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J. Rauch

M. Reed

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ÉCOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES

91128 PALAISEAU CEDEX - FRANCE

Tél. (6) 941.82.00 - Poste N°
Télex : ECOLEX 691 596 F

SÉMINAIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES 1985 - 1986

CLASSICAL, CONORMAL, SEMILINEAR WAVES.

by J. RAUCH and M. REED

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§ 1. INTRODUCTION.

We discuss, in more detail, the conormal progressing waves constructed by Bony [1] . The goal is to show that if the wave (≡ solution) is classical in the past, it remains so in the future. Given a certain minimal regularity this is true provided the notion of classicality is slightly extended so as to yield a class closed under nonlinear functions.

We begin with a brief review of the notions of conormal distributions see [4, §18.2]. If $\Sigma \subset \mathbb{R}^d_x$, $x = x_1, x_2, \dots, x_d$ is a regular hypersurface and $\mathcal{V}$ in the Lie algebra of vector fields tangent de Σ then a distribution defined on a neighborhood of Σ is said to be conormal if for some $s \in \mathbb{R}$, and all finite sets $V_1 \dots V_N \in \mathcal{V}$

$$(1) \qquad V_1 \dots V_N\, u \in H^s_{loc} .$$

We abbreviate by $\mathcal{V}^\infty u \in H^s$ and denote the space by $H^{s,\infty}_\Sigma$. In coordinates with $\Sigma = \{x_1=o\}$, such distributions have partial Fourier transforms with respect to x_1 which are symbols

$$u = \int a(x_2 \dots, x_d, \xi_1)\, e^{i x_1 \xi_1} d\xi_1$$

$a \in S^\mu(\mathbb{R}^d \times \mathbb{R})$. The correspondance $s \to \mu$ is made exact if in (1) the space H^s_{loc} is replaced by the Besov space $({}^\infty H_s)_{loc}$ with $s = -\mu - 1/2$. The resulting class is denoted I^μ_Σ . Recall

$${}^\infty\|u\|_s \sim \max_j \|u_j\|_{L^2}\, 2^{js}$$

when $u_j \sim u\chi_{2^j \leqslant |\xi| \leqslant 2^{j+1}}$ is the normal dyadic component of u . Since $H^{s'} \subset {}^\infty H_s \subset H^{s''}$ for all $s' > s > s''$, with a little loss one can pass from the I^μ to the $H^{s,\infty}$ and back.

example. If u is piecewise smooth in Ω , that is $C^\infty(\Omega \setminus \Sigma)$ and for each $x \in \Sigma$ the restriction of u to each side of Σ has a C^∞ extension to a neighborhood of x , then $u \in I^\mu_\Sigma$ with $s = m + 1/2$ where m is the smallest integer such that $u \notin C^m$.

This example is the most classical of all conormal distributions.

We state our results for first order hyperbolic systems. The extension to higher order is immediate, and some of the results are valid for simply characteristic hypersurfaces and general L ([8]).

Fix L a first order linear strictly hyperbolic system in $\Omega \subset \mathbb{R}^d$. A regular characteristic hypersurface Σ is given in the open set Ω and also a timelike function t with Ω in the domain of determineacy of $\Omega^- \equiv \Omega \cap \{t < 0\}$. We are interested in solutions $u \in L^\infty_{loc}(\Omega)$ to

$$(2) \qquad L\,u + f(x,u) = g$$

where $f \in C^\infty(\Omega \times \mathbb{C}^k, \mathbb{C}^k)$.

<u>Theorem 1</u>. (Bony [1]). Propagation of conormal regularity.

If $s > d/2$, $u \in^\infty H_s(\Omega) \cap I^\mu_\Sigma(\Omega^-)$, and, $g \in I^\mu_\Sigma(\Omega)$. Then $u \in I^\mu_\Sigma(\Omega)$.

<u>Remark</u>. The proof in [1] for $H^{s,\infty}_\Sigma$ is valid for the I^μ_Σ.

Our first "classical conormal" result is in the piecewise C^∞ category.

<u>Theorem 2</u>. Propagation of piecewise smooth solutions.

If an addition, g is piecewise smooth and u is piecewise smooth in $\Omega-$, then u is piecewise smooth.

