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APPLICATION OF THE MICROLOCAL THEORY OF SHEAVES TO THE STUDY OF $\mathcal{O}_X$. [1]

par M. KASHIWARA[2] - P. SCHAPIRA[3]

(1) With the exception of some proofs which are included here, and some remarks which are omitted, this text is essentially the same as a paper to appear in "Proceedings of the 1984 A.M.S. symposium on pseudo-differential operators with applications to partial differential equations" .

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We shall recall some constructions and results of [9] with emphasis on the applications to the study of the microlocalization of the sheaf $\mathcal{O}_X$ of holomorphic functions on a complex manifold X along real submanifolds of X .

I - Micro-support

Let X be a real manifold of class C^α , with $1 \leqslant \alpha \leqslant \infty$ or $\alpha = \omega$ (i.e. : X real analytic). We denote by $\pi : T^*X \longrightarrow X$ the cotangent bundle to X , by ω_X the canonical 1-form on T^*X . If Y is a submanifold of X we denote by T^*_YX the conormal bundle to Y . In particular T^*_XX denotes the zero section of T^*X , that one identifies with X .
We denote by $D^+(X)$ the derived category of the category of complexes, bounded from below, of sheaves of abelian groups on X .
Thus an object $\underline{F}$ of $D^+(X)$ is represented by a complex of sheaves :

$$\underline{F} \cong \cdots \longrightarrow \underline{F}^i \longrightarrow \underline{F}^{i+1} \longrightarrow \cdots$$

with $\underline{F}^i = 0$ for $i \ll 0$. Moreover two complexes which are quasi -isomorphic are identified in $D^+(X)$, and any object $\underline{F}$ may be represented by a complex of flabby sheaves.

Example 1.1. : Let A be a locally closed subset of the real manifold X .
The sheaf $\underline{\mathbb{Z}}_A$ on X satisfies :

$$(\underline{\mathbb{Z}}_A)_x = \mathbb{Z} \quad \text{if } x \in A$$

$$(\underline{\mathbb{Z}}_A)_x = 0 \quad \text{if } x \notin A$$

Assume A is closed in X . We have an exact sequence :

$$0 \longrightarrow \underline{\mathbb{Z}}_{(X\setminus A)} \rightarrow \underline{\mathbb{Z}}_X \rightarrow \underline{\mathbb{Z}}_A \longrightarrow 0$$

Thus we have an isomorphism in $D^+(X)$:

$$0 \longrightarrow \underline{\mathbb{Z}}_{(X \setminus A)} \longrightarrow \underline{\mathbb{Z}}_X \longrightarrow 0 \qquad \simeq \quad \underline{\mathbb{Z}}_A \ [-1]$$

(one identifies a sheaf $\underline{F}$ with the complex $\dots \longrightarrow 0 \longrightarrow \underline{F} \longrightarrow 0 \longrightarrow \dots$ where $\underline{F}$ is in degree 0, and $\underline{F}\,[d]$ is the shifted complex : $\underline{F}[d]^i = \underline{F}^{i+d}$) .

Now we return to the situation where X is a real manifold of class C^α .

Definition 1.2. : Let $\underline{F} \in \mathrm{Ob}(D^+(X))$. The micro-support of $\underline{F}$, denoted $SS(\underline{F})$, is the subset of T^*X defined by :

$p \notin SS(\underline{F}) \Longleftrightarrow$ there exists an open neighborhood U of p in T^*X such that for any $x_1 \in X$, any real function ϕ of class C^α , defined in a neighborhood of x_1 with $\phi(x_1) = 0$, $d\phi(x_1) \in U$, we have :

$$(\mathbb{R}\Gamma_{\{x;\phi(x) \geqslant 0\}}(\underline{F}))_{x_1} = 0$$

Recall that if Z is a locally closed subset of X (here $Z = \{x;\phi(x) \geqslant 0\}$), the complex $\mathbb{R}\Gamma_Z(\underline{F})$ is calculated by representing $\underline{F}$ by a complex of flabby sheaves and applying the functor $\Gamma_Z(\cdot)$, where $\Gamma_Z(\underline{F})$ is the subsheaf of $\underline{F}$ of sections with support in Z . In this paper, we shall write $H^j_Z(\underline{F})$ instead of $H^j(\mathbb{R}\Gamma_Z(\underline{F}))$.

Roughly speaking, when $\underline{F}$ is a sheaf, $p \notin SS(\underline{F})$ means that $\underline{F}$ has no section, and no "cohomology" supported by "half-spaces" whose conormal lies in a neighborhood of p .

