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NON LINEAR EVOLUTION EQUATIONS
CAUCHY PROBLEM AND SCATTERING THEORY

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Exposé I

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(0) Introduction

In this talk, I should like to present a general framework to study the Cauchy problem and the theory of Scattering for a class of non linear evolution equations which occur in Physics. The equations I have in mind are among others :

The non linear Schrödinger (NLS) equation

$$i \frac{d\varphi}{dt} = -\frac{1}{2} \Delta\varphi + f_0(\varphi), \qquad (0.1)$$

The Hartree equation

$$i \frac{d\varphi}{dt} = -\frac{1}{2} \Delta\varphi + \varphi(V * |\varphi|^2), \qquad (0.2)$$

The non linear Klein Gordon (NLKG) equation

$$\Box\varphi + m^2 \varphi + f_0(\varphi) = 0 , \qquad (0.3)$$

where φ is a complex valued function defined in $n+1$ dimensional space time $\mathbb{R}^{n+1}$, Δ is the Laplace operator in $\mathbb{R}^n$, $\Box = \frac{\partial^2}{\partial t^2} - \Delta$,

f_0 is a non linear function of φ , a typical example being

$$f_0(\varphi) = \lambda_1 |\varphi|^{p_1 - 1} \varphi + \lambda_2 |\varphi|^{p_2 - 1} \varphi , \qquad (0.4)$$

and V is a real even function of space.

The general framework will also apply, at least as far as the Cauchy problem is concerned, to the Yang Mills equations, and to various systems of equations of the previous type coupled between themselves or with otherwise linear equations such as the Maxwell and Dirac equation, with reasonable (but in general non linear) couplings. Rather than giving a list of equations and systems and of the available results for them, I shall concentrate on the abstract theory, and illustrate each of its steps with the example of the NLS equation and to a lesser extent of the NLKG equation. The theory has reached roughly the same stage of developement for these two equations as regards both the Cauchy problem and the theory of scattering. The case of the NLS equation is slightly more difficult (although less complicated) than that of the NLKG equation, essentially because there exists a larger wealth of results and especially estimates for the corresponding free equation in the latter than in the former case. Also the result for the NLS equation are more recent and less well known. The Hartree equation can be treated exactly in the same way as the NLS equation, with similar results. Other equations are in a much less advanced stage of development. Systems including the Dirac equation are plagued by the non positivity of the energy, already at the stage of the Cauchy problem, while systems involving the Maxwell and a fortiori the Yang Mills equations exhibit long range effects which make the theory of scattering much more difficult.

The general framework was originally developed by Segal [8] and in a less general form by Browder [1] following earlier work by Jörgens [5] on the NLKG equation. There is a vast amount of literature on the NLKG equation. The main results concerning scattering were obtained by Strauss [10] and Moravetz and Strauss [7] , following earlier work of Segal, and with later contribution of several authors. The corresponding results for the NLS equation were obtained in [3] and [6] . The Cauchy problem for the Yang Mills equation was solved locally in [9] and globally in [4] in dimensions $1+1$ and $2+1$ and in [2] in dimension $3+1$. There is no complete theory of scattering for that case.

(1) The Cauchy problem at finite times

The equations considered here can be written in general as

$$\frac{du}{dt} = Ku + f(t,u) \tag{1.1}$$

where u is a function from space time to a finite dimensional vector space, K a linear differential operator which we assume for simplicity to be time independent and f a non linear interaction term, in general of lower degree than K in the space derivatives. For instance, for the NLS equation, one can take $u = \varphi$, $K = \frac{i}{2}\Delta$ and $f(t,u) = -if_0(\varphi)$. For the NLKG equation, one can take $u = \binom{\varphi}{\psi}$, $K = \begin{pmatrix} 0 & 1 \\ \Delta - m^2 & 0 \end{pmatrix}$ and $f(u) = \begin{pmatrix} 0 \\ -f_0(\varphi) \end{pmatrix}$ so that ψ is simply $\frac{d\varphi}{dt}$ as a consequence of (1.1). The Cauchy problem consists in solving the equation (1.1) with initial condition $u(t_0, x) = u_0(x)$ for some $t_0 \in \mathbb{R}$ and some prescribed u_0. It is convenient and traditional to split that problem into two separate ones. The first problem is the <u>local</u> Cauchy problem and consists in solving (1.1) in some small interval around t_0. The second problem is the global Cauchy problem and consists in extending the solutions thereby obtained to all times. We shall consider these two problems successively.

(1.a) The local Cauchy problem

A general abstract method for studying the Cauchy problem was proposed by Segal in 1963 [8]. It consists in recasting the Cauchy problem in the form of an integral equation and solving that equation by a fixed point method. For that purpose, one introduces the one parameter group of operators $U(t) = \exp(tK)$ formally generated by K (to be called henceforth the free evolution group) and rewrites the Cauchy problems in the form

$$u(t) = U(t-t_0)u_0 + \int_{t_0}^{t} d\tau\, U(t-\tau) f(\tau, u(\tau)) \equiv A(t_0, u_0)u(t) \tag{1.2}$$

One then looks for solutions of (1.2) in the space $\mathscr{C}(I,X)$ of continuous function of the time t in some interval $I \subset R$ to a suitable Banach space X. For comp_act I, $\mathscr{C}(I,X)$ is itself a Banach space with norm

$$|u|_I \equiv \sup_{t \in I} \|u(t)\|_X \quad , \tag{1.3}$$

where $||.||_X$ denotes the norm in X. We denote by $B(I,\rho)$ the ball of radius ρ in $\mathscr{C}(I,X)$. If one can choose X in such a way that

(1) $U(.)$ is a continuous one parameter group of bounded operators in X, with

$$\|U(t)\|_{X \leftarrow X} \leq \mu(|t|) \;, \tag{1.4}$$

(2) f is a continuous function from $\mathbb{R} \times X$ to X satisfying a Lipschitz condition

$$\|f(t,u_1) - f(t,u_2)\|_X \leq C(I,\rho)\, \|u_1 - u_2\|_X \tag{1.5}$$

for all $u_i \in X$, $\|u_i\|_X \leq \rho$ and all $t \in I$, , then one sees easily that for $u_0 \in X$ and for $I = [t_0 - T\,,\, t_0 + T]$ with T sufficiently small (depending on u_0), the operator $A(t_0 \mu_0)$ leaves invariant a ball $B(I,\rho)$ in $\mathscr{C}(I,X)$ containing the free term $U(.-t_0)u_0$ and is a contraction in that ball, so that the equation (1.2) has a unique solution in $B(I,\rho)$ by the contraction mapping principle, thereby providing a solution of the local Cauchy problem. The method extends in a straightforward way to the case where $U(.)$ is only a semi-group, or where $U(.)$ is replaced by a two parameter group or semi group of bounded propagators $U(t,s)$, corresponding to the case where the operator K

is time dependent.