In the linear theory a distribution $u \in I^\mu_\Sigma$ is called classical if the symbol $a \in S^m_{phg}$ with Rem $= \mu$. That is, if there exist $a_{m-j}(x',\xi_1)$ homogeneous of degree m-j with

$$(3) \qquad a \sim \sum_{j=0}^{\infty} a_{m-j}(x',\xi_1)\ .$$

<u>example</u> 1. If u is piecewise smooth and C^ℓ then

$$u \sim \sum_{j=\ell+1}^{\infty} [\partial_1^j u(0,x')]\ (x_1)_+^j/j! \ + C^\infty$$

where [ ] denotes "jump in". Then

$$a(x',\xi_1) \sim \sum_{j=\ell+1}^{\infty} [\partial_1^j\ u(0,x')]\ e^{-i\pi(j+1)/2}\ (\xi-i0)^{-j-1}$$

<u>example 2</u>. If $\mu \notin \mathbb{Z}$, $\mathrm{Re}\, m = \mu$, the general example is

$$u \sim \sum_{\pm} \Sigma \alpha_j^{\pm}(x')(x_1 \pm i0)^{-m+j+1} \quad , \quad \alpha_j^{\pm} \in C^{\infty}(\mathbb{R}^{d-1}).$$

For $\mu \in \mathbb{Z}$, there are annoying logarithms which appear in the description in x-space. In the second example note that the distributions are continuous exactly for $\mu < -1$, and it is in that range that nonlinear operations are natural. The necessity of expansions (3) in the linear theory is easy to understand since if one starts with $(x_1+i0)^m$, $\mathrm{Re}\, m > 0$ and multiplies by a smooth function φ , one has

$$\varphi.(x_1+i0)^m \sim \Sigma \frac{\varphi^j(0,x')}{j!} x_1^{\ j}(x_1+i0)^m \sim \Sigma \varphi^j(0,x')(x_1+i0)^{m+j} \quad .$$

In the ξ variables $(x_1+i0)^{\ell}$ appearshomogeneous of degree $-\ell-1$. Thus if a term homogeneous of degree h appears in a symbol, one expects to encouter terms of order $h-n$, $n \in \mathbb{N}_+$. For non linear problems one must multiply. Multiplying functions homogeneous of degree ℓ_1 and ℓ_2 yields one of degree $\ell_1 + \ell_2$. In Fourier the degrees are $-\ell_1-1$, $-\ell_2-1$, and $-\ell_1-\ell_2-1$. Thus if homogenerties h_1 and h_2 occur in symbols one expects to encounter $h_1 + h_2 + 1$. This leads to the following extension of the notion of classical conormal distribution, adapted to nonlinear problems.

<u>Definition</u>. Suppose $H \subset \{\mathrm{Re}\, z < 1\}$ satisfies

(i) $H + \mathbb{Z}_- \subset H$

(ii) $H + H + 1 \subset H$

(iii) For any $M \in \mathbb{R}$, $\{h \in H : \mathrm{Re}\, h \geq M\}$ is finite.

A distribution u defined an a neighborhood of Σ is said to be classical of type H , $u \in I_{\Sigma}^{H}$, if

$$a \sim \sum_{h\in H} a_h(x,\xi)$$

in the sense that for any $M \in \mathbb{Z}$.

$$(a - \sum_{\mathrm{Re}\, h > M} a_h) \in S^M$$

<u>example</u>. The smallest H containing m with $\mathrm{Re}\, m < -1$ is the set of all $n_1 m + (n_1-1) - n_2$ with $n_1 \geq 1$, $n_2 \geq 0$ integers.

Theorem 3. Propagation of classical conormal regularity.
Suppose u as in Theorem 2 and H as in the definition.
If in addition $g \in I^H_\Sigma(\Omega)$, $u \in I^H_\Sigma(\Omega^-)$ then $u \in I^H_\Sigma(\Omega)$.

Remark. The properties (i), (ii), (iii) of H imply that Re H < -1 . That this is needed to obtain good results is seen in the following example.

example. $u_t + 2u^3 = 0$, $u(0,.) = (x_1)^m_+$.

(4) $u = (x_1)^m_+ (1+t(x_1)^{2m}_+)^{-1/2} \sim x_1^m \sum_0^\infty \{-t(x_1)^{2m}_+\}^n \binom{-1/2}{n}$ has all terms of the same regularity if Re m = 0 , which corresponds to Re H = - 1. On the other hand for Re m > 0 ,(hence Re H < -1) the expansion (4) is perfectly classical.