Similarly if $u : \underline{F} \to \underline{G}$ is a morphism in $D^+(X)$, we define $SS(u)$ as $SS(\underline{H})$, where $\underline{H}$ is the simple complex associated to the double complex $\underline{F} \longrightarrow \underline{G}$ (i.e. : the "mapping cone" of u).

It follows immediately by the definition that :

- $SS(\underline{F})$ is a closed cone in T^*X ,
- $SS(\underline{F}) \cap T^*_X X = \mathrm{supp}(\underline{F})$, where $\mathrm{supp}(F) = \overline{\bigcup_j \mathrm{supp} H^j(\underline{F})}$ is the support of the complex $\underline{F}$,
- If $0 \longrightarrow \underline{F}_1 \longrightarrow \underline{F}_2 \longrightarrow \underline{F}_3 \longrightarrow 0$ is an exact sequence of sheaves (or more generally if we have a distinguished triangle $\underline{F}_1 \longrightarrow \underline{F}_2 \longrightarrow \underline{F}_3 \longrightarrow \underline{F}_1[+1]$ in $D^+(X)$), then :

$$SS(\underline{F}_i) \subset SS(\underline{F}_j) \cup SS(\underline{F}_k) \quad \text{if } \{i,j,k\} = \{1,2,3\}$$

- One proves, using contact transformations (cf.§2 below) that $SS(\underline{F})$ is an involutive subset of T^*X , assumming of course $\alpha \geqslant 2$.

Example 1.3 i) Let Y be a closed submanifold of X . Then :

$$SS(\underline{\mathbb{Z}}_Y) = T^*_Y X$$

ii) Let ϕ be a real C^1-function, $\overline{Y^+} = \{x;\ \phi(x) \geqslant 0\}$, $Y^+ = \{x;\phi(x) > 0\}$ and assume $d\phi \neq 0$ on $\{x;\phi(x) = 0\}$. Then :

$$SS(\underline{\mathbb{Z}}_{\overline{Y^+}}) = (T^*_X X \cap \overline{Y^+}) \cup \{(x;\ \lambda d\phi(x))\ ;\ \phi(x) = 0\ , \lambda \geqslant 0\}$$

$$SS(\underline{\mathbb{Z}}_{Y^+}) = (T^*_X X \cap \overline{Y^+}) \cup \{(x;\ \lambda d\phi(x))\ ;\ \phi(x) = 0\ , \lambda \leqslant 0\}$$

Example 1.4 : Let X be a complex manifold, $\mathcal{M}$ a coherent module over the sheaf of rings $\mathcal{D}_X$ of holomorphic differential operators (of finite order) (cf. [17]) . Then :

$$SS(\mathbb{R}\,\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) = char(\mathcal{M})$$

where $char(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$.

II - CONTACT TRANSFORMATIONS

In this section we assume $\alpha \geqslant 2$.
Let Ω be a subset of T^*X , and set :

$$N(\Omega) = \{\underline{F} \in Ob(D^+(X))\ ;\ SS(\underline{F}) \cap \Omega = \emptyset\}$$

We introduce $D^+(X;\Omega)$, the localization of $D^+(X)$ by $N(\Omega)$ (cf.[4]).
Then :

$$Ob(D^+(X;\Omega)) = Ob(D^+(X))$$

but a morphism $u : \underline{F} \longrightarrow \underline{G}$ in $D^+(X)$ becomes an isomorphism in $D^+(X;\Omega)$ if $SS(u) \cap \Omega = \emptyset$.

Example 2.1. : Let $X = \mathbb{R}$, and let x be a coordinate on X , (x,ξ) the associated coordinates on T^*X . Let $\Omega = \{(x,\xi)\ ;\ \xi > 0\}$. Then :

$$\underline{\mathbb{Z}}_{\{0\}} \cong \underline{\mathbb{Z}}_{\{x;x\geqslant 0\}} \cong \underline{\mathbb{Z}}_{\{x;x<0\}}[1] \text{ in } D^+(X;\Omega) .$$

Now let X and Y be two manifolds of the same dimension, Ω_X and Ω_Y two conic open subsets of T^*X and T^*Y respectively, $\phi : \Omega_X \xrightarrow{\sim} \Omega_Y$ a (homogeneous) contact transformation. Let $\Lambda_\phi \subset \Omega_X^a \times \Omega_Y$ be the associated Lagrangean manifold ; Ω_X^a is the image of Ω_X by the antipodal map of T^*X , and :

$$\Lambda_\phi = \{(x,y;\xi,\eta) \in T^*(X \times Y) \ ; \ (y,\eta) = \phi(x,-\xi)\}$$

Let q_1 and q_2 be the projections from $X \times Y$ to X and Y respectively. To $\underline{K} \in \mathrm{Ob}(D^+(X \times Y))$ we associate the functor $\psi_{\underline{K}}$ from $D^+(X)$ to $D^+(Y)$:

$$\psi_{\underline{K}}(\underline{F}) = \mathbb{R}\, q_{2!}(\underline{K} \overset{\mathbb{L}}{\otimes} q_1^{-1}\underline{F})$$

(Recall that $q_{2!}$ means the direct image by q_2 with proper supports, and $\mathbb{L}$ and $\mathbb{R}$ mean the left and right derived functors).