The theory just described applies to the equations mentioned in the introduction for suitable choices of the space X. However, it imposes restrictions on that choice that are both unnecessarily strong and inconvenient for various purposes. This is especially true for the NLS equation, as we shall see below. It is therefore useful, and actually possible, to develop a more general formalism, where we relax the assumption that the free group U(.) be bounded in X. For that purpose, we assume that there exists a "large" space Z (in practice one can often take $Z = \mathcal{D}'(\mathbb{R}^n)$) such that f be a continuous map from $\mathbb{R} \times X$ to Z and U(.) a one parameter continuous group in Z, so that the integrand $U(t - \tau) f(\tau, u(\tau))$ in (1.2) is well defined (in Z) for $u \in \mathcal{C}(I,X)$ and τ in I. We then replace the separate assumptions (1.4) and (1.5) on U and f by a joint assumption on the pair(U,f). Since in addition we are interested in the integral in (1.2) rather than in the integrand, we formulate this assumption on the integral itself. There is however one complication coming from the fact that U is no longer assumed to be bounded in X, namely the fact that $v \in \mathcal{C}(I,X)$ no longer implies that the function $(t,s) \to U(t-s)v(s)$ belongs to $\mathcal{C}(\mathbb{R} \times I,X)$. We take that fact into account by stating the basic assumption in terms of integrals more general than that occuring in (1.2), and formally defined by

$$G([s_1,s_2],u)(t) \equiv \int_{s_1}^{s_2} d\tau\, U(t-\tau)\, f(\tau, u(\tau)). \tag{1.6}$$

It then turns out that the main results relative to the local Cauchy problem can be derived from the following assumption on G.

Assumption 1.1. For any interval I, any $t \in R$ and any $u \in \mathcal{C}(I,X)$, the function $\tau \to U(t-\tau)\, f(\tau,u(\tau))$ is locally Bochner integrable from I to X, and the function $(s_1, s_2, t) \to G([s_1,s_2],u)(t)$ is continuous from $I \times I \times \mathbb{R}$ to X. For any bounded intervals I and J, for any $\rho > 0$, for any u_1 and $u_2 \in B(I,\rho)$, G satisfies the following Lipschitz condition

$$|G(I,u_1) - G(I,u_2)|_J \leqslant C(I,J,\rho)|u_1 - u_2|_I , \qquad (1.7)$$

where $C(I,J,\rho)$ is separately non decreasing in I, J and ρ, and tends to zero when I tends to zero for fixed J and ρ.

Note that the monotonicity of $C(I,J,\rho)$ in each of its arguments is quite natural, from the very nature of the estimate (1.7). The purpose of introducing the interval J is to control the t-dependence of G, which is not controlled otherwise since we no longer assume U to be bounded in X. The assumption 1.1 is easily seen to be satisfied under the separate assumptions made in Segal's theory, and in particular (1.7) follows easily from (1.4) and (1.5). The next level of generality, which is sufficient to cover all the applications considered later, is the situation where there exists in addition to X an auxiliary Banach space $\bar{X}$ such that for each $t \neq 0$, $U(t)$ is a bounded operator from $\bar{X}$ to X, continuous in t for $t \neq 0$, and satisfying an estimate

$$\|U(t)\|_{X \leftarrow \bar{X}} \leqslant \mu(|t|) \qquad (1.\bar{4})$$

for some function μ continuous for $t \neq 0$ and integrable at zero, and where f is a continuous function from $\mathbb{R} \times X$ to $\bar{X}$ satisfying a Lipschitz condition

$$\|f(t,u_1) - f(t,u_2)\|_{\bar{X}} \leqslant C(I,\rho)\, \|u_1 - u_2\|_X \qquad (1.\bar{5})$$

for $u_i \in X, ||u_i||_X \leq \rho$ and $t \in I$.

The reasons for stating the assumption 1.1 in the form given rather to assume the above separate properties on U and f are twofold. On the one hand, that form might be actually weaker than the separate assumptions on (U,f) in more complicated situations, and in any case it is exactly what is needed for the local Cauchy problem. On the other hand, it anticipates on the form of the assumption needed to study the local problem at infinity, which is the basic step in the theory of scattering.

Under the assumption 1.1, one derives by various arguments revolving around the contraction mapping principle the basic results concerning the the local Cauchy problem : existence of solutions in a small time interval, uniqueness of solutions in an arbitrary interval and continuity of the solutions with respect to the initial data in the neighborhood of a given solution.

<u>Proposition 1.1</u> Let the assumption 1.1 hold, let $t_0 \in \mathbb{R}$. assume (for simplicity) that the interaction is source free, namely $f(t,0) = 0$. Then

(1) For any $\rho > 0$, there exists $T \equiv T(t_0,\rho)$ such that (with $I \equiv [t_0 - T, t_0 + T]$) for any $u_0 \in X$ such that $U(.-t_0)u_0 \in B(I,\rho)$, the equation (1.2) has a unique solution in $B(I, 2\rho)$.

(2) for any interval $I \ni t_0$ and any $u_0 \in X$ such that $U(.-t_0)u_0 \in \mathscr{C}(I,X)$, the equation (1.2) has at most one solution in $\mathscr{C}(I,X)$.

(3) Let I and J be compact intervals, $I \subset J$, $t_0 \in I$, $u_0 \in X$ such that $U(.-t_0)u_0 \in \mathscr{C}(J,X)$, and $u \in \mathscr{C}(I,X)$ a solution of (1.2). Then there exists a neighborhood $\mathcal{U}$ of (t_0, u_0) such that for any (t'_0,u'_0) in $\mathcal{U}$, the equation (1.2) with (t_0,u_0) replaced by (t'_0,u'_0) has a unique solution u' in $\mathscr{C}(I,X)$, and that solution depends continuously on (t'_0,u'_0). (The relevant topology on (t_0,u_0) is that induced by the topology of $\mathbb{R}$ for t_0 and by the norm $|U(.-t_0)u_0|_J$ on u_0. The relevant topology on u is that induced by the norm $\sup_{s\in I} |U(.-s)u(s)|_J$).

