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SEMINAIRE EQUATIONS AUX DERIVEES PARTIELLES 1986 - 1987

ON THE SINGULARITIES OF HARMONIC MAPS FROM A DOMAIN IN $\mathbb{R}^3$ INTO S^2

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On the singularities of harmonic maps from a domain in $\mathbb{R}^3$ into S^2

by

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I report here on a joint work with H. Brézis and E. Lieb about the singularities of minimizing harmonic maps from a domain in $\mathbb{R}^3$ into the Euclidean sphere in $\mathbb{R}^3$.

Part A

I. Introduction.

Let $S^2 = \{x = (x_1,x_2,x_3) \in \mathbb{R}^3 \mid |x|^2 = x_1^2+x_2^2+x_3^2 = 1\}$, let Ω be a bounded regular open set in $\mathbb{R}^3$ and let g be a smooth (i.e. $C^{2,\alpha}$) map from $\partial\Omega$ into S^2. Let

$$\mathcal{E} = \{u = (u^1,u^2,u^3) \in H^1(\Omega;\mathbb{R}^3) \mid u=g \text{ on } \partial\Omega \text{ and } u(x) \in S^2 \text{ for a.e.} x\}$$

For u in $\mathcal{E}$ we define

$$E(u) = \int_\Omega |\nabla u|^2 ,$$

where $|\nabla u|^2 = \sum_{\substack{1\le i\le 3\\ 1\le j\le 3}} \left(\frac{\partial u^i}{\partial x_j}\right)^2$. We shall say that u in $\mathcal{E}$ is a minimizing map if

$$E(u) = \operatorname*{Inf}_{\varphi \in \mathcal{E}} E(\varphi) . \tag{1}$$

For φ in E we define the regular set of φ by

$$R(\varphi) = \{x \in \bar{\Omega} \mid \varphi \text{ is } C^\infty \text{ in a neighborhood of } x \text{ in } \bar{\Omega}\}$$

and the singular set of φ by

$$S(\varphi) = \bar{\Omega} \setminus R(\varphi) .$$

R. Schoen and K. Uhlenbeck [1] [1.4] have proved (see also M. Giaquinta - E. Giusti [6] and J. Jost - M. Meier [10] for related problems) that, if u is a minimizing map, then $S(u) \subset \Omega$ and is finite. For a point x_0 in $S(u)$, let Σ be a small sphere centered at x_0 ; u restricted to Σ is a continuous map from Σ into S^2 and so has a degree d in $\mathbb{Z}$; clearly d is independant of Σ provided that the radius of Σ is small enough; this number d will be called the degree of the singularity x_0 . Our first result is

<u>Theorem 1</u> [4]

Let u be a minimizing map and x_0 be in $S(u)$. Then the degree of the singularity x_0 is $+1$ or -1 and more precisely, near x_0 ,

(2) $\varphi(x) \simeq \pm R(x-x_0)/|x-x_0|$ where R is a rotation.

A sketch of a proof of Theorem 1 will be given in section II.

Remark 2

a. R. Hardt - D. Kinderlehrer - F.H. Lin [9] had proved that there exists some constant which does not depend on g and on Ω which bounds the absolute value of the degree of any singularity of any minimizing map.

b. The significance of (2), following R. Schoen - K. Uhlenbeck [13] and L. Simon [15] is (where we have taken $x_0 = 0$)

(3) $$\lim_{\varepsilon \to 0^+} \left\| u(\varepsilon x) \mp R\left(\frac{x}{|x|}\right) \right\|_{H^1(B)} = 0$$

and

(4) $$\lim_{\varepsilon \to 0^+} \| u(\varepsilon x) \mp Rx \|_{C^2(S^2)} + \| D_\rho (u(\varepsilon x)) \|_{C^1(S^2)} = 0$$

where $B = \{x \in \mathbb{R}^3 \mid |x| < 1\}$ and D_ρ is the partial differentiation in spherical coordinates of $\mathbb{R}^3$ with respect to $\rho = |x|$. In fact, R. Gulliver - B. White [7] have improved (4): they prove that there exists some strictly positive (which does not depend on u) and a constant C such that