Theorem 2.2. : Let $p_X \in \Omega_X$, $p_Y = \phi(p_X) \in \Omega_Y$.
There exists $\underline{K} \in \mathrm{Ob}(D^+(X \times Y))$ such that $\psi_{\underline{K}}$ induces an equivalence of categories :

$$\psi_{\underline{K}} : D^+(X;p_X) \xrightarrow{\sim} D^+(Y;p_Y)$$

If $\Lambda_\phi = T^*_Z(X \times Y)$, where Z is a submanifold of $X \times Y$, the sheaf $\underline{K}$ will satisfies :

$$\underline{K} \cong \underline{\mathbb{Z}}_Z\,[d] \text{ in } D^+(X \times Y \ ; \ p_X^a \times p_Y)$$

for some shift d .

Now assume X and Y are complex manifolds of complex dimension n , and ϕ is a complex contact transformation. Assume for simplicity that $\Lambda_\phi = T^*_Z(X \times Y)$, for Z a complex submanifold.

Theorem 2.3. : Assume :

$$\underline{K} \cong \underline{\mathbb{Z}}_Z\,[n-\mathrm{codim}_{\mathbb{C}} Z] \text{ in } D^+(X \times Y \ ; \ p_X^a \times p_Y)$$

Then one can find (non unique) isomorphisms :

$$\psi_{\underline{K}}(\mathcal{O}_X) \cong \mathcal{O}_Y \text{ in } D^+(Y,p_Y)$$

We can describe this isomorphism as follows. Let $d = \operatorname{codim}_{\mathbb{C}} Z$.

$$\psi_{\underline{K}}(\mathcal{O}_X) \cong \mathbb{R}q_{2!}\,(q_1^{-1}\mathcal{O}_X|_Z)[n-d]$$

Let us choose a relative differential form v on Z above Y :

$$v \in \Omega_Z^{(2n-d)} \otimes_{\mathcal{O}_Y} \Omega_Y^{(n)\otimes -1}$$

We have natural morphisms :

$$q_1^{-1}\mathcal{O}_X|_Z \longrightarrow \mathcal{O}_Z \text{ (restriction)}$$

$$\mathcal{O}_Z \longrightarrow \Omega_Z^{(2n-d)} \otimes_{\mathcal{O}_Y} \Omega_Y^{(n)\otimes -1} \qquad \text{(multiplication by v)}$$

$$\mathbb{R}q_{2!}\,(\Omega_Z^{(2n-d)} \otimes_{\mathcal{O}_Y} \Omega_Y^{(n)\otimes -1})\ [n-\operatorname{codim}_{\mathbb{C}} Z] \longrightarrow \mathcal{O}_Y \text{ (integration)}$$

This defines the morphism :

$$\psi_{\underline{K}}(\mathcal{O}_X) \longrightarrow \mathcal{O}_Y .$$

If Z is a hypersurface defined by the equation $\varphi = 0$ (with $d\varphi \neq 0$ on Z) we may take $v = w/d\varphi$, where w is a volume element in $\Omega_{X\times Y}^{(2n)}$.

III - MICROLOCALIZATION

We still assume X of class C^α ,$\alpha \geqslant 2$. Let M be a submanifold, $\underline{F} \in \mathrm{Ob}(D^+(X))$. Recall (cf.[17]) that Sato's microlocalization of $\underline{F}$ along M , denoted $\mu_M(\underline{F})$, is an object of $D^+(T_M^*X)$, whose stalk at $(x^o,\xi^o) \in T_M^*X$ is given by :

$$H^j(\mu_M(\underline{F}))_{(x^o,\xi^o)} = \varinjlim_{U,G} H^j(\mathbb{R}\Gamma_{U\cap G}(U,\underline{F}))$$

where U runs over the set of open neighborhoods of x^o in X , and G runs over the set of closed wedges in X along M whose polar (in a system of local coordinates) is a neighborhood of ξ^o . Remark that : $\mu_M(\underline{F})|_{T_X^*X} \cong \mathbb{R}\Gamma_M(\underline{F})$

Example 3.1. : Assume $M = \{x \in X ; \varphi(x) = 0\}$ with $d\varphi \neq 0$ on M .
Then if $x^o \in M$:

$$\mu_M(\underline{F})_{(x_o,d\varphi(x_o))} = (\mathbb{R}\Gamma_{\{x;\varphi(x)\geqslant 0\}}(\underline{F}))_{x_o} .$$

Example 3.2. Let M be a real analytic manifold of dimension n , X a complexification of M . The sheaf C_M of Sato's microfunctions on M is defined by :

$$C_M = \mu_M(\mathcal{O}_X) \otimes \underline{\omega}_M [n]$$

where $\underline{\omega}_M$ is the orientation sheaf on M , (cf.[17]).