Once classical conormal waves are constructed it is natural to study their interaction as in [2,3] . In fact, one of the main goals is to create a multidimensional piecewise C^∞ theory so the theorems of Rauch-Reed [5] on the creation of singularities can be extended to higher dimensions. So far, only the case of two speed operators has been carried out [6], and there one has no such creation !

In §2 we prove a mild extension of Theorem 2 by an argument different from that used to obtain Theorem 3. In §3 we sketch the calculus avalable for studying the symbol of u in the special case of u piecewise smooth. This also gives us the chance to describe the idea of the proof of Theorem 3 which will be published elsewhere [8] .

§ 2. PROOF OF THEOREM 2.

We prove a slightly stronger result then Theorem 2. If u is a conormal solution we show that if u is smooth on the closure of one side of Σ in t < 0 , this property persists in t > 0 . Toward this end let $\mathcal{O}$ be a neighborhood of Σ in Ω with $\mathcal{O} \setminus \Sigma$ consisting of two components $\mathcal{O}_\ell$ and $\mathcal{O}_r$ (left and right), and $\mathcal{O}_\ell$ in the domain of determinary of $\mathcal{O}_\ell \cap \{t<0\}$.

Theorem 4. Suppose u is a conormal solution as in Theorem 1 and that in addition $g \in C^\infty(\overline{\mathcal{O}}_\ell)$ and $u \in C^\infty(\overline{\mathcal{O}}_\ell \cap \Omega-)$. Then $u \in C^\infty(\overline{\mathcal{O}}_\ell)$.

Proof. The proof consists of a local argument in a neighborhood of $x \in \Sigma \cap \{t = 0\}$ and a patching. We omit the latter step.

We will show that if $\frac{d}{2} < s < -\mu - 1/2$ then for any $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$

(5) $$(x_1\partial_1, \partial_2, \ldots, \partial_d)^\alpha \ \partial_1^n \ u \in H^s_{loc}(\overline{\mathcal{O}}_\ell) \ ,$$

symbolically $\partial_1^n u \in H^{s,\infty}_\Sigma(\overline{\mathcal{O}}_\ell)$. From Bony's theorem we know $u \in H^{s,\infty}_\Sigma(\Omega)$. Thus the case $n = 0$ is done.

We make a change of the $\mathbb{C}^k$ variable u , and multiply the equation by an inevitable matrix to convert L to an operator of the form (see [5])

(6) $$\begin{bmatrix} 0 & 0 & \cdots\cdots & 0 \\ 0 & & & \\ \vdots & & [\ *\ ] & \\ 0 & & & \end{bmatrix}_{k\times k} \partial_1 + \Sigma A_j(x)\partial_j + B$$

where the $(k-1) \times (k-1)$ matrix [*] is non singular.

To prove (5) for $n = 1$, first solve the final $(k-1)$ equations of (2) for $(\partial_1 u_2, \partial_1 u_2, \ldots \partial_1 u_k)$ to see that $\partial_1(u_2,\ldots,u_k) \in H^{s,\infty}_\Sigma(\mathcal{O}_\ell)$. To interpret the first equation, let $X \equiv \Sigma(A_j)_{1,1} \ \partial_j$, $b \equiv B_{1,1}$ to find

(7) $$(X+b)u_1 + f_1(x,u) = g_1 - \mathcal{V}(u_2,\ldots,u_d)$$

Differentiating with respect to x_1 yields

(8) $$(X+b+\frac{\partial f_1}{\partial u_1}) \frac{\partial u}{\partial x_1} + [X+b, \partial_1]\, u_1 + \sum_{j>1} \frac{\partial f}{\partial u_j} \partial_1 u_j \in H^{s,\infty}_\Sigma(\mathcal{O}_\ell)$$

The commutator $[X,\partial_1]$ is tangential and we already know u , $\partial_1(u_2\ldots u_k) \in H^{s,\infty}_\Sigma(\mathcal{O}_\ell)$ so (8) takes the form

(9) $$(X + H^{s,\infty}_\Sigma(\mathcal{O}_\ell))\ \partial_1 u \in H^{s,\infty}_\Sigma(\mathcal{O}_\ell)$$

