

# SÉMINAIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES – ÉCOLE POLYTECHNIQUE

A. MELIN

## **Some mathematical problems in inverse potential scattering**

*Séminaire Équations aux dérivées partielles (Polytechnique)* (1986-1987), exp. n° 20,  
p. 1-6

<[http://www.numdam.org/item?id=SEDP\\_1986-1987\\_\\_\\_\\_A19\\_0](http://www.numdam.org/item?id=SEDP_1986-1987____A19_0)>

© Séminaire Équations aux dérivées partielles (Polytechnique)  
(École Polytechnique), 1986-1987, tous droits réservés.

L'accès aux archives du séminaire Équations aux dérivées partielles (<http://sedp.cedram.org>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

ÉCOLE POLYTECHNIQUE  
CENTRE DE MATHÉMATIQUES  
91128 PALAISEAU CEDEX - FRANCE

---

Tél. (6) 941.82.00 - Poste N°  
Télex : ECOLEX 691596 F

SEMINAIRE EQUATIONS AUX DERIVEES PARTIELLES 1986 - 1987

SOME MATHEMATICAL PROBLEMS IN INVERSE POTENTIAL SCATTERING.

par A. MELIN



## Some mathematical problems in inverse potential scattering.

ANDERS MELIN

Department of Mathematics  
University of Lund

We shall consider the Schrödinger operator  $H_v = -\Delta + v(x)$  in  $\mathbf{R}^n$  when  $n = 3, 5, \dots$ . We shall work with real-valued potentials in the class  $\mathcal{V}$ . This is defined by the inequalities

$$(1) \quad \int (1 + |x|)^{|\alpha| - (n-2)} |D^\alpha v(x)| dx < \infty.$$

The notation  $\|v\|$  will be used for different semi-norms in  $\mathcal{V}$ .

**Remark.** By making changes of scales we find that  $H_v$  is unitarily equivalent to  $t^{-2}H_{v_t}$ , where  $v_t(x) = t^2 v(tx)$ . The semi-norms with  $|\alpha| = n - 2$  are not changed then, and we let  $\|\nabla^{n-2} v\|_1$  denote the sum of such semi-norms. We shall say that  $v$  is small if this number is small. If  $v$  is small, then  $H_v$  is conjugated to  $H_0$  by an operator which is an isomorphism in  $L^p$  when  $1 \leq p \leq \infty$ , and there are no bound states then. This result will follow from our constructions of intertwining operators.

### Intertwining operators.

Most of the ideas behind the construction of intertwining operators can be found in Faddeev's papers [2-3], where he introduced direction-dependent Green's functions. Since we are able to give explicit formulas for these functions, it will be possible for us to obtain precise information about the intertwining operators. This will allow us to generalize to higher dimensions the approach to inverse scattering that we developed in Melin [4].

A continuous linear transformation  $A : C_0^\infty(\mathbf{R}^n) \rightarrow \mathcal{D}'(\mathbf{R}^n)$  is called an intertwining operator (between  $H_v$  and  $H_0$ ) if  $H_v A = A H_0$ . We shall always identify such operators with their distribution kernels, and the operator equation above can then be written as a differential equation

$$(2) \quad (\Delta_x - \Delta_y - v(x))A(x, y) = 0.$$

In order to describe the properties of these operators we introduce the space  $\mathcal{M}$  of locally integrable functions  $U(x, y)$  on  $\mathbf{R}^n \times \mathbf{R}^n$  such that

$$\|U\|_{\mathcal{M}} = \max\left\{\sup_x \int |U(x, y)| dy, \sup_y \int |U(x, y)| dx\right\} < \infty.$$

Then  $\mathcal{M}$  is a Banach algebra under the composition

$$(U_1 \circ U_2)(x, y) = \int U_1(x, z)U_2(z, y) dz.$$

Every  $U \in \mathcal{M}$  defines a linear operator  $U : L^p \rightarrow L^p$ , when  $1 \leq p \leq \infty$ , and

$$(3) \quad \|U\|_{L^p \rightarrow L^p} \leq \|U\|_{\mathcal{M}}.$$

**THEOREM 1.** Assume that  $v$  is small (not necessarily real-valued). Then there exists an intertwining operator  $A = I + U$  with  $\|U\|_{\mathcal{M}} < 1$ .  $H_v$  is then conjugated to  $H_0$  by an operator which is an isomorphism in any  $L^p$  with  $1 \leq p \leq \infty$ .