In order to state our next result, let us recall the classical notion of "Maslov index" associated to three Lagrangean planes, (cf. [6], [11] , [13]). Let (E,σ) be a (real) symplectic vector space, λ_1, λ_2, λ_3 three Lagrangean planes (plane = linear subspace). The index $\tau(\lambda_1, \lambda_2, \lambda_3)$ is the signature of the quadratic form q on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ defined by :

$$q(x_1,x_2,x_3) = \sigma(x_1,x_2) + \sigma(x_2,x_3) + \sigma(x_3,x_1), \quad (x_1,x_2,x_3) \in \lambda_1 \oplus \lambda_2 \oplus \lambda_3 .$$

Theorem 3.3. : Let X and Y be two real manifolds of the same dimension, $\Omega_X \subset T^*X$ and $\Omega_Y \subset T^*Y$ two conic open subsets, $\phi : \Omega_X \longrightarrow \Omega_Y$ a contact transformation. Let M and N be two submanifolds of X and Y respctively, and assume that ϕ induces an isomorphism :

$$T^*_M X \cap \Omega_X \xrightarrow[\phi]{\sim} T^*_N Y \cap \Omega_Y .$$

Let $p_X \in \Omega_X$, $p_Y = \phi(p_X) \in \Omega_Y$ and let $\psi_{\underline{K}}$ be the equivalence of categories given by theorem 2.2. Then we have an isomorphism :

$$\mu_M(\underline{F})_{p_X} \cong \mu_N(\psi_{\underline{K}}(\underline{F}))_{p_Y} [d]$$

for some shift d .

Moreover assume $\Lambda_\phi = T^*_Z(X \times Y)$ for Z a submanifold of $X \times Y$, and $\underline{K} = \underline{\mathbb{Z}}_Z$ in $D^+(X \times Y \,;\, p^a_X \times p_Y)$. Then :

$$d = \frac{1}{2} (\dim M - \dim N + \dim X - \dim Z + \tau)$$

where $\tau = \tau(\lambda_o(p_Y), \phi^*(\lambda_o(p_X)), \lambda_N(p_Y))$,

$\lambda_o(p_Y) = T_{p_Y} \pi^{-1}\pi(p_Y)$,

$\phi^* (\lambda_o(p_X)) = d\phi(p_X) (T_{p_X} \pi^{-1}\pi(p_X))$,

$$\left| \lambda_N(p_Y) = T_{p_Y} \overset{*}{T}_N Y \right.$$

When X and Y are complex manifolds and φ is a complex contact transformation, we may combine theorems 2.3 and 3.3. First we introduce another index.

Let X be a complex manifold, M a real submanifold. Let $p \in \overset{*}{T}_M X$ .We set :

$$s(M,p) = \frac{1}{2}\tau(\lambda_M(p),\ i\,\lambda_M(p),\ \lambda_o(p))$$

(where $\lambda_M(p) = T_p \overset{*}{T}_M X$, $\lambda_o(p) = T_p \overset{-1}{\pi}\ \pi(p)$, and τ is the index associated to the real symplectic structure of T^*X , that is, to $2\mathrm{Re}\ d\omega_X$).

Remark that $s(M,p)$ is an integer. In fact $s(M,p)$ may also be obtained as follows.

Let (E,σ) be a complex symplectic vector space. One says that a real subspace λ is $\mathbb{R}$-Lagrangean if λ is Lagrangean for Re σ . For such a plane one says that λ is I-symplectic if Im σ is non degenerate on λ (cf. [18]). This is equivalent to saying that $(E,\frac{1}{i}\sigma)$ is a complexification of $(\lambda,\frac{1}{i}\sigma_{|\lambda})$.

Let ρ be a complex isotropic subspace of E . The symplectic forms σ induces a symplectic structure on the space $\rho^{\perp}/\rho$. To an $\mathbb{R}$-Lagrangean plane λ in E one associates the $\mathbb{R}$-Lagrangean plane λ^{ρ} of $\rho^{\perp}/\rho$ by setting :

$$\lambda^{\rho} = ((\lambda \cap \rho^{\perp}) + \rho)/\rho$$

Now take

$$\rho = \lambda \cap i\,\lambda$$

Then λ^{ρ} is $\mathbb{R}$-Lagrangean and I-symplectic in $\rho^{\perp}/\rho$.