With the normal form (6), all the surfaces x_1 = const. are characteristic and X is the ray direction along these surfaces [7, lemma 2.3]. Thus (9) is a linear ordinary differential equation for $\partial_1 u$ along rays. Since backward rays from $\mathcal{O}_\ell \cap \{t > 0\}$ enter $\mathcal{O}_\ell \cap \{t < 0\}$ where $\partial_1 u \in H^{s,\infty}_\Sigma$, it follows that $\partial_1 u \in H^{s,\infty}_\Sigma(\mathcal{O}_\ell)$. In fact, this conclusion is a consequence of Theorem 1, and can be obtained directly by the usual commutation arguments since $[X,\mathcal{V}] \in \mathcal{V}$. This proves (5) for $n = 1$

For $n > 1$ we reason by induction, assuming $\partial^j_{x_1} u \in H^{s,\infty}_\Sigma(\mathcal{O}_\ell)$ for

$j \leq n-1$. Solving the last $k-1$ equations for $\partial_{x_1}(u_2 \cdots u_k)$ and then applying $\partial_{x_1}^{n-1}$ yields $\partial_{x_1}^{n}(u_2,\ldots,u_d) \in H_{\Sigma}^{s,\infty}(\mathcal{O}_\ell)$. Applying $\partial_{x_1}^{n}$ to (7) and noting that the induction hypothesis yields

$$[X+b\ ,\ \partial_{x_1}^{n}]\ u \in H_{\Sigma}^{s,\infty}(\mathcal{O}_\ell)$$

$$\partial_{x_1}^{n} f_1 = \frac{\partial f_1}{\partial u_1}\ \partial_{x_1}^{n}\ u_1 + H_{\Sigma}^{s,\infty}(\mathcal{O}_\ell)$$

we find a transport equation

$$(X + H_{\Sigma}^{s,\infty}(\mathcal{O}_\ell))\partial_{x_1}^{n}\ u_1 \in H_{\Sigma}^{s,\infty}(\mathcal{O}_\ell)$$

and it follows as for $n = 1$, that $\partial_{x_1}^{n}\ u_1 \in H_{\Sigma}^{s,\infty}(\mathcal{O}_\ell)$. #

§ 3 . SYMBOLIC CALCULUS, IDEA OF GENERAL PROOF.

In this section we recall, from [7] the fact that once you know that u is piecewise smooth it is not hard to find rules for computating the jumps $[\partial_{x_1}^{\ell} u]$ across $\Sigma = \{x_1=0\}$. For example, given $u \in H_{\Sigma}^{s,\infty}(\Omega)$, $s > 1/2$ we know that $u_{|\Sigma} \in C^{\infty}(\Sigma)$. Given this trace, the jump in $\partial_{x_1} u$ is computed in the coordinates where L has form (6) according to

$$[\partial_{x_1}(u_2,\ldots,u_k)] = 0$$

$$(X + b + \frac{\partial f_1}{\partial u_1})\ \ [\partial_{x_1} u]\ = [\partial_{x_1} g]\ .$$

As in the last section, we can integrate this linear ordinary differential equations along rays on Σ to determine $[\partial_{x_1} u_1]$ from its values in $t < 0$. Then we have

$$u = u(0,x^1) + [\partial_1 u(0,x^1)]\ \ (x_1)_+ + r$$

with $r \in C^1$ along with all of its tangential derivatives. Thus $\partial r/\partial x_1|_{\Sigma} \in C^{\infty}(\Sigma)$. Given this trace, one can find the next jump $[\partial_{x_1}^{2} u]$, and so on.

The proof in the general case follows these lines. For $u \in I_{\Sigma}^{\mu}$, it follows that $\partial_{x_1}^{j} u|_{\Sigma} \in C^{\infty}(\Sigma)$ for $j \leq \ell(\mu)$. Given these traces, the principal symbol a_m is computed from its values in $t < 0$. Then

$$u = u(0,x^1) + \int a_m(x',\xi_1)\, e^{ix_1\xi_1}\, d\xi_1 + r$$

One then shows, and this is the technically difficult step, that $r \in I_\Sigma^{\mu+1}$, that is one derivative better. Given the traces of r for $j \leqslant \ell(\mu+1)=\ell(\mu)+1$, one can calculate symbol(s) of the next order, and so on.

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J. RAUCH