(5) $$\| u(\varepsilon x) \mp Rx \|_{C^2(S^2)} + \| D_\varphi (u(\varepsilon x)) \|_{C^1(S^2)} \leqq C\rho^\lambda \ .$$

c. R. Cohen et. al. [4] have observed numerically that if $\Omega = B$, $u(x) = P\left((P^{-1}\left(\frac{x}{|x|}\right))^2\right)$ $\left(\text{resp. } u(x) = P\left(2P^{-1}\left(\frac{x}{|x|}\right)\right)\right)$

where $P: \mathbb{C} \to S^2$ is the usual stereographic projection - see (15) - , and if $g = u$ on ∂B , then u is not a minimizing map.

For our next result we take

$$\Omega = B = \{x \in \mathbb{R}^3 \mid |x| < 1\} , \; g(x) = x .$$

We prove in [4]

Theorem 3

$\frac{x}{|x|}$ is a minimizer.

Two proofs of Theorem 3 will be given in section III.

Remark 4.

It is in fact possible to prove (see [4]) that $\frac{x}{|x|}$ is the unique minimizing map. Uniqueness follows also from Theorem 3 and A. Baldes [1]

II. Sketch of a proof of Theorem 1

We are going to prove

Theorem 5

If $\Omega = B$ and if $g\left(\frac{x}{|x|}\right)$ is a minimizer then either $g \equiv$ const. or there exists a rotation R such that $g(x) = \pm Rx$ for any x in S^2 . Clearly Theorem 1 follows from Theorem 5 and [13].

Sketch of a proof of Theorem 5.

We take $\Omega = B$, $u(x) = g\left(\frac{x}{|x|}\right)$ and we assume that u is a minimizer; in particular u satisfies the Euler - Lagrange equation

$$-\Delta u = u|\nabla u|^2 ,$$

hence g is a harmonic map from S^2 into S^2 .

Let d be the degree of the continuous map $g: S^2 \to S^2$. Since every harmonic map from S^2 into S^2 of degree 0 is a constant map (see e.g. [12]) we have

(6) $d = 0 \Rightarrow g$ is a constant map.

We are going to prove

(7) $d = \pm 1 \Rightarrow$ there exists a rotation R such that $g(x) = \pm Rx \ \forall x \in S^2$,

and

(8) $|d| \geqq 2$ is impossible.

Theorem 5 follows from (6), (7) and (8).

Proof of (7)

Let a be a point in S^2 ; let ε be in $(0, 1)$ and let $T_\varepsilon^a: \bar{\Omega}\setminus\{a\} \to S^2$ be defined by the condition that x belongs to the segment $[\varepsilon a, T_\varepsilon^a x]$. Note that

(9) $$T_\varepsilon^a x = x \qquad \forall\, x \in S^2$$

We define $u_\varepsilon^a : \bar{\Omega}\ \{a\} \to S^2$ by

(10) $$u_\varepsilon^a(x) = g(T_\varepsilon^a x)$$

It is easy to check that u_ε^a is in H^1 and it follows from (9) that

(11) $$u_\varepsilon^a(x) = g(x) \qquad \forall\, x \in \partial\,\Omega\ .$$

Hence $u_\varepsilon^a \in \mathcal{E}$ and so we have

(12) $$E(u) \leqq E(u_\varepsilon^a)\ .$$

A straightforward computation leads to

(13) $$E(u_\varepsilon^a) = E(u) - \varepsilon(a.\int_{S^2} \sigma|\nabla_T g(\sigma)|^2 d\sigma) + o(\varepsilon)\ ,$$

where ∇_T is the tangential gradient.
Since (12) is true for any a in S^2 and any ε in $(0,1)$ we have

(14) $$\int_{S^2} \sigma|\nabla_T g(\sigma)|^2 d\sigma = 0$$

Finally using the description of harmonic maps from S^2 into S^2 it follows (see [4]) from (14) that if $|d| = 1$ then there exists a rotation R such that $gx = \pm\, Rx$.

Remark 6.