Let λ_o be a complex Lagrangean plane of E . One defines an Hermitian form γ_λ on λ_o^{ρ} by setting :

$$\gamma_\lambda(u,v) = \sigma(u,\bar{v})$$

where $u,v \in \lambda_o^{\rho}$ and $\bar{v}$ is the complex conjugate to v in the isomorphism :

$$\mathbb{C} \otimes_{\mathbb{R}} \lambda^{\rho} \simeq \rho^{\perp}/\rho$$

Then one proves :

Lemma 3.4 : One has :

$$\operatorname{sgn}\ \gamma_\lambda = \frac{1}{2}\ \tau(\lambda, i\lambda, \lambda_o)$$

Remark that :

$$\operatorname{rank}(\gamma_\lambda) = \dim_{\mathbb{C}} \lambda_o^\rho - \dim_{\mathbb{R}} (\lambda^\rho \cap \lambda_o^\rho) = \frac{1}{2} \dim_{\mathbb{C}} E - \dim_{\mathbb{C}}(\lambda \cap i\lambda) - \dim_{\mathbb{R}} (\lambda \cap \lambda_o) + 2 \dim_{\mathbb{C}}(\lambda \cap i\lambda \cap \lambda_o)$$

since $\lambda^\rho \cap \lambda_o^\rho = (\lambda \cap \lambda_o)/(\lambda \cap i\lambda \cap \lambda_o)$

It remains to apply this lemma with $E = T_p T^* X$, $\lambda = \lambda_M(p)$, $\lambda_o = \lambda_o(p)$.

Corollary 3.5 : Let X and Y be two complex manifolds, ϕ a complex contact transformation from $\Omega_X \subset T^*X$ to $\Omega_Y \subset T^*Y$. Let M and N be two real submanifolds of X and Y respectively, and assume ϕ induces an isomorphism $T^*_M X \cap \Omega_X \xrightarrow[\phi]{\sim} T^*_N Y \cap \Omega_Y$. Then :

i) the function $s(M,p) - s(N,\phi(p))$ is locally constant on $T^*_M X \cap \Omega_X$,

ii) locally on Ω_X , we may quantize ϕ as an isomorphism :

$$\phi_* \ \mu_M(\mathcal{O}_X) \simeq \mu_N(\mathcal{O}_Y)\ [d]$$

where $d = \frac{1}{2}\ [(\dim M + s(M,p)) - (\dim N + s(N,\phi(p))]$ and $p \in \Omega_X \cap T^*_M X$.

Proof : Set $q = \phi(p)$.

First let as show that the function

$\tau = \tau(\lambda_o(q), \phi^*(\lambda_o(p)), \lambda_N(q))$ is locally constant with respect to $p \in T^*_M X \cap \Omega_X$. Let $\nu(q)$ be a complex Lagrangean plane of $T_q T^* X$,$\nu(q)$ being transversal to each of $\lambda_o(q)$, $\phi^*(\lambda_o(p))$, $\lambda_N(q)$. By the cocycle property of the Maslov index we have :