The basic problem one encounters when trying to apply the abstract theory to a specific equation is of course to choose the space X. For that purpose, it is useful to look ahead and anticipate on the treatment of the global Cauchy problem. There, an essential role is played by the conservation laws associated with the equation (1.1). The net effect of these conservation laws is the existence of a space X_2 such that any solution with initial data in X_2 is a priori controlled in X_2. In general, the dominant part of the conserved quantities comes from the free part of the equation, and therefore is quadratic. As a consequence, in general X_2 is a Hilbert space. Furthermore, the free group U is in general bounded in X_2. X_2 is the natural space where to look for global solutions, and therefore also the natural candidate for X. Failure to choose $X = X_2$ will be a source of difficulties at the stage of the global Cauchy problem.

In the subsequent applications we shall use mainly the usual L^q spaces with norm $||.||_q$, $(1 \leq q \leq \infty)$ and the Sobolev space W^{kq} (k integer, $k \geq 0$, $1 \leq q \leq \infty$) defined by

$$W^{kq} = \{u : \sum_{\alpha : 0 \leq |\alpha| \leq k} ||D^\alpha u||_q \equiv ||u||_{kq} < \infty\} \tag{1.6}$$

where $\alpha = (\alpha_1, \dots \alpha_n)$ denotes a multiindex of space derivatives, and $|\alpha| = \alpha_1 + \dots + \alpha_n$.

We shall also use the notation H^k for $W^{k,2}$. Sobolev spaces satisfy various embedding relations. In particular $H^1 \subset L^q$ for $2 \leq q \leq \frac{2n}{n-2}$ $(n \geq 3)$, and $H^k \subset L^\infty$ for $k>n/2$.

Furthermore, H^k is an algebra (for the pointwise operations) for $k > n/2$.

We now show how the abstract theory can be applied to the NLS equation. The free group is $U(t) = \exp(i\frac{t}{2}\Delta)$ and satisfies the following two properties, the only ones to be used at this stage :

. $U(t)$ is unitary in H^k for any $k \geq 0$.

. $U(t)$ is bounded and strongly continuous in t from $L^{\bar{q}}$ to L^q (more generally from $W^{k,\bar{q}}$ to $W^{k,q}$), where $2 \leq q \leq \infty$ and for any such q, $\bar{q}$ denotes the dual exponent, $1/q + 1/\bar{q} = 1$, with an estimate

$$\| U(t) \|_{L^q \leftarrow L^{\bar{q}}} \leq (2\pi |t|)^{-\delta(q)} \tag{1.7}$$

with

$$\delta(q) = \frac{n}{2} - \frac{n}{q} \tag{1.8}$$

Under suitable assumptions the space X_2 suggested by the conservation laws turns out to be H^1, resulting from the conservation of the L^2 norm and the energy. Unfortunately, there does not seem to be any simple way to take $X = H^1$, except for $n = 1$, and we must make another choice for X. There is a large amount of freedom in that choice. "Large" spaces (typically such that $X \supset H^1$) have the obvious advantage that one can handle a larger set of initial data, while "small" spaces (typically such that $X \subset H^1$) provide additional smoothness of the solutions. Additional differences will appear at the stage of globalization. We give below for illustration three examples where the assumption (1.1) is satisfied and for which the local theory applies. For each of them, we indicate the choice of the space and the relevant assumptions on the interaction f_0.

<u>Case 1</u> (small space) n arbitrary, $X = H^k$, $k > n/2$, $f_0 \in \mathcal{C}^{k+1}$ with $f_0(0) = 0$. This case is covered by the theory in Segal's original form. Note that there is no restriction on the behaviour of f_0 at infinity, as is typical of theories where X consists of bounded functions.

<u>Case 2</u> (large space) $n \geq 2$, $X = L^{r_1} \cap L^{r_2}$ with

$$\frac{1}{2} - \frac{1}{n} < \frac{1}{r_2} < \frac{1}{r_1} \leq \frac{1}{2} \tag{1.9}$$

so that $X \supset H^1$ by the Sobolev embedding theorem ; $f_0 \in \mathscr{C}^1$ with $f_0(0)=0$, and in addition

$$|f_0'(z)| \leq C\left(|z|^{p_1-1} + |z|^{p_2-1}\right) \tag{1.10}$$

with

$$\frac{r_1}{\bar{r}_2} \leq p_1 \leq p_2 \leq \frac{r_2}{\bar{r}_1} \ . \tag{1.11}$$

The conditions (1.9) and (1.11) imply

$$0 \leq p_1 - 1 \leq p_2 - 1 < \frac{4}{n-2} \tag{1.12}$$

and an additional coupled restriction between p_1 and p_2 to the effect that they are not too far from each other. In particular for a single power p, one can take any $p \in [1, 4/(n-2))$ for a suitable choice of X. That case requires the full generality of the abstract theory. The assumption 1.1 is satisfied via (1.4) and (1.5) with the choice $\bar{X} = L^{\bar{r}_1} \cap L^{\bar{r}_2}$. In particular (1.4) follows from (1.7).

A similar theory can be made for n = 1, but there is hardly any point in doing so since for n = 1 one can work directly with $X = X_2 = H^1$.

Case 3 (intermediate example) $n = 3$, $X = H^1 \cap L^\infty$, $f_0 \in \mathscr{C}^2$ with $f_0(0) = f'_0(0) = 0$. That case has been studied in some detail because of the special interest in dimension 3 and in working with the largest convenient space of bounded functions. That theory falls again in the general case,(1.4) and (1.5) now hold with $X = H^1 \cap W^{1,q}$ for an arbitrary $q \in (3,6)$.

We conclude this section with some general comments on smoothness. The question is to ascertain whether solutions of (1.2) in $\mathscr{C}(I,X)$ have additional smoothness properties in space and time if the initial data are smooth. At the abstract level, where no mention is made of space variables, smoothness in space is naturally replaced by the property that the solution remains in the domain of some power of the free generator, while smoothness in time is expressed in terms of differentiability.