The distribution  $A = A(x, y)$  is constructed by successive approximations, i.e.

$$(4) \quad A(x, y) = \sum_0^{\infty} U_N(x, y),$$

where

$$U_0(x, y) = \delta(x - y), \quad (\Delta_x - \Delta_y)U_N(x, y) = v(x)U_{N-1}(x, y).$$

In order to solve the equation to the right one makes use of special fundamental solutions  $E_\theta$  for the ultra-hyperbolic operator. One defines  $U_N$  as  $E_\theta * (vU_{N-1})$ , where  $(vU_N)(x, y) = v(x)U_N(x, y)$ .

**THEOREM 2.** There is a unique fundamental solution  $E_\theta$  for  $\Delta_x - \Delta_y$  when  $\theta \in S^{n-1}$  such that

- (i)  $\langle y - x, \theta \rangle \geq 0$  in the support of  $E_\theta$ ;
- (ii)  $E_\theta(x + z, y + z) \rightarrow 0$  in  $\mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n)$  as  $|z| \rightarrow \infty$ ,  $z/|z| \rightarrow \pm\theta$ ;
- (iii)  $E_\theta = \sum c_{\alpha\beta} \partial_x^\alpha \partial_y^\beta h_{\alpha\beta}$ , where  $\sup_x \int_{|x-y| < C} |h(x, y)| dy < \infty$  for every  $C$ .

We shall now give some estimates for the  $U_N = U_{N, \theta}$  from which it follows in particular that the series (4) is always convergent in the distribution sense at least.

**THEOREM 3.** There is a continuous function  $c(\lambda)$  on  $\overline{\mathbf{R}}_+$  such that  $c(\lambda) \rightarrow 0$  as  $\lambda \rightarrow \infty$  and

$$(5) \quad \sup_\theta \|e^{-\lambda \langle y - x, \theta \rangle} U_{N, \theta}\|_{\mathcal{M}} \leq \left( c(\lambda) \|\nabla^{n-2} v\|_1 \right)^N$$

when  $\lambda \geq 0$ ,  $N > 0$ .

**Remark .** The set

$$\{U; e^{-\lambda \langle y - x, \theta \rangle} U(x, y) \in \mathcal{M}\}$$

is a Banach-algebra which operates on

$$L_{\lambda, \theta}^p = \{f; e^{\lambda \langle y, \theta \rangle} f(y) \in L^p\}.$$

**COROLLARY 4.**  $H_v$  is conjugated to  $H_0$  by an isomorphism in  $L_{\lambda, \theta}^p$ , when  $\lambda$  is large and  $1 \leq p \leq \infty$ .

We can say much more about the  $U_N$ . It is convenient to introduce the following definition.

**DEFINITION 5.**  $\mathcal{M}_\theta$  is the set of all  $U \in \mathcal{M}$  which satisfy the following conditions.

- (i)  $\langle y - x, \theta \rangle \geq 0$  in the support of  $U$ ;
- (ii) The repeated commutators of  $U$  with the  $\partial_i$  are in  $\mathcal{M}$ ;
- (iii)  $U$  satisfies the following conditions at  $\infty$

$$\int |U(x, y)| dy \rightarrow 0 \text{ as } |x| \rightarrow \infty, x/|x| \rightarrow \theta$$

and

$$\int |U(x, y)| dx \rightarrow 0 \text{ as } |y| \rightarrow \infty, y/|y| \rightarrow -\theta.$$

By carefully estimating the  $U_N$  we can prove the following result.

