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THE DIRICHLET PROBLEM FOR THE BIHARMONIC EQUATION IN A LIPSCHITZ DOMAIN

by Carlos E. KENIG

(Following the joint paper [9] of B.E.J. Dahlberg, C.E. Kenig and G.C. Verchota)

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In this work we give optimal estimates for the Dirichlet problem for the biharmonic operator Δ^2 on an arbitrary bounded Lipschitz domain D in $\mathbb{R}^n$, with the boundary values having first derivatives in $L^2(\partial D)$, and with the normal derivative being in $L^2(\partial D)$.

In recent years, considerable attention has been given to the Dirichlet and Neumann problems for Laplace's equation in a Lipschitz domain D, with $L^p(\partial D)$ data, and optimal estimates. We now know optimal estimates for both these problems in the optimal range of p's and we also have good representation formulas for the solution in terms of layer potentials. (See [4], [5], [10], [11], [13] and [7]).

In this work we initiate the corresponding study for the Dirichlet problem for the biharmonic operator Δ^2. The main idea in our work is to reduce the Dirichlet problem for the biharmonic operator, to bilinear estimates for harmonic functions in D. These bilinear estimates are Lipschitz domain generalizations of a (weak) version of the fact that the paraproduct ([3]) of two L^2 functions is in L^1. Our estimates are obtained by using the results in [6] and [11], integration by parts and the results of Coifman, McIntosh and Meyer [2].

For C^1 domains in the plane, J. Cohen and J. Gosselin ([1]) have established results analogous to ours, in L^p, $1 < p < \infty$, by the method of multiple layer potentials. G. Verchota ([14]) has shown how to modify our approach to obtain L^p results, $1 < p < \infty$, for C^1 domains in $\mathbb{R}^n$.

Our main result is

<u>Theorem</u> : Let D be a bounded connected Lipschitz domain in $\mathbb{R}^n$, with connected boundary.

(a) Let $f \in L^2_1(\partial D)$, $g \in L^2(\partial D)$. Then there exists a unique function u in D such that

(i) $\Delta^2 u = 0$ in D

(ii) $\lim_{\substack{X\to Q\\ X\in\Gamma(Q)}} u(X) = f(Q)$ a.e. $(d\sigma)$

(iii) $\lim_{\substack{X\to Q\\ X\in\Gamma(Q)}} \vec{n}_Q \cdot \nabla u(X) = g(Q)$ a.e. $(d\sigma)$

(iv) $\|M(\nabla u)\|_{L^2(\partial D,d\sigma)} < C\{\|f\|_{L^2_1(\partial D)} + \|g\|_{L^2(\partial D,d\sigma)}\}$

where $L_1^2(\partial D)$ denotes the space of all functions with one derivative in $L^2(\partial D)$, $\vec{n}_Q$ the unit normal at $Q \in \partial D$, $\Gamma(Q)$ the non-tangential region $\{X \in D : |X-Q| < (1+\alpha)\mathrm{dist}(X,\partial D)\}$, and $M(\nabla u)$ is the non-tangential maximal function $M(\nabla u)(Q) = \sup_{X\in\Gamma(Q)} |\nabla u(X)|$.

(b) There exists $\varepsilon = \varepsilon(D) > 0$ such that the above result holds with 2 replaced by p, where $2-\varepsilon < p < 2 + \varepsilon$.

(c) Given $p < 2$ there exists a bounded Lipschitz domain $D \subset \mathbb{R}^2$, with connected boundary, and a biharmonic function u in D, with $M(u) \in L^p(\partial D)$, $M(\nabla u) \in L^p(\partial D)$, $u = 0$ on ∂D, $\frac{\partial u}{\partial \vec{n}} = 0$ on ∂D, but $u \not\equiv 0$ on D.

<u>Remarks</u> : Part (c) shows that the results in part (b) for $p < 2$ are sharp in the class of all Lipschitz domains. What happens for $p > 2$ remains an open problem. The results in parts (a) and (b) deal with non-tangentiel convergence. There are also corresponding Sobolev space results. For example B. Dahlberg and C. Kenig ([8]) have shown that the solution in (a) belongs to the Sobolev space $H^{3/2}(D)$.