A general study along these lines has been made by Segal in the original version of the theory. It does not extend in any obvious way to the more general framework given here. Actually we shall see below on the example of the NLS equation that serious difficulties are likely to occur. The difficulty is already apparent when considering the differential equation (1.1), where smoothness would be expected to mean for instance that $\frac{du}{dt}$ and Ku belong to X, while the interaction term is a priori expected to belong to the auxiliary space $\bar{X}$ (if any), which may be very different from X.

At the concrete level of a given PDE, smoothness in space can be analyzed in a much more flexible way by asking whether a given solution $u \in \mathscr{C}(I,X)$ of (1.2) with suitable initial data is such that Du remains in X for some class of differential operators D in the space variables, not restricted to the single operator K.

In any case, at the abstract as well as at the concrete level, the proof of smoothness often boils down to studying a linearized version of the equation (1.2) in a neighborhood of a given solution. Typically, for a concrete PDE, if D is a differential operator commuting with U(.) and f a local function of u, one has to study the linear inhomogeneous equation

$$Du(t) = U(t-t_0)\,Du_0 + \int_{t_0}^{t} d\tau \; U(t-\tau)\left\{ \frac{\partial f}{\partial u}(\tau, u(\tau))\, Du(\tau) + \text{lower order terms} \right\} \tag{1.13}$$

for v = Du considered as an unknown function, and prove that it has a solution for v with more smoothness than was anticipated for Du. As we shall see below, this technical step, and thereby the smoothness problem, has a close connection with the problem of globalization in small spaces.

(1.b) The global Cauchy problem

The natural question that arises next is whether the solutions obtained at the previous step can be continued for all times, namely the problem of globalization. It appears already on elementary examples of ODE that global solutions may fail to exist. For instance, the differential equation

$$\frac{dy}{dt} = y^2$$

with initial condition $y(0) = y_0 > 0$ is solved by $y(t) = y_0(1 - t\, y_0)^{-1}$, which blows up at time $t = y_0^{-1}$. When the local existence result of Section 1.a is available, one is tempted to construct global solutions by iterating the local construction : one solves successively the Cauchy problem with initial times $t_j (j = 0,1,2..)$ and initial data $u(t_j)$ in time intervals $[t_j, t_j+1]$ with $t_{j+1} = t_j + T_j$. The reason why that method fails to yield global solutions is simple : at each step of the resolution, the norm of $u(t_j)$ can increase by some factor $\lambda > 1$. The time T_j of local resolution, on the other hand, in general decreases as a negative power of that norm.(typically a power $-(p-1)$ if f_0 has degree p). As a consequence the T_j form a convergent geometric series, and the solution cannot be continued beyond $\bar{t} = t_0 + \sum_{j=0}^{\infty} T_j$. In addition, this argument shows that the solution ceases to exist because its norm tends to infinity. It also points out to a possible way to circumvent the difficulty, namely the method of a priori estimates : that method consists in proving that any solution of the equation (1.2) is a priori bounded for each time, in terms of the initial data, in the norms which are relevant to solve the local problem. If this is the case, the T_j that occur in the previous iteration, which are expressed in terms of the appropriate norm of $u(t_j)$, are controlled by the a priori estimate and therefore do not tend to zero for any finite t_j, thereby preventing the convergence of the previous series.

In the original version of the theory (with U a bounded group in X) it is sufficient to obtain an a priori estimate of the norm of u(t) in X, namely to prove that there exists a fonction $M(t_0, u_0, t)$ such that any solution of (1.2) satisfies

$$\|u(t)\|_X \leq M(t_0, u_0, t) \tag{1.14}$$

for all t for which it is defined. In the generalized framework, the previous condition has to be replaced by the following condition · For any compact interval J containing t_0, there exists $M(t_0,u_0,J)$ such that any solution $u \in \mathcal{C}(I,X)$ of (1.2) in an interval I, $t_0 \in I \subset J$, satisfies

$$\sup_{s \in I} |U(\cdot - s)u(s)|_J \equiv \sup_{\substack{s \in I \\ t \in J}} \|U(t-s)u(s)\|_X \leq M(t_0, u_0, J). \tag{1.15}$$

The basic tool that enters in the derivation of the a priori estimates (1.14) or (1.15) consists of the conservation laws associated with the equation. In a number of interesting cases, these conservation laws follow by a straighforward application of Noether's theorem from the fact that (1) the equation under consideration is the Euler-Lagrange equation associated with some Lagrangian and (2) the Lagrangian is invariant under some transformation group. This is the case in particular for the NLS equation (0.1) and for the NLKG equation (0.3) provided f_0 is of the form

$$f_0(z) = \frac{\partial V}{\partial \bar{z}} \tag{1.16}$$

for some real function $V(z) = V(|z|)$ depending only on $|z|$ (derivatives being taken with respect to z and $\bar{z}$, regarded as independent variables). The Lagrangian densities are then

$$\mathcal{L} = \frac{i}{2}(\bar{\varphi}\,\partial_t\varphi - \varphi\,\partial_t\bar{\varphi}) - \frac{1}{2}|\nabla\varphi|^2 - V(\varphi) \tag{1.17}$$

for the NLS equation, and

$$\mathcal{L} = \partial_\mu\bar{\varphi}\,\partial^\mu\varphi - m^2|\varphi|^2 - V(\varphi) \tag{1.18}$$

for the NLKG equation.

These Lagrangians are invariant under gauge transformation of the first kind $\varphi \to e^{i\theta}\varphi$, and in addition under transformations of the Galilei group for the NLS equation and of the Poincaré group for the NLKG equation.

