathbf{R}^n \times \overline{\mathbf{R}}_+ \times S^{n-1}) \to \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n) \ .$$

It is easy to see that γ_* is surjective , and $r = 0$ in the support of any distribution in its kernel.

DEFINITION 8. *We shall say that $B \in \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n)$ is a θ-admissible kernel if $B = \gamma_* b$, where b has the form*

$$b(x,r,\omega) = Y_+(\langle \omega, \theta \rangle) b_+(x,r,\omega) + Y_-(\langle \omega, \theta \rangle) b_-(x,r,\omega) \ .$$

Here $Y_\pm$ are the Heaviside functions on the positive and negative half-lines, and we require the functions $b_\pm$ to be continuous in $r \geq 0$ and smooth in the other variables.

If B is θ-admissible then we use the notation $\operatorname{tr} B$ for the function $b(x,0,\omega)$. (Note that $B(x,y) = |x-y|^{1-n}$ gives $\operatorname{tr} B = $ *constant*.)

THEOREM 9. *Assume that B_1 and B_2 are θ-admissible kernels which are properly supported. Then $B_1 \circ B_2$ is also θ-admissible and $\operatorname{tr}(B_1 \circ B_2)$ vanishes.*

It is easy to see from the explicit formulas for the $U_N = U_{N,\theta}$ in (4) , that $A_\theta - I$ is θ-admissible and

$$\operatorname{tr}(A_\theta - I) = c_n Y_+(\langle \omega, \theta \rangle) \int_{\langle z, \omega \rangle = 0} \langle \omega, \partial_z \rangle^{n-2} v(x+z)\, dz \ . \tag{10}$$

Since

$$Q_\theta = A_\theta^* A_\theta = \delta(x-y) + U_{1,\theta} + U_{1,\theta}^* + \cdots \ ,$$

an application of Theorem 9 shows that

$$\operatorname{tr}(Q_\theta - I) = c_n \operatorname{sgn}(\langle \omega, \theta \rangle) \int_{\langle z, \omega \rangle = 0} \langle \omega, \partial_x \rangle^{n-2} v(x+z)\, dz \ . \tag{11}$$

This gives us the following result.

THEOREM 10. *If $Q_\theta = A_\theta^* A_\theta$ and v is small, then*

$$v(x) = c_n \int_{S^{n-1}} \operatorname{sgn}(\langle\omega,\theta\rangle)\langle\omega,\partial_x\rangle \operatorname{tr}(Q_\theta - I)(x,\omega)\, d\omega \; . \tag{12}$$

PROOF: The theorem follows from (11) and the formula

$$\begin{aligned}\int_{S^{n-1}} \Big(\int_{\langle z,\omega\rangle=0} \langle\omega,\partial_x\rangle\langle\omega,\partial_z\rangle^{n-2} v(x+z)\, dz\Big)\, d\omega \\ = \int_{S^{n-1}} \Big(\int_{\langle z,\omega\rangle=0} \Delta^{(n-1)/2} v(x+z)\, dz\Big)\, d\omega \\ = c_{1,n} \int_{\mathbf{R}^n} |z|^{-1}\Delta^{(n-1)/2} v(x+z)\, dz = c_{2,n} v(x) \; .\end{aligned}$$

Remark. Assume that v is small and $\theta \in S^{n-1}$. Then we can write the scattering matrix as a product $N_+ N_-$, where N_+ and N_- are upper and lower triangular w.r.t. the ordering of the sphere induced by θ. Moreover, Q_θ can be easily computed from $N_+^* N_+$ then.

Remark. In the coordinates x, r, ω the fundamental solution takes the form

$$E_\theta = c_n' \int_{\langle\omega,\theta\rangle\geq 0} L_\omega \, d\omega \; ,$$

where

$$(\gamma_*^{-1} L_\omega)(x, r, \phi) = \delta^{(n-2)}(\langle x,\omega\rangle)\delta(\phi - \omega) \; .$$

This gives the right formula also in the case when $n = 1$. Thus we may hope to obtain more analogies with the one-dimensional situation by using the blow up by γ. We note also that the fundamental solution is independent of r when expressed in the new coordinates

We shall finish this section by discussing the "miracle", which was discovered by R.G. Newton (see Newton [**6–7**] and also Cheney [**1**]). We write $A_\theta = I + U_\theta$ and introduce the functions

$$\beta_\pm(x,\theta,k) = e^{-ik\langle x,\theta\rangle}(W_\pm \mathcal{F}^*)(x, k\theta) - 1 \; .$$

Here $\mathcal{F}$ denotes the Fourier transform and we identify operators with their distribution kernels. We also introduce the functions

$$\eta_\pm(x,\theta,s) = \int_{\langle z,\theta\rangle=0} U_{\pm\theta}(x, s\theta + z + x)\, dz \; .$$

The expressions for the wave operators in terms of the A_θ show that $\eta_\pm$ and $\beta_\pm$ are related by the following formula

$$\eta_\pm(x,\theta,s) = (2\pi)^{-1} \int \beta_\pm(x,\theta,k) e^{-iks}\, dk \; . \tag{13}$$

By using the more or less explicit formulas for the $U_{N,\theta}$ one can prove that

$$\eta_\pm(x,\theta,s) = \big(a_\pm(x,\theta,s) + s^{1/2} b_\pm(x,\theta,s)\big) Y_\pm(s) \; , \tag{14}$$

where $a_\pm$ and $b_\pm$ are smooth functions. The "miracle" can then be expressed in the following way.

THEOREM 11. *Assume that $v \in \mathcal{V}$ is small. Then $\eta_\pm$ has the form (14) with $a_\pm$ and $b_\pm$ smooth , and*

$$\langle \theta, \partial_x \rangle a_\pm (x, \theta, 0) = \pm c_n v(x) \ , \tag{15}$$

where c_n is a positive constant which depends on n only.

This result is obtained from the explicit formula for U_θ as a sum of the U_N. Since the U_N become smoother when N increases it is clear that the left-hand side of (15) is only depending on a finite number of the U_N. The theorem has therefore a sense also in the case of large potentials.

Remark. If v is small, then $W_\pm$ are unitary, and $W_+ S = W_-$. We may write this equation as

$$(W_+ \mathcal{F}^*) \tilde{S} = W_- \mathcal{F}^* \ , \tag{16}$$

where $\tilde{S}$ is the scattering matrix, and the second arguments in $W_\pm \mathcal{F}^*$ are described in terms of polar coordinates k, θ. Using (13) and taking the Fourier transform w.r.t. k on (16) we find that the scattering matrix gives us an integral equation of Marchenko type for $\eta_\pm$. The arguments are the same as in Cheney [**1**,Theorem 6.3].

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