Part (b) is an automatic real variable consequence of part (a). (See [9] for the details).

We will now sketch the proof of the existence part of part (a), in the special case when the domain D is given by $D = \{(x,y) : y > \varphi(x)\}$, where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function. Because of the results in [11], it is enough to consider the case when $f \equiv 0$.

Let G(X,Y) be the green function for Δ ou D. Since $u|_{\partial D} = 0$, we should have

$$u(X) = \int_D G(X,Y)\, \Delta u(Y)dY .$$

Since u is biharmonic, $\Delta u(Y) = w(Y)$, should be harmonic, and we make the guess that $W(X) = \frac{\partial}{\partial y} V(X)$, where $X = (X,y)$, and v is a harmonic function in D, with $L^2(\partial D,d\sigma)$ data. In fact, we claim that the operator

$$T : v|_{\partial D} \to \frac{\partial u}{\partial \vec{n}}\Big|_{\partial D}$$

is an invertible map of $L^2(\partial D,d\sigma)$ onto $L^2(\partial D,d\sigma)$. In fact, using the Green's potential representation, Fubini's theorem, and the fact that $\frac{\partial}{\partial \vec{n}} G(.,Y)$ is the density of harmonic measure at $Y \in D$ (see[4]), we have

$$\int_{\partial D} v\, Tv = \int_D v(Y) \frac{\partial}{\partial Y_n} v(Y)\, dY = \frac{1}{2} \int_{\mathbb{R}^{n-1}} v(X,\varphi(X))^2\, dx \geq C \int_{\partial D} v^2 d\sigma$$

This shows that if $T : L^2(\partial D,d\sigma) \to L^2(\partial D,d\sigma)$ is bounded, it will have a bounded inverse. To establish the boundedness of T , it is enough to show that if $u(X) = \int_D G(X,Y) \frac{\partial}{\partial Y_n} v(Y)dY$, then

$\| M(\nabla u)\|_{L^2(\partial D,d\sigma)} \leq C\|v\|_{L^2(\partial D,d\sigma)}$. This also shows the estimate in (a).

But, u is the sum of a harmonic function H and a Newtonian potential $N(X) = \int_D \frac{1}{|X-Y|^{n-2}} \frac{\partial}{\partial Y_n} v(Y)\, dY$. If we can show that

$\| M(\nabla N)\|_{L^2(\partial D,d\sigma)} \leq C\| v\|_{L^2(\partial D,d\sigma)}$, then, as the boundary values of H and N coincide, $\| H\|_{L^2_1(\partial D)} \leq C\|v\|_{L^2(\partial D,d\sigma)}$. But then, by the results in [11] , since H is harmonic,

$$\| N(\nabla H)\|_{L^2(\partial D)} \leq C\|v\|_{L^2(\partial D,d\sigma)} ,$$

and we would be done. We have therefore reduced ourselves to establishing the following lemma.

<u>Lemma</u> : Let v be harmonic in D , and define $N(X) = \int_D \frac{1}{X-Y^{n-2}} \frac{\partial}{\partial Y_n} v(Y)dY$.

Then,

$$\| M(\nabla N)\|_{L^2(\partial D,d\sigma)} \leq C\|v\|_{L^2(\partial D,d\sigma)}$$

<u>Proof</u> : Let B be the fundamental solution for the biharmonic equation Δ^2 , i.e. $\Delta_y B(X-Y) = \frac{1}{|X-Y|^{n-2}}$ $X \neq Y$ (for example, if $n \geq 5$, $B(Y) = C_n|Y|^{4-n}$).

Let e_j , $j = 1$, n be the standard basis of $\mathbb{R}^n$. We recall the definition of the Riesz transforms, $R_j v$ of v , $j = 1$, , $n-1$. They are harmonic functions, which together with v satisfy the generalized Cauchy-Riemann equations, i.e. $\frac{\partial v}{\partial X_n} = - \sum_{j=1}^{n-1} \frac{\partial}{\partial X_j} R_j v$ (see[12]).