For any d in $\mathbb{Z}$ there are harmonic maps g from S^2 into S^2 of degree d which satisfy (14); hence we cannot use the same testing functions u_ε^a to prove (8). In fact in order to prove (8) we are going to split the singularity of degree d (if $d \geqq 2$) into d singularities of degree $+1$.

<u>Proof of (8)</u>

Let $P : \mathbb{C} \to S^2$ be the stereographic projection defined by

(15) $$P(z) = (1+|z|^2)^{-1} (2x, 2y, 1-|z|^2)$$

where $z = x+iy$. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f = P^{-1} o g o P$; let ε be in (o,∞) ; and let $\alpha : [\varepsilon,1] \to [0,\infty)$ be any smooth function such that $\alpha(\varepsilon) = 1$, $\alpha(1) = 0$, and $\alpha(t) > 0$ for $t \neq 1$. We define now $u_\varepsilon : \Omega \to S^2$ by

$$u_\varepsilon(x) = P\left\{ \frac{1}{\alpha\left(\frac{\varepsilon}{|x|}\right)} f\left(P^{-1}\left(\frac{x}{|x|}\right)\right)\right\} \quad \text{if} \quad |x| > \varepsilon$$

$$u_\varepsilon(x) = P(\infty) \quad \text{if} \quad |x| \leqq \varepsilon .$$

Note that $u_\varepsilon = g$ on $\partial\Omega$, and the singular set of u_ε is

(16) $$S(u_\varepsilon) = \{\varepsilon P(z) \mid f(z) = 0\} .$$

So if f has d distinct zeros, then u_ε has d singularities. Since u is a minimizer we have

(17) $$E(u) \leqq E(u_\varepsilon) .$$

Note that

(18) $$E(u) = 8\pi|d| .$$

A straightforward computation (see [3]) leads to

(19) $$E(u_\varepsilon) = 8\pi(|d|-\varepsilon) + 16\varepsilon \int_\varepsilon^1 dt \int_{\mathbb{R}^2} \frac{\alpha'(t)^2 \; |f(z)|^2 dx\, dy}{(\alpha(t)^2+|f(z)|^2)^2(1+|z|^2)^2} .$$

Hence using (17), (18) and (19) we have

(20) $$\frac{\pi|d|}{2} \leq \int_\varepsilon^1 dt \int_{\mathbb{R}^2} \frac{\alpha'(t)^2 \; |f(z)|^2 dx\, dy}{(\alpha(t)^2+|f(z)|^2)^2(1+|z|^2)^2} .$$

We now take $\varepsilon \to 0$ and after choosing the "best" α (i.e. the α which minimizes the right hand side of (20) when $\varepsilon = 0$) we get

(21) $$\left(\frac{\pi|d|}{2}\right)^{1/2} \leq \int_0^1 dt \left\{ \int_{\mathbb{R}^2} \frac{|f(z)|^2 \; dx\, dy}{(t^2+|f(z)|^2)^2(1+|z|^2)^2} \right\}^{1/2} .$$

Unfortunately if, for example, $f(z) = z^2$ then (21) is true; this in fact quite natural since z^2 has a double zero and so (see (16)) u_ε has only one singularity. The singularity of u has not been split. In order to avoid this difficulty we remark that if R is a rotation and if $u_R = R \circ u$, $g_R = R \circ g$ then, clearly, u_R is a minimizer for the boundary condition g_R ; hence we have also (see (21))

(22) $$\left(\frac{\pi|d|}{2}\right)^{1/2} \leq \int_0^1 ds \left\{ \int_{\mathbb{R}^3} \frac{|f_R(z)|^2 \; dx\, dy}{(S^2+|f_R(z)|^2)^2(1+|z|^2)^2} \right\}^{1/2}$$

where $f_R = P^{-1} o g_R o P$.

We now average (22) over all rotations and after some computations (see [4]) we get

$$(23) \qquad |d| < 2 ,$$

hence the assertion (8)

III. Proofs of Theorem 3

We give in this section two proofs of Theorem 3.