THEOREM 6. Assume that  $v \in \mathcal{V}$ . Then there is a continuous family

$$A_\theta = I + U_\theta \in I + \mathcal{M}_\theta, \theta \in S^{n-1}$$

and a continuous family  $R_\theta(x, y)$  with all derivatives in  $L^1(\mathbf{R}^n \times \mathbf{R}^n)$  so that

$$(6) \quad A_\theta^{-1} H_v A_\theta = H_0 + R_\theta.$$

Moreover,  $A_\theta^{-1} \in I + \mathcal{M}_\theta$ . The  $R_\theta$  are of finite rank and vanish identically if  $v$  is small.

**Remark.** The  $A_\theta$  have properties similar to elliptic pseudo-differential operators of order 0, and the kernel of  $U_\theta^N$  becomes more and more smooth as  $N$  tends to  $\infty$ . The  $U_\theta$  will have a small spectral radius in  $\mathcal{M}$ . Using the polar decomposition of  $A_\theta$  it will then be possible for us to construct a unitary operator  $T \in I + \mathcal{M}$  so that

$$(7) \quad T^* H_v T = H_0 + K,$$

where  $K$  is a finite sum of tensor products  $f \otimes \bar{f}$ , and all derivatives of  $f$  are in  $L^1$ .

### The scattering matrix.

In this section we assume that  $v \in \mathcal{V}$  is real-valued. The scattering matrix can be expressed in terms of the operators  $A_\theta$  in Theorem 6. This is true at least in the case of a small potential.

THEOREM 7. Assume that  $v$  is small. Then the scattering matrix can be identified with the function

$$(8) \quad \{(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^n; |\xi| = |\eta|\} \ni (\xi, \eta) \rightarrow (\widehat{vA}_{-\eta/|\eta|})(\xi, \eta).$$

We shall give a heuristic motivation for this. Consider an initial state  $u$  with  $\hat{u}$  supported outside the origin and near the ray defined by  $\theta$ . Then the wave  $u(t) = e^{-itH_v} u$  travels to infinity in the direction of  $-\theta$  as  $t \rightarrow -\infty$ , and it follows from the definition of  $\mathcal{M}_\theta$  that  $A_\theta u(t)$  is close to  $u(t)$ , when  $t$  is close to  $-\infty$ . But  $A_\theta u(t) = e^{-itH_v} u$  by the intertwining property, so we obtain an approximation for  $W_- u$  by  $A_\theta u$ . A strict proof involves a partition of unity w.r.t. the frequency variables and gives us formulas for the wave operators

$$W_\pm = \lim_{t \rightarrow \pm\infty} e^{itH_v} e^{-itH_0},$$

and then also for the scattering operator  $S = W_+^* W_-$ .

In the case of a large potential the presence of bound states causes some problems. However, the construction of the intertwining operators shows that we can always define  $A_\theta(1 - \chi(\varepsilon D))$  where  $\chi$  is some cut-off function with compact support and  $\varepsilon$  is small. The conclusion of Theorem 7 is then still true if we just restrict the map (8) to the set where the energy  $|\xi|^2 = |\eta|^2$  is large. Since we know that  $A_\theta - I \in \mathcal{M}$ , we can also easily conclude from this result that  $v$  can be recovered from  $S$ .

### Trace formulas and the miracle.

We start with some general remarks. To a real-valued potential  $v \in \mathcal{V}$  we have associated distributions such as  $W_{\pm}$ ,  $A_{\theta}$  and  $S$ ,  $Q_{\theta} = A_{\theta}^* A_{\theta}$ . The objects in the first category are intertwining operators between  $H_v$  and  $H_0$ , while the objects in the second category commute with  $H_0$ . This means that they solve the ultra-hyperbolic equation when they are identified with their distribution kernels. The distributions we consider have their leading singularities concentrated near the diagonal in  $\mathbf{R}^n \times \mathbf{R}^n$ , and we shall see that all relevant information is sitting in these singularities. For the case of simplicity we assume here that the potential is small in order to have some objects globally defined.