Of special interest are the conservation laws that give rise to quantitites having some positivity properties, so that they can be used to control some norm of the solutions. For the NLS equation, gauge invariance yields the conservation of the L^2 norm and time translation invariance yields the conservation of the energy :

$$\|\varphi(t)\|_2 = \|\varphi_0\|_2 \quad , \quad E(\varphi(t)) = E(\varphi_0) \tag{1.19}$$

where

$$E(\varphi) = \frac{1}{2}\|\nabla\varphi\|_2^2 + \int dx\, V(\varphi) \tag{1.20}$$

These two conservation laws together imply at least formally, that the solutions are uniformly bounded in $X_2 = H^1$ provided $\varphi_0 \in L^2$ and $E(\varphi_0)$ is finite, and provided V satisfies some lower boundedness condition that prevents the kinetic and potential energies to become separately infinite for fixed total energy $E(\varphi)$. A sufficient condition to that effect is that V satisfies the condition

$$V(\rho) \geqslant -C(\rho^2 + \rho^{1+p_3}) \tag{1.21}$$

with $p_3 < 1 + 4/n$.
Similarly, for the NLKG equation, gauge invariance yields the conservation of the charge, which is not positive, while the energy is given by

$$E(\varphi) = \|\dot{\varphi}\|_2^2 + \|\nabla\varphi\|_2^2 + m^2 \|\varphi\|_2^2 + \int dx\, V(\varphi). \tag{1.22}$$

Energy conservation then implies formally that for any solution φ of the NLKG, $u \equiv (\varphi, \dot{\varphi})$ is uniformly bounded in $X_2 = H^1 \oplus L^2$ under a suitable lower boundedness condition on V. The relevant condition in that case is

$$V(\rho) \geqslant -C\rho^2. \tag{1.23}$$

One is then faced with the task of proving the (so far formal) conservation laws in the functional framework where one solves the local Cauchy problem. Two difficulties may occur, depending on the relation of X with X_2 : The first one is that the conservation laws are easily derived in differential form from the equation (1.1), whereas solutions of (1.2) in $\mathcal{C}(I,X)$ may not be sufficiently smooth for (1.1) to hold in a sufficiently strong sense. That difficulty can be circumvented by a cut off and limiting procedure. One regularizes both the equation and the initial data by introducing suitable cut offs, one proves that the solutions of the regularized equation satisfy a regularized form of the conservation law, and one removes the cut off by a limiting procedure. A more serious difficulty occurs if X is a "large" space, in the sense that $X \not\subset X_2$. In that case, one does not even know in advance that a solution in $\mathcal{C}(I,X)$ with initial data in $X \cap X_2$ remains in X_2 for all times where it is defined, and one must prove this fact together with the conservation law. For that purpose, the cut off and limiting procedure has to be supplemented with a weak compactness argument in X_2, in which the conservation law for the regularized solutions is used to prove their boundedness in X_2 uniformly with respect to the cut off.

Once it is proved that the solutions in $\mathscr{C}(I,X)$ with initial data in $X \cap X_2$ satisfy the relevant conservation laws, and are estimated in X_2 in terms of the initial data by virtue of these conservation laws, the last step in the globalization problem is to derive the a priori estimates (1.14) or (1.15) from the conservation laws. Here again the situation depends critically on the relation between X and X_2, but now in contrast with the previous step, the trouble comes from "small" spaces. For "large" spaces $X \supset X_2$, boundedness in X_2 immediately implies the required estimates in X. For small spaces $X \subset X_2$ on the contrary, additional estimates are needed. In the typical case when X is a Sobolev space one generally needs to control suitable L^q norms of derivatives D of higher order than occurs in the conservation laws This can (or cannot) be done by starting from equations of the type (1.13) for such derivatives and using the estimates provided by the conservation laws to obtain sublinear integral inequalities for their relevant norms, from which the required a priori estimates follow by Gronwall's inequality. This step is technically very similar to that required for the proof of smoothness properties of the solutions. As mentioned earlier smoothness is closely connected with globalisation in small spaces.

The NLS equation provides enlightening examples of the various possibilities described above, depending on the choice of X. We consider again the three examples given above and indicate briefly the corresponding results for the global Cauchy problem. In all cases we make again the assumptions needed for the local Cauchy problem and we assume (1.10 and (1.21) with $p_3 < 1 + 4/n$. We present the three cases in the order of increasing difficulty, which is not the same as before, as just explained. In all three cases $X_2 = H^1$.

Case 2 (large space) $n \geq 2$, $X = L^{r_1} \cap L^{r_2}$. Then for all $\varphi_0 \in X_2 (\subset X)$ (1.2) has a unique solution in $\mathscr{C}(\mathbb{R}, X_2)$, and that solution is uniformly bounded in X_2 (and therefore in X).

Case 3 (intermediate space) $n = 3$, $X = H^1 \cap L^\infty$. Assume in addition that f_0 satisfies the condition

$$|f_0'(z)| \leq C \; |z|^{p_2 - 1} \qquad \text{for} \qquad |z| \geq 1 \tag{1.24}$$

with

$$0 \leq p_2 - 1 < \frac{4}{n-2} \tag{1.25}$$

Then (1.2) has a unique solution in $\mathscr{C}(\mathbb{R}, X)$ and that solution is uniformly bonded in X_2. Actually it turns out that the solution is also uniformly bonded in L^∞ and therefore in X provided the free term $U(.-t_0)u_0$ satisfies that property.

Note that the condition (1.25) which was needed in Case 2 already at the local stage is needed also here, but only at the global stage. This indicates that the splitting in two stages is somewhat artificial.

Case 1 (small spaces) $X = H^k$ with $k > \frac{n}{2}$. Here comes the surprise : The globalisation proof just sketched breaks down for $n \geq 8$. For $n \leq 7$, one can prove that (1.2) has a unique solution in $\mathscr{C}(\mathbb{R}, X)$ by adding the assumptions $f_0'(0) = 0$ and (1.24), (1.25) for $2 \leq n \leq 7$, plus additional mild restrictions on the behaviour of f_0'' and f_0''' for $4 \leq n \leq 7$ (that result is new).

Comparing the situation in Cases 1 and 2 leads to suspect that for high dimensions, the global solutions obtained in Case 1 may fail to remain smooth even for smooth initial data, and therefore that the smoothness problem is much more complicated in the general framework of Section 1.a than in Segal's original theory.