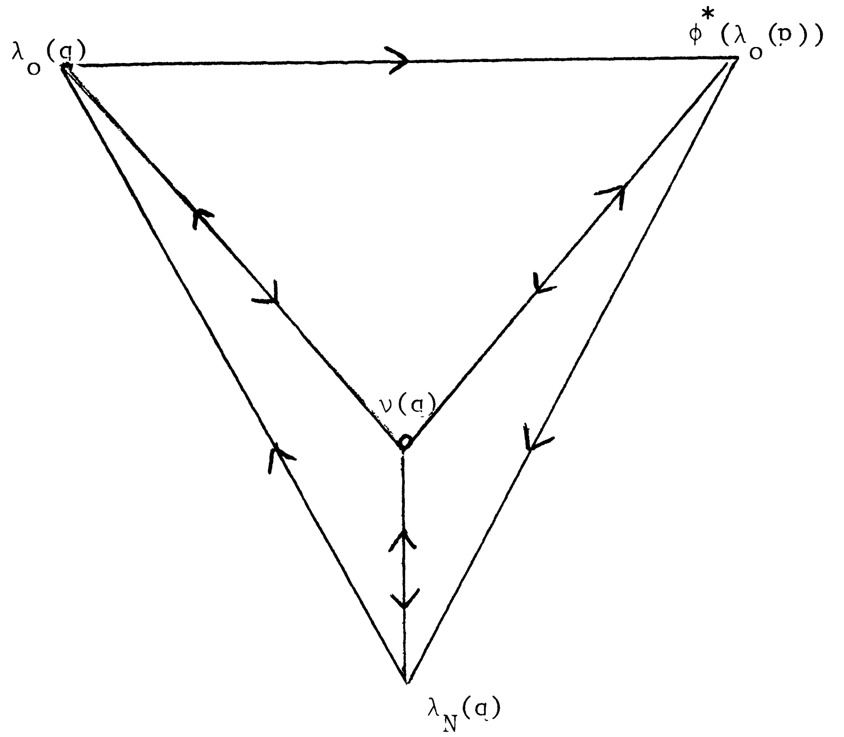
$$\tau = \tau_1 + \tau_2 + \tau_3$$

$$\tau_1 = \tau(\lambda_o(q), \phi^*(\lambda_o(p)), \nu(q))$$

$$\tau_2 = \tau(\phi^*(\lambda_o(p)), \lambda_N(q), \nu(q))$$

$$\tau_3 = \tau(\lambda_N(q), \lambda_o(q), \nu(q)) .$$

This relation can be vizualized by the diagram :



Since $\lambda_o(q)$, $\phi^*(\lambda_o(p))$, $\nu(q)$ are complex Lagrangean planes, $\tau_1 = 0$. Since dim $\phi^*(\lambda_o(p)) \cap \lambda_N(q)$ (= $\dim(\lambda_o(p) \cap \lambda_M(p))$) and $\dim(\lambda_o(q) \cap \lambda_N(q))$ are locally constant, we find that τ_2 and τ_3 are locally constant. Now we have, writing λ_M(resp. λ_N) instead of $\lambda_M(p)$ (resp. $\lambda_N(q)$) :

$$2(s(M,p) - s(N,q)) = \tau(\lambda_M, i\lambda_M, \lambda_o(p)) - \tau(\lambda_N, i\lambda_N, \lambda_o(q))$$

$$= \tau(\lambda_N, i\lambda_N, \phi^*(\lambda_o(p)) - \tau(\lambda_N, i\lambda_N, \lambda_o(q))$$

$$= \tau(i\lambda_N, \phi^*(\lambda_o(p)), \lambda_o(q)) - \tau(\lambda_N, \phi^*(\lambda_o(p)), \lambda_o(q))$$

$$= 2\,\tau(\lambda_o(q), \phi^*(\lambda_o(p)), \lambda_N)$$

(We used the fact that the multiplication by i transforms τ to $-\tau$).

IV - APPLICATIONS

In this section we shall illustrate Corollary 3.5. Thus X is a complex manifold, M a real submanifold of class C^2. Let $p \in T^*_M X$. We set :

$$E(p) = T_p T^* X\ ,\quad \sigma(p) = d\omega(p)$$
$$\lambda_M(p) = T_p\ T_M^* X$$
$$\lambda_o(p) = T_p\ \pi^{-1}\pi(p)$$
$$\dim_{\mathbb{C}} X = n$$
$$\dim(\lambda_M(p) \cap \lambda_o(p)) = m\ (= \operatorname{codim}_{\mathbb{R}} M)$$
$$\dim_{\mathbb{C}}(\lambda_M(p) \cap i\lambda_M(p) \cap \lambda_o(p)) = \delta(p)$$
$$\dim_{\mathbb{C}}(\lambda_M(p) \cap i\lambda_M(p)) = d(p)$$

We have already defined the integer $s(M,p)$ as $\frac{1}{2}\ \tau(\lambda_M(p),\ i\lambda_M(p),\ \lambda_o(p))$ where τ is the index associated to the symplectic form $\operatorname{Re}\ \sigma(p)$ on $E(p)$. Now we define $s^+(M,p)$ and $s^-(M,p)$ by :

$$s^+(M,p) - s^-(M,p) = s(M,p)$$
$$s^+(M,p) + s^-(M,p) = n - m + 2\delta(p) - d(p)$$

Remark that :

$$\delta(p) = \operatorname{codim}_{\mathbb{C}}(T_{\pi(p)}M + i\ T_{\pi(p)}M)$$

This number is of course equal to zero if M is a real hypersurface. More generally $\delta(p) = 0$ is equivalent to saying that the submanifold M is non characteristic in $X^{\mathbb{R}}$ for the Cauchy-Riemann system $\bar{\partial}$.