Using the summation convention, the integrand for the Newtonian potentiel we are considering is

$$\frac{\partial}{\partial Y_j} \frac{\partial}{\partial Y_j} B \frac{\partial}{\partial Y_n} v + \frac{\partial}{\partial Y_n^2} B \frac{\partial v}{\partial Y_n} =$$

$$= \frac{\partial}{\partial Y_j} \frac{\partial}{\partial Y_j} B \frac{\partial}{\partial Y_n} v - \frac{\partial}{\partial Y_j} \frac{\partial}{\partial Y_n} B \frac{\partial}{\partial Y_j} v + \frac{\partial}{\partial Y_j} \frac{\partial}{\partial Y_n} B \frac{\partial}{\partial Y_j} v$$

$$- \frac{\partial^2}{\partial Y_n^2} B \frac{\partial}{\partial Y_j} R_j v =$$

$$= < (- \frac{\partial}{\partial Y_1} \frac{\partial}{\partial Y_n} B , \dots , - \frac{\partial}{\partial Y_{n-1}} \frac{\partial}{\partial Y_n} B, \frac{\partial}{\partial Y_j} \frac{\partial}{\partial Y_j} B), \nabla v > +$$

$$+ < \frac{\partial}{\partial Y_j} \frac{\partial}{\partial Y_n} B e_n , \nabla R_j v > - < \frac{\partial^2}{\partial Y_n^2} B e_j , \nabla R_j v > =$$

$$= < \vec{\alpha} , \nabla v > + < \vec{\beta}_j , \nabla R_j v > ,$$

where $< , >$ is the inner product in $\mathbb{R}^n$, and

$$\vec{\alpha} = (- \frac{\partial}{\partial Y_1} \frac{\partial}{\partial Y_n} B , \dots\dots , - \frac{\partial}{\partial Y_{n-1}} \frac{\partial}{\partial Y_n} B, \frac{\partial}{\partial Y_j} \frac{\partial}{\partial Y_j} B) ,$$

$$\vec{\beta}_j = \frac{\partial}{\partial Y_j} \frac{\partial}{\partial Y_n} B e_n - \frac{\partial^2}{\partial Y_n^2} B e_j .$$

Note that $\vec{\alpha}$, $\vec{\beta}_j$, $j = 1$, , $n-1$ are divergence free vectors, and so, by integration by parts,

$$N(X) = \int_{\partial D} [- \vec{n}_j(Q) \frac{\partial}{\partial Q_j} \frac{\partial}{\partial Q_n} B(Q-X) - \vec{n}_n(Q) \frac{\partial}{\partial Q_j} \frac{\partial}{\partial Q_j} B(X-Q)] . v(Q) d\sigma(Q) +$$

$$+ \int_{\partial D} [\vec{n}_n(Q) \frac{\partial}{\partial Q_n} \frac{\partial}{\partial Q_j} B(Q-X) - \vec{n}_j(Q) \frac{\partial^2}{\partial Q_n^2} B(Q-X)] \; R_j v(Q) \; d\sigma(Q)$$

Because of [6] and classical arguments (see [13] for the details in a similar situation), $\| R_j v\|_{L^2(\partial D, d\sigma)} \leqslant C \|v\|_{L^2(\partial D, d\sigma)}$. Thus, N(X) is simply a sum of boundary potentiels of the form $\int_{\partial D} \frac{\partial}{\partial Q_j} \frac{\partial}{\partial Q_n} B(X-Q) f(Q) d\sigma(Q)$, where $\| f\|_{L^2(\partial D, d\sigma)} \leqslant C \|v\|_{L^2(\partial D, d\sigma)}$. The fact that $M(\nabla N)$ is in $L^2(\partial D, d\sigma)$ now follows from the results of Coifman, McIntosh and Meyer [2] .

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