We conclude this section with some brief comments on the necessity of the lower bound (1.21) or (1.23) on V for the existence of global solutions. A condition of this type is indeed necessary. In fact, if

$$V(\rho) = -C\,\rho^{p_3+1} \tag{1.26}$$

with $C > 0$ and $p_3 \geq 1+4/n$ for the NLS equation, $p_3 > 1$ for the NLKG equation, then one can show that solutions of (1.1) blow up in a finite time if the initial data are large. The general method to prove such a result consists in choosing cleverly a suitable norm of the solution and deriving for that norm, from the differential equation (1.1), a differential inequality which implies some form of blow up. In the case of the NLS equation, a suitable norm is the moment of inertia of solution $\frac{1}{2}\|x\varphi\|_2^2$. For interactions of the type (1.26) with $p_3 > 1+4/n$, one can show that

$$\frac{d^2}{dt^2}\,\frac{1}{2}\,\|x\varphi\|_2^2 \leq E(\varphi) \tag{1.27}$$

so that $\|x\varphi\|_2$ vanishes for some finite time if $E(\varphi) < 0$. (Remember that $E(\varphi)$ is constant), a condition which for any (suitably regular) φ_0 is always fulfilled by $\lambda\varphi_0$ for sufficiently large λ, since the negative potential energy increases faster than quadratically in λ. The fact that $\|x\varphi\|_2 \to 0$ implies that the kinetic energy tends to infinity, since

$$\|x\varphi\|_2\ \|\nabla\varphi\|_2 \geqslant \frac{n}{2}\,\|\varphi\|_2^2 \tag{1.28}$$

and $\|\varphi\|_2$ is constant. It also suggests that the solution actually "blows in" by concentrating at a single point. By a slightly more refined argument, one can see that concentration is most likely to occur at the center of mass $\bar{x}$ of the solution, defined by $\bar{x}\|\varphi\|_2^2 = \langle \varphi, x\varphi \rangle$.

In the case of the NLKG equation, a suitable norm to consider is simply $\|\varphi\|_2$. For interactions of the type (1.26) with $p_3 = 1 + 2\delta > 1$ one can show that $F \equiv \|\varphi\|_2^{-\delta}$ satisfies the inequality

$$\frac{d^2 F}{dt^2} \leq \delta(\delta+1) E \; F^{-(1+2/\delta)} \tag{1.29}$$

which for $E < 0$ implies that $\|\varphi(t)\|_2$ becomes infinite in a finite time.

(2) Scattering theory

We now turn to the problem of the asymptotic behaviour in time of the solutions of the equation (1.1). Such problems are in general difficult, but some progress can be made in situations where in addition to the given evolution equation, there exists another, simpler, evolution equation to the solutions of which one can compare those of the original equation. That comparison is the basic purpose of Scattering theory. In the present case, the simpler equation to which (1.1) will be compared is the free equation

$$\frac{du}{dt} = K u \tag{2.1}$$

The first step consists in looking for dispersive solutions of (1.1), namely for solutions that behave asymptotically in time like solutions of (1.2). If u is such a solution one expects that there exists $u_{\pm} \in X$ so that $u(t) \sim U(t)u_{\pm}$ as $t \to \pm\infty$ or more precisely that $\tilde{u}(t) \to u_{\pm}$

as $t \to \pm\infty$, where $\tilde{u}$ is defined by

$$\tilde{u}(t) = U(-t)\, u(t) \tag{2.2}$$

If this is the case, one obtains formally from (1.2)

$$u_{\pm} = \tilde{u}(t_0) + \int_{t_0}^{\pm\infty} d\tau\, U(-\tau)\, f(\tau, u(\tau)) \tag{2.3}$$

and therefore

$$u(t) = U(t)\, u_{\pm} + \int_{\pm\infty}^{t} d\tau\, U(t-\tau)\, f(\tau, u(\tau)). \tag{2.4}$$

Dispersive solutions are obtained by solving (2.4) for u, namely by solving the Cauchy problem with infinite initial time. More generally, it is useful to study the equation

$$u(t) = U(t)\, \tilde{u}_0 + \int_{t_0}^{t} d\tau\, U(t-\tau) f(\tau, u(\tau)) \equiv A_0\, (t_0, \tilde{u}_0) u(t) \tag{2.5}$$

for given $\tilde{u}_0$ and t_0 in a neighborhood of $\pm\infty$, and to derive results on that equation that are uniform in t_0 and have some continuity in t_0 in the neighborhood of $\pm\infty$.

In terms of scattering theory, the maps $u_{\pm} \to u(0)$, where u is solution of (2.4), are simply the wave operators, the construction of which is therefore a by-product of the solution of the Cauchy problem at infinity. That problem will be considered in Section 2a, at a level of abstraction comparable with that of the Cauchy problem at finite times.

The second and more difficult step in the theory of Scattering is to ascertain whether all solutions of the equation (1.1) or rather (1.2) are actually dispersive. This is the problem of asymptotic completeness. It is much more dependent on the specific equation under consideration, and the available treatment both for the NLS and NLKG equations, are heavily based on the conservation laws associated with those equations. That problem will be considered in Section 2b.

(2a) The Cauchy problem at infinity.

We want to solve the equation (2.5) with t_0 at or in a neighborhood of infinity (We restrict our attention to $t \to +\infty$ for definiteness). For that purpose, we split again the problem in two steps: the first one is the local Cauchy problem at infinity and consists in solving (2.5) in some interval $[T, \infty)$ with T sufficiently large. The second one consists in extending the solutions thereby obtained to all values of time and is therefore covered by the results of Section 1b. We therefore concentrate on the first step, which we treat again by a contraction method similar to that of Section 1a. We now look for solutions of (2.5) in a space $\mathcal{X}_0(I) \subset \mathcal{C}(I,X)$, which for unbounded I includes some time decay in its definition, so that the integrals that occur in (2.4),(2.5) are convergent at infinity for $u \in \mathcal{X}_0(.)$ and exhibit some uniformity properties in t an t_0. There are various ways to formulate that time decay. The simplest one is to take a family of semi norms $||.||_\alpha$, $\alpha \in A$, on X, all bounded by the norm in X and including the norm in X itself, and a family of continuous functions m_α from $\mathbb{R}$ to $[1,\infty)$ such that for any compact interval I, $\sup\limits_{\alpha} \sup\limits_{t\in I} m_\alpha(t) < \infty$, and to define for any interval I

$$\mathcal{X}_0(I) = \{ u \in \mathcal{C}(I,X) : |u|_{0I} < \infty \} \tag{2.6}$$

where

$$|u|_{0I} = \sup_{\alpha} \sup_{t\in I} m_\alpha(t) \, \|u(t)\|_\alpha \tag{2.7}$$

Such spaces will be referred to as uniform spaces. For compact I, $\mathcal{X}_0(I) = \mathcal{C}(I,X)$ with $|u|_{0\ I} \geq |u|_I$, while the inclusion $\mathcal{X}_0(I) \subset \mathcal{C}(I,X)$ is strict for unbounded I.