Example 4.1. : Assume M is a real hypersurface, $\varphi(x) = 0$ an equation of M , with $d\varphi \neq 0$ on M . Let :

$$T_x^{\mathbb{C}}M = \{v \in T_x X\ ;\ < v\ ,\ \partial\varphi(x) > = 0\ \}$$

where $\partial\varphi$ is the differential of φ with respect to the holomorphic variables. Let L_φ be the Levi form of φ on $T_x^{\mathbb{C}}M$. Recall that if $(x_1,\dots,x_n)$ is a system of holomorphic coordinates on X , $(\bar{x}_1,\dots,\bar{x}_n)$ the complex conjugate coordinates, then L_φ is represented by the matrix $\left(\frac{\partial^2\varphi}{\partial x_i \partial \bar{x}_j}\right)$ $(1 \leqslant i,j \leqslant n)$ on $T_x^{\mathbb{C}}M$.

Proposition 4.2 : In the situation of Example 4.1, $s^+(M,d\varphi(x))$ and $s^-(M,d\varphi(x))$ are respectively the number of positive and negative eigenvalues of L_φ on $T_x^{\mathbb{C}}M$.

The proof follows from Lemma 3.4 and [18, Proposition 1.6] .
In order to formulate our next result, let us denote by $\nu(p)$ the complex line of $T_p T^* X$ generated by the Euler vector field, i.e. by $H(\omega_X)$, where ω_X is the (complex) canonical 1-form on T^*X, and H the symplectic isomorphism

$T^*T^*X \cong TT^*X$.

Proposition 4.3 : Let M be a real submanifold of class C^2 of X , $p \in T^*_M X$.
Assume :

$$\dim_{\mathbb{R}} (T_p T^*_M X \cap \nu(p)) = 1$$

Then $H^j(\mu_M(\mathcal{O}_X))_p = 0$ for $j \notin [m + s^-(M,p) - \delta(p), n - s^+(M,p) + \delta(p)]$.

Sketch of the proof

We may interchange $(T^*X, T^*_M X , p)$ with $(T^*Y, T^*_N Y , q)$ by a complex contact transformation, where now N is a real hypersurface of the complex manifold Y .
Assume we know that :

$$H^j(\mu_N(\mathcal{O}_Y))_q = 0 \quad \text{for} \quad j \notin [1 + s^-(N,q) + \alpha \ , n - s^+(N,q) + \beta \ ]$$

Applying corollary 3.5, we find :

$$H^j(\mu_M(\mathcal{O}_X))_p = 0 \quad \text{for} \quad j \notin [1 + s^-(N,q) + \alpha - \gamma \ , n - s^+(N,q) + \beta - \gamma \ ]$$

where :

$$\gamma = \frac{1}{2}(s(M,p) - s(N,q) - m + 1)$$

Let us write $s(M)$, $s(N)$, λ_M, λ_N, etc... instead of $s(M,p)$, $s(N,q)$, $\lambda_M(p)$, $\lambda_N(q)$ etc... . We have :

$$1 + s^-(N) + \alpha - \gamma = \alpha + \frac{1}{2}(1 + s^+(N) + s^-(N) - s(M) + m)$$

Since $s^+(N) + s^-(N) = n - 1 - \dim_{\mathbb{C}}(\lambda_N \cap i\,\lambda_N) = n - 1 - d(p)$, we find :

$$\begin{aligned} 1 + s^-(N) + \alpha - \gamma &= \alpha + \frac{1}{2}(n + m - s(M) - d(p)) \\ &= \alpha + s^-(M) + m - \delta(p) \end{aligned}$$

Similarly :

$$n + s^+(N) + \beta - \gamma = n - s^+(M) + \delta(p) + \beta$$

If we choose N such that $s^-(N,q) = 0$, then we can take $\alpha = 0$: in fact

$H^o(\mu_N(\mathcal{O}_Y))_q = 0$ by the principle of holomorphic extension. Similarly if we choose N such that $s^+(N,q) = 0$ then we can take $\beta = 0$: this follows from a theorem of Malgrange [14] which asserts that $H^j_Z(\mathcal{O}_X) = 0$ for $j > n$ and any locally closed subset Z in X , hence in particular $H^j(\mu_N(\mathcal{O}_X)) = 0$ for $j > n$. This completes the proof.

Proposition 4.4. : Let M be a real submanifold of class C^2 , $p_o \in T^*_M X$.
Assume :

i) $\dim_{\mathbb{R}} (T_{p_o} T^*_M X \cap \nu(p_o)) = 1$

ii) $s^-(M,p) - \delta(p)$ is constant in a neighborhood of p_o .

Set :

$$j_o = \operatorname{codim} M + s^-(M,p_o) - \delta(p_o) .$$

Then $H^j(\mu_M(\mathcal{O}_X))_{p_o} = 0$ for $j \neq j_o$, and for $j = j_o$ this space is infinite dimensional.