1. First proof of Theorem 3

This proof relies on Theorem 1 and [13] - [14]. We consider $\Omega = B$ and a smooth map $g: \partial\Omega \to S^2$ of degree one (one can take, for example, $g(x) = x$) . Let u be a minimizer; since the degree of g is not zero, $S(u)$ cannot be emty. Let x_0 be in $S(u)$; by [14] $x_0 \in \Omega$. It follows from Theorem 1 that there exists a rotation R such that $u(x) \cong \pm R\left(\frac{x-x_0}{|x-x_0|}\right)$ near x_0 ; but using [13] we know that the homogeneous tangent map: $\Omega \to S^2$, $x \to \pm R\left(\frac{x}{|x|}\right)$, has to be a minimizer with respect to its own boundary conditions and since E is invarient under isometry we have Theorem 3.

2. Second proof of Theorem 3.

This proof is more direct. Here Ω is the unit ball B and g is the identity. Let

$$\mathcal{E}' = \{u \in \mathcal{E} \mid S(u) \subset \Omega \text{ and } S(u) \text{ is finite}\} .$$

F. Bethuel - X. Zheng [1] have proved that $\mathcal{E}'$ is dense in $\mathcal{E}$
Hence, in order to prove Theorem 3 we have only to prove

$$(24) \qquad E(u) \geqq 8\pi \qquad \forall\, u \in \mathcal{E}' .$$

(Note that $E\left(\frac{x}{|x|}\right) = 8\pi$) .
For u in $\mathcal{E}$ we define a vector field $\vec{D}$ in $L^1(\Omega)^3$ by

$$\vec{D} = (u.(u_y x u_z),\ u.(u_z x u_x),\ u.(u_x x u_y))$$

The usefulness of $\vec{D}$ comes from the following two facts (see [4]):

$$(25) \qquad 2|\vec{D}| \leqq |\nabla u|^2 \qquad \forall\, u \in \mathcal{E}$$

$$(26) \qquad \operatorname{div} \vec{D} = 4\pi \sum_{n=1}^{p} k_n \delta_{a_n} \qquad \forall\, u \in \mathcal{E}' ,$$

where in (26) $\{a_n \,/\, 1 \leqq n \leqq p\} = S(u)$, k_n is the degree of u at a_n and δ_{a_n} is the Dirac mass at the point a_n . Let $\theta : \bar{\Omega} \to \mathbb{R}$ be such that $|\theta(x)-\theta(y)| \leqq |x-y|$ and let u be in $\mathcal{E}'$; it follows from (25) and (26) that

$$E(u) \geqq 2 \int |\vec{D}| \geqq 2 \int \vec{D}.\nabla\theta = 2 \{ \int_{\partial\Omega} (\vec{D}.\vec{\nu})\theta - \sum_{n=1}^{p} k_n \theta(a_n) \} .$$

But $\vec{D}.\vec{\nu} = 1$ on $\partial\Omega$ since $u = g$ on $\partial\Omega$ and so we have

$$(27) \qquad E(u) \geqq 8\pi (\int_{\partial\Omega} \theta d\mu - \int_{\bar{\Omega}} \theta d\nu)$$

where $\mu = \frac{d\sigma}{4\pi}$ and $\nu = \sum_{n=1}^{p} k_n \delta_{a_n}$. Note that $\sum_{n=1}^{p} k_n = 1$. We now use

Lemma 7.

Let (M,d) be a compact metric space, let μ be a probability measure on M and let $\nu = \sum_{n=1}^{p} k_n \delta_{a_n}$ where $a_1, \dots, a_p$ are p points of M, the k_i belong to $\mathbb{Z}$ and satisfy $\sum_{i=1}^{p} k_i = 1$. Then

$$\operatorname{Max} \{ \int \theta d\mu - \int \theta d\nu \mid \theta \in \mathrm{Lip}_1 \} \geq \operatorname*{Min}_{c \in M} \int d(x,c) d\mu(x)$$

where $\mathrm{Lip}_1 = \{ \theta \in C(M;\mathbb{R}) \mid |\theta(x) - \theta(y)| \leq d(x,y) \ \forall (x,y) \in M^2 \}$.