A more complicated family of spaces is obtained by taking, in addition to some family of semi norms of the previous type, another family of semi norms $||\cdot||_\beta$, $\beta \in B$, all bounded by the norm in X, and for each β a real number $q(\beta) \geq 1$, and defining $\mathcal{X}_0$ (I) by (2.6), with

$$|u|_{0I} = \mathrm{Max}\left\{ \sup_\alpha \sup_{t\in I} m_\alpha(t) \|u(t)\|_\alpha \, , \, \sup_\beta \| \, \|u(\cdot)\|_\beta \, \|_{q(\beta),\, I} \right\}, \quad (2.8)$$

where $||\cdot||_{q,I}$ denotes the norm in $L^q(I,\, dt)$. Such spaces will be referred to as integral spaces. One could of course define more general spaces with weighted L^q norms in the time variable.

It then turns out that the main results relative to the Cauchy problem at infinity can be derived from a set of abstract assumptions that are very similar to the assumption 1.1. We restrict our attention to the case of uniform spaces, where the assumptions can be stated entirely in terms of the integrals $G([s_1,s_2],\, u)$ defined by (1.6). For any interval $I \subset R$ we denote by $\bar{I}$ its closure in $\bar{R} = R \cup \{+\infty, -\infty\}$ with the obvious topology. In particular if $I = [T,\infty)$, then $\bar{I} = [T,\infty]$ we denote by $B_0(I,\rho)$ the ball of radius ρ in $\mathcal{X}_0(I)$.

Assumption 2.1 For any closed interval $I \subset \mathbb{R}$, any $t \in \mathbb{R}$ and any $u \in \mathcal{X}_0(I)$, the function $\tau \to U(t-\tau)\, f(\tau, u(\tau))$ is Bochner integrable from I to X and the function $(s_1,s_2) \longrightarrow G([s_1,s_2],u)$ is continuous from $\bar{I} \times \bar{I}$ to $\mathcal{X}_0(\mathbb{R})$. For any closed interval I, for any $\rho > 0$, for any u_1 and u_2 in $B_0(I,\rho)$, G satisfied the following Lipschitz condition

$$|G(I,u_1) - G(I,u_2)|_{0\mathbb{R}} \leq C_0(I,\rho)\, |u_1 - u_2|_{0I} \quad (2.9)$$

where $C_0\ (I,\rho)$ is separately non increasing in I and ρ and tends to zero when I tends to zero for fixed ρ .
(By "I tends to zero" we mean that both ends of I tend to a common value, possibly $+\infty$ or $-\infty$).

Under the assumption 2.1, one derives the basic results concerning the Cauchy problem at infinity by arguments similar to those used in the study of the local Cauchy problem at finite times, and based again on the contraction mapping principle. These results include the existence of asymptotic states (i.e. of the limits $u_{\pm} = \lim u(t)$ as $t \to \pm \infty$) for dispersive solutions (now technically defined as solutions in $\mathcal{X}_0(R)$), the existence an uniqueness of solutions of (2.5) in a neighborhood of infinity, and continuity of the solutions with respect to the initial time and initial data. In order to state the results, it is convenient to define the space

$$X_0 = \{ u \in X \ : \ U(\cdot)u \in \mathcal{X}_0(\mathbb{R}) \} \tag{2.10}$$

with norm

$$\| u \|_0 = | U(\cdot) u |_{0\mathbb{R}} \tag{2.11}$$

Clearly X_0 is a Banach space continuously embedded in X.

<u>Proposition 2.1</u> Let the assumption 2.1 hold. Assume (for simplicity) that the interaction is source free, namely $f(t,0) = 0$. Then
(1) Let $I = [T,\infty)$, $t_0 \in \bar{I}$, $\tilde{u}_0 \in X_0$ and let $u \in \mathcal{X}_0(I)$ be a solution of (2.5). Then $\tilde{u} \in \mathcal{C}(\bar{I},X_0)$ (see(2.2). In particular there exists u_+ in X_0 such that $\tilde{u}(t) \to u_+$ in X_0 when $t \to \infty$.

(2) For any $\rho > 0$, there exists $T_0(\rho) < \infty$ such that for any $t_0 \in \bar{I}$, where $I = [T_0(\rho), \infty)$ and for any $\tilde{u}_0 \in X_0$ with $||\tilde{u}_0||_0 \leq \rho$, the equation (2.5) has a unique solution in $B_0(I, 2\rho)$. That solution is actually unique in $\mathcal{X}_0(I)$.

(3) In the situation of (2), the maps $(t_0, \tilde{u}_0) \to u$ and $(t_0, \tilde{u}_0) \to \tilde{u}$ are continuous, with the topology of $I \times X_0$ on $(t_0, \tilde{u}_0)$, of $\mathcal{X}_0(I)$ on u and of $\mathcal{C}(\bar{I}, X_0)$ on u.

Similar results can be obtained with the integral spaces of the type (2.6)-(2.8), although the abstract formulation is slightly more complicated.

As a by product of the local theory at infinity, one obtains in general a proof of existence of global solutions and of asymptotic completeness(namely all solutions are dispersive) for small initial data. Contrary to the global existence proof of section 1b, that proof does not depend on a priori estimates of the solutions. In fact, the applicability of the contraction argument leading to proposition 2.1 part(2) relies on the possibility of making $C_0(I,\rho)$ small. That can be achieved in general not only by taking I small for given ρ, but also by taking $I = \mathbb{R}$ and ρ small.

<u>Proposition 2.2</u> Let the assumption 2.1 hold and assume in addition that $C_0(\mathbb{R},\rho) \to 0$ when $\rho \to 0$. Assume that $f(t,0) = 0$. Then there exists $\rho > 0$ such that for any $\tilde{u}_0 \in X_0$ with $||\tilde{u}_0||_0 \leq \rho$ and any $t_0 \in \bar{\mathbb{R}}$, the equation (2.5) has a unique solution in $\mathcal{X}_0(R)$ (if t_0 is finite, the solution is actually unique in $\mathcal{C}(\mathbb{R},X)$).

When trying to apply the abstract theory to a specific equation, the basic problem one encounter is to choose the spaces $\mathcal{X}_0(.)$. That choice is dictated by the available decay estimates on the free group U(.), and the need to prove the basic estimate (2.9). In particular the time decay to be included in the definition of $\mathcal{X}_0(.)$ can be at most that of the solutions of the free equation (2.1). On the other hand, as in linear scattering theory, some decay of the interaction in space is needed. For interactions of the typical form (0.4), that condition takes the form of lower bounds on the allowed values of p_1, p_2.