It is natural to blow up the singularities around the diagonal. We therefore consider the map

$$(9) \quad \gamma : \mathbf{R}^n \times \overline{\mathbf{R}}_+ \times S^{n-1} \ni (x, r, \omega) \rightarrow (x, x + r\omega) \in \mathbf{R}^n \times \mathbf{R}^n .$$

This map induces a map between distributions

$$\gamma_* : \mathcal{D}'(\mathbf{R}^n \times \overline{\mathbf{R}}_+ \times S^{n-1}) \rightarrow \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n) .$$

It is easy to see that  $\gamma_*$  is surjective, and  $r = 0$  in the support of any distribution in its kernel.

**DEFINITION 8.** We shall say that  $B \in \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n)$  is a  $\theta$ -admissible kernel if  $B = \gamma_* b$ , where  $b$  has the form

$$b(x, r, \omega) = Y_+(\langle \omega, \theta \rangle) b_+(x, r, \omega) + Y_-(\langle \omega, \theta \rangle) b_-(x, r, \omega) .$$

Here  $Y_{\pm}$  are the Heaviside functions on the positive and negative half-lines, and we require the functions  $b_{\pm}$  to be continuous in  $r \geq 0$  and smooth in the other variables.

If  $B$  is  $\theta$ -admissible then we use the notation  $\text{tr } B$  for the function  $b(x, 0, \omega)$ . (Note that  $B(x, y) = |x - y|^{1-n}$  gives  $\text{tr } B = \text{constant}$ .)

**THEOREM 9.** Assume that  $B_1$  and  $B_2$  are  $\theta$ -admissible kernels which are properly supported. Then  $B_1 \circ B_2$  is also  $\theta$ -admissible and  $\text{tr}(B_1 \circ B_2)$  vanishes.

It is easy to see from the explicit formulas for the  $U_N = U_{N, \theta}$  in (4), that  $A_{\theta} - I$  is  $\theta$ -admissible and

$$(10) \quad \text{tr}(A_{\theta} - I) = c_n Y_+(\langle \omega, \theta \rangle) \int_{\langle z, \omega \rangle = 0} \langle \omega, \partial_z \rangle^{n-2} v(x + z) dz .$$

Since

$$Q_{\theta} = A_{\theta}^* A_{\theta} = \delta(x - y) + U_{1, \theta} + U_{1, \theta}^* + \dots ,$$

an application of Theorem 9 shows that

$$(11) \quad \text{tr}(Q_{\theta} - I) = c_n \text{sgn}(\langle \omega, \theta \rangle) \int_{\langle z, \omega \rangle = 0} \langle \omega, \partial_z \rangle^{n-2} v(x + z) dz .$$

This gives us the following result.

THEOREM 10. If  $Q_\theta = A_\theta^* A_\theta$  and  $v$  is small, then

$$(12) \quad v(x) = c_n \int_{S^{n-1}} \operatorname{sgn}(\langle \omega, \theta \rangle) \langle \omega, \partial_x \rangle \operatorname{tr}(Q_\theta - I)(x, \omega) d\omega .$$

PROOF: The theorem follows from (11) and the formula

$$\begin{aligned} \int_{S^{n-1}} \left( \int_{\langle z, \omega \rangle = 0} \langle \omega, \partial_x \rangle \langle \omega, \partial_z \rangle^{n-2} v(x+z) dz \right) d\omega \\ = \int_{S^{n-1}} \left( \int_{\langle z, \omega \rangle = 0} \Delta^{(n-1)/2} v(x+z) dz \right) d\omega \\ = c_{1,n} \int_{\mathbf{R}^n} |z|^{-1} \Delta^{(n-1)/2} v(x+z) dz = c_{2,n} v(x) . \end{aligned}$$

**Remark.** Assume that  $v$  is small and  $\theta \in S^{n-1}$ . Then we can write the scattering matrix as a product  $N_+ N_-$ , where  $N_+$  and  $N_-$  are upper and lower triangular w.r.t. the ordering of the sphere induced by  $\theta$ . Moreover,  $Q_\theta$  can be easily computed from  $N_+^* N_+$  then.