We apply this Lemma to $M = \bar{\Omega}$ with the usual distance. It then follows from (27) that

$$E(u) \geq 2 \operatorname*{Min}_{c \in \bar{\Omega}} \int_{\partial\Omega} |x-c| \, d\sigma(x) \; ; \tag{28}$$

but the right hand side of (27) is $2 \int_{\partial\Omega} |x| d\sigma(x)$ i.e. 8π. Hence Theorem 3.

We finally sketch a proof of Lemma 7. By approximation we may assume that $\mu = \frac{1}{q} \sum_{j=1}^{q} \delta_{b_j}$. Let $\mu' = q\mu$ and let

$$\nu'_+ = \sum_{k_n > 0} k_n q \delta_{a_n} = \sum_{i=1}^{\ell} \delta_{P_i} \quad \text{where} \quad \ell = q \left(\sum_{k_n > 0} k_n \right)$$

$$\nu'_- = - \sum_{k_n < 0} k_n q \delta_{a_n} = \sum_{i=q+1}^{\ell} \delta_{N_i}$$

$$I = \operatorname{Max} \{ \int \theta d\mu - \int \theta d\nu \mid \theta \in \mathrm{Lip}_1 \}$$

$$\gamma_1 = \mu' + \nu'_-$$

$$\gamma_2 = \nu'_+$$

and finally let $N_j = b_i$ for $j \in [1,q]$.

We have

$$qI = \text{Max} \{ \int \theta d\gamma_1 - \int \theta d\gamma_2 \mid \theta \in \text{Lip}_1 \}$$

It follows from Kantorovich's theorem [11] that

$$(29) \qquad qI = \min_{m \in \mathcal{M}} \int_{M \times M} d(x,y) \, dm(x,y) \ ,$$

where $\mathcal{M}$ is the set of positive measure on $M \times M$ such that $\pi_1 m = \gamma_1$ and $\pi_2 m = \gamma_2$ if we denote by π_1 (resp. π_2) the projection on the first factor (resp. second factor) of $M \times M$. Note that

$$\mathcal{M} = \{ \sum_{\substack{1 \leq i \leq l \\ 1 \leq j \leq l}} t_{ij} \delta_{N_i} \otimes \delta_{P_j} \mid t_{ij} \geq 0 \forall i \forall j \ , \ \sum_{i=1}^{\ell} t_{ij} = 1 \forall j \ , \ \sum_{j=1}^{\ell} t_{ij} = 1 \forall i \} .$$

$\mathcal{M}$ is a convex set. Let $\mathcal{M}'$ be the set of extremal points of $\mathcal{M}$; we have

$$(30) \quad \min_{m \in \mathcal{M}} \int_{M \times M} d(x,y) \, dm(x,y) = \min_{m \in \mathcal{M}'} \int_{M \times M} d(x,y) \, dm(x,y) \ .$$

The set $\mathcal{M}'$ is described by Birkhoff's theorem [3]:

$$(31) \qquad \mathcal{M}' = \{ \sum_{i=1}^{\ell} \delta_{N_i} \otimes \delta_{P_{\sigma(i)}} \mid \sigma \in \Sigma_\ell \} \ ,$$

where Σ_ℓ is the set of permutations of $\{1, \ldots ,\ell\}$.

From (29), (30) and (31) we have

(32) $$qI = \min_{\sigma \in \Sigma_\ell} \sum_{i=1}^{\ell} d(N_i, P_{\sigma(i)}) .$$

Using a theorem in Graph Theory due to Y.O. Hamidoune - M. Las Vergnas [8] we know that for any σ in Σ_ℓ there exists i_0 in $[1,1]$ such that (see [4]):

(33) $$\Sigma\, d(N_i, P_{\sigma(i)}) \geqq \sum_{j=i}^{q} d(P_{i_0}, N_j) = q \int_M d(P_{i_0}, x)\, d\mu .$$

Lemma 7 follows from (32) and (33).

Remark

In [4] there is also an alternative argument to the use of [8].

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