Sketch of the proof.
As for the proof of Proposition 4.3, we may interchange, by a complex contact transformation ϕ, $(T^*X, T^*_M X , p_o)$ and $(T^*Y, T^*_N Y, q_o)$, where now N is a real hypersurface, and $s^-(N,q_o) = 0$. On the other hand one proves easily that under the hypothesis ii), $s^-(N, \phi(p)) - \delta(\phi(p))$ is locally constant. Since N is a hypersurface, $\delta(\phi(p)) = 0$, and $s^-(N,q) \equiv 0$ for q in a neighborhood of q_o . Thus N is the boundary of a pseudo-convex open set (cf. Hörmander [5, Theorem 2.6.12]), and we get :

$$H^j(\mu_N(\mathcal{O}_Y))_{q_o} = 0 \text{ for } j \neq 1$$

The same calculation as in the proof of Proposition 4.3 gives the result.

Remark 4.5 : Let us denote by $\dot{T}^*X$ the bundle $T^*X - T^*_X X$ and by $\dot{\pi}$ the projection $\dot{T}^*X \longrightarrow X$. Let M be a real submanifold of class C^2 , of codimension m , and assume $\delta(p) = 0$ that is, $TM + i\, TM = TX$. Let $\underline{\omega}_M$ be the orientation sheaf on M . We have a triangle (cf. [17]) :

$$\mathcal{O}_X|_M \longrightarrow \mathbb{R}\Gamma_M(\mathcal{O}_X) \otimes \underline{\omega}_M[m] \longrightarrow \mathbb{R}\dot{\pi}_* \mu_M(\mathcal{O}_X) \otimes \underline{\omega}_M[m] \longrightarrow \mathcal{O}_X|_M[1]$$

Applying Proposition 4.3, we get $H^j(\mu_M(\mathcal{O}_X)) = 0$ for $j < m$, and we have an exact sequence :

$$0 \longrightarrow (\mathcal{O}_X)_{|M} \longrightarrow H^m_M(\mathcal{O}_X) \otimes \underline{\omega}_M \longrightarrow \dot{\pi}_* H^m(\mu_M(\mathcal{O}_X)) \otimes \underline{\omega}_M \longrightarrow 0$$

Thus :

i) Assume $s^-(M,p) > 0 \quad \forall p \in \dot{T}^*_M X$. Then :

$$H^m_M(\mathcal{O}_X) \otimes \underline{\omega}_M \cong \mathcal{O}_X|_M$$

ii) Assume there exists a closed convex proper cone Γ in $T^*_M X$ such that $s^-(M,p) > 0 \quad \forall p \notin \Gamma$. Then any section u of $H^m_M(\mathcal{O}_X)$ is represented by (i.e. : is the "boundary value of") a function f holomorphic in a wedge along M whose polar is contained in Γ (f is unique modulo $\mathcal{O}_X|_M$) .

iii) Assume now that on an open subset U of $\dot{T}^*_M X$, we have $\lambda(p) \cap i\,\lambda(p) = 0$. Let γ denote the natural map $\dot{T}^*_M X \longrightarrow S^*_M X = \dot{T}^*_M X/\mathbb{R}^+$. Then the sheaf $\gamma_*(H^{m+s^-(M,p)}(\mu_M(\mathcal{O}_X)))$ is flabby on $\gamma(U)$ (here $p \in U$), since it is isomorphic to the sheaf of boundary values of holomorphic functions on the boundary of a strictly pseudo-convex open set (corollary 3.5). This allows us to obtain results of the type "Edge of the wedge theorem", but we leave the exact formulation to the reader (for example, cf.[3]).

Remark 4.6 : Assume M is real analytic and $\delta(p) = 0$. Then $H^m_M(\mathcal{O}_X)$ is isomorphic to the sheaf of hyperfunctions on M solution of the induced Cauchy-Riemann system. More generally, under the same hypotheses, $\mu_M(\mathcal{O}_X)$ [m] is isomorphic to the complex of microfunction solutions of the induced Cauchy-Riemann system, (cf. Kashiwara-Kawai [7] , [8]). In this situation the result of Proposition 4.3 is not new, and follows from [17, chapter III, Theorem 2.3.10] , (cf. also Naruki [16]). In fact an extensive literature exists on this subject, starting may be with H. Lewy [12] and A. Andreotti - H. Grauert [1] . Let us only quote some of the most recent works related to Proposition 4.3 or Remark 4.5 : Nacinovich [15] , Baouendi - Chang - Trèves [2] , Sjöstrand [19] Tajima [20] .

Remark 4.7 : We give an application of Proposition 4.4 to the study of systems of microdifferential equations with simple characteristics in [10] .

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