**Remark.** In the coordinates  $x, r, \omega$  the fundamental solution takes the form

$$E_\theta = c'_n \int_{\langle \omega, \theta \rangle \geq 0} L_\omega d\omega ,$$

where

$$(\gamma_*^{-1} L_\omega)(x, r, \phi) = \delta^{(n-2)}(\langle x, \omega \rangle) \delta(\phi - \omega) .$$

This gives the right formula also in the case when  $n = 1$ . Thus we may hope to obtain more analogies with the one-dimensional situation by using the blow up by  $\gamma$ . We note also that the fundamental solution is independent of  $r$  when expressed in the new coordinates

We shall finish this section by discussing the “miracle”, which was discovered by R.G. Newton (see Newton [6-7] and also Cheney [1]). We write  $A_\theta = I + U_\theta$  and introduce the functions

$$\beta_\pm(x, \theta, k) = e^{-ik\langle x, \theta \rangle} (W_\pm \mathcal{F}^*)(x, k\theta) - 1 .$$

Here  $\mathcal{F}$  denotes the Fourier transform and we identify operators with their distribution kernels. We also introduce the functions

$$\eta_\pm(x, \theta, s) = \int_{\langle z, \theta \rangle = 0} U_{\pm\theta}(x, s\theta + z + x) dz .$$

The expressions for the wave operators in terms of the  $A_\theta$  show that  $\eta_\pm$  and  $\beta_\pm$  are related by the following formula

$$(13) \quad \eta_\pm(x, \theta, s) = (2\pi)^{-1} \int \beta_\pm(x, \theta, k) e^{-iks} dk .$$

By using the more or less explicit formulas for the  $U_{N,\theta}$  one can prove that

$$(14) \quad \eta_\pm(x, \theta, s) = (a_\pm(x, \theta, s) + s^{1/2} b_\pm(x, \theta, s)) Y_\pm(s) ,$$

where  $a_\pm$  and  $b_\pm$  are smooth functions. The “miracle” can then be expressed in the following way.

**THEOREM 11.** Assume that  $v \in \mathcal{V}$  is small. Then  $\eta_{\pm}$  has the form (14) with  $a_{\pm}$  and  $b_{\pm}$  smooth, and

$$(15) \quad \langle \theta, \partial_x \rangle a_{\pm}(x, \theta, 0) = \pm c_n v(x),$$

where  $c_n$  is a positive constant which depends on  $n$  only.

This result is obtained from the explicit formula for  $U_{\theta}$  as a sum of the  $U_N$ . Since the  $U_N$  become smoother when  $N$  increases it is clear that the left-hand side of (15) is only depending on a finite number of the  $U_N$ . The theorem has therefore a sense also in the case of large potentials.

**Remark.** If  $v$  is small, then  $W_{\pm}$  are unitary, and  $W_+ S = W_-$ . We may write this equation as

$$(16) \quad (W_+ \mathcal{F}^*) \tilde{S} = W_- \mathcal{F}^*,$$

where  $\tilde{S}$  is the scattering matrix, and the second arguments in  $W_{\pm} \mathcal{F}^*$  are described in terms of polar coordinates  $k, \theta$ . Using (13) and taking the Fourier transform w.r.t.  $k$  on (16) we find that the scattering matrix gives us an integral equation of Marchenko type for  $\eta_{\pm}$ . The arguments are the same as in Cheney [1, Theorem 6.3].

#### REFERENCES

1. M.Cheney, *Inverse scattering in dimension 2*, J. Math. Phys. **25** (1984), 94-107.
2. L.D. Faddeev, *The inverse problem in quantum theory of scattering*, J. Math. Phys. **4** (1963), 72-104.
3. L.D. Faddeev, *Inverse problem of quantum scattering theory, II*, J. Sov. Math. **5** (1976), 334-396.
4. A. Melin, *Operator methods for inverse scattering on the real line*, Comm. in Partial Differential Eqs. **10** (1985), 677-766.
5. A.Melin, *Intertwining methods in the theory of inverse scattering*, to appear in Int. J. of Quantum Chemistry.
6. R.G. Newton, *Inverse scattering II. Three dimensions*, J. Math. Phys. **21** (1980), 1698-1715.
7. R. G. Newton, *An inverse spectral problem in three dimensions*, SIAM-AMS Proceedings **14**, 81-90.