We now show how the abstract theory can be applied to the NLS equation. The basic decay estimate on the free group is (1.7) We consider only the cases 2 and 3 of section 1.a.

<u>Case 2</u> $X = L^{r_1} \cap L^{r_2}$. For the norms $||.||_\alpha$ and the estimating functions $m_\alpha(t)$ of the abstract theory, we take the norms in L^r, $r_1 \leq r \leq r_2$, with $m_r(t) = (1 + |t|)^{\bar{\delta}(r)}$ and $\bar{\delta}(r) = \mathrm{Min}(\delta(r), \delta)$ for some δ, $0 < \delta < 1$, and with $\delta(r)$ defined by (1.8), so that for any interval I

$$|u|_{0I} = \operatorname*{Sup}_{t \in I} \operatorname*{Sup}_{r_1 \leq r \leq r_2} (1 + |t|)^{\bar{\delta}(r)} \, \|u(t)\|_r \tag{2.12}$$

One can then show that the assumption 2.1 holds provided f_0 satisfies the assumptions of section 1.a (namely $f_0 \in \mathcal{C}^1$, $f_0(0) = 0$ and (1.10), (1.11) and in addition

$$p_1 > \mathrm{Max}\left(\delta^{-1}, 1 + \frac{2}{n}(1+\delta)\right). \tag{2.13}$$

The condition on p_1 resulting from the best choice of δ can be rewritten as

$$p_1 \, \delta(p_1 + 1) > 1 \tag{2.14}$$

or equivalently

$$p_1 > \frac{1}{2n}\left\{n + 2 + ((n+2)^2 + 8n)^{1/2}\right\}, \tag{2.15}$$

namely $p_1 > 1 + \sqrt{2}$ for $n = 2$, $p_2 > 2$ for $n = 3$, etc.

The same equation can be treated in integral spaces of the type (2.6) , (2.8) under exactly the same assumptions on f_0 and with essentially the same results. For that purpose, one takes for the norms $||.||_\alpha$ and associated function m_α again the L^r norms for $r_1 \le r \le r_2$, with $m_r(t) = 1$, and for the norms $||\ .||_\beta$ and associated exponents $q(\beta)$ again the same norms with $q(r) = (1+\varepsilon)\ \bar{\delta}(r)^{-1}$ for some ε sufficently small, such that

$$r_1 \geqslant \mathrm{Max}\left\{\delta^{-1}\left(1+\varepsilon\left(1-\delta(r_1)\right)\right),\ 1+\frac{2}{n}\left(1+\delta+\varepsilon(1-\delta)\right)\right\} \qquad (2.13)_\varepsilon$$

A nice application of the previous integral spaces can be made by using an inequality of Strichartz which states that for all $\varphi \in L^2$, $U(.)\,\varphi \in L^q(R^{n+1})$ for $q = 2 + 4/n$. Together with elementary arguments, this implies that the previous integral spaces with $\delta = n/(n+2)$ and $\varepsilon = 1$ are such that $X_0 \supset H^1$. In particular, if one is in a situation where energy conservation holds and where the H^1 norm of solutions is bounded, namely if f_0 satisfies (1.16) and (1.21), then one obtains from Proposition 2.2 a proof of asymptotic completeness for small data in H^1. For interactions of the type (0.4) with a single power p and positive λ , this covers the case when $4/n \le p - 1 < 4/(n-2)$.

<u>Case 3</u> $n = 3$, $X = H^1 \cap L^\infty$. In that case one can define $\mathcal{X}_0(.)$ is a way similar to that of case 2, namely

$$|u|_{0I} = \sup_t \mathrm{Max}\left\{ \|u(t)\|_{1,2}\ ,\ \sup_{2\le r\le\infty} (1+|t|)^{\bar{\delta}(r)}\ \|u(t)\|_r \right\} \qquad (2.16)$$

with again $\bar{\delta}(r) = \mathrm{Min}(\delta(r), \delta)$ for some δ, $0 < \delta \le \frac{n}{2}$.

One can then show that the assumption 2.1 holds if in addition to the assumptions made in section 1.a for the local Cauchy problem, f_0 satisfies the condition

$$p_1 > \mathrm{Max}\left\{1 + \varepsilon^{-1}, \; 1 + \frac{2}{n}(1+\varepsilon)\right\} \tag{2.17}$$

That condition is stronger than (2.13), indicating that the present choice of X is less natural than the previous one.
When combined with the assumptions and results of Section 2.a, the results of this section provide some additional information. For instance, starting with initial data $\tilde{u}_0 \in X_0$ at some sufficiently large positive time t_0, one can construct a solution of the Cauchy problem that is dispersive at $+\infty$ and continue that solution to all times. In particular if $t_0 = +\infty$, $\tilde{u}_0 = u_+$, the wave operator Ω_+ $u_+ \to u(0)$ is well defined. Such solutions need not be dispersive at $t \to \infty$ however, i.e. asymptotic completeness may fail to hold. Another result of interest may be the extension of conservation laws to infinite times. For instance, for the NLS equation in case 2, one can show under the assumptions made in sections 2.a and 1.b that the dispersive solutions obtained in Proposition 2.1 satisfy $\tilde{u} \in \mathcal{C}(\bar{I}, H^1)$ and satify the identities

$$\|\varphi\|_2 = \|\varphi_+\|_2 \quad , \quad E(\varphi) = \frac{1}{2} \|\nabla\varphi_+\|_2^2$$

the second of which is strongly reminiscent of the intertwining property of the wave operators familiar in linear scattering theory.

Similar results hold for the NLKG equation.

(2b) Asymptotic completeness

Once one knows how to solve the Cauchy problem at infinity (for initial data in X_0) and in particular how to construct the wave operators $\Omega_\pm : u_\pm \to u(0)$ (as operators from X_0 to X_0), the next question to ask is whether all solutions of the equation (2.5) with finite t_0 are dispersive, namely lie in $\mathcal{X}_0(R)$, for all initial data in X_0 or at least in some space Σ densely and continuously embedded in X_0. If in addition Σ is stable under $\Omega_\pm$, this property implies asymptotic completeness in Σ.

Such a property holds only in special cases and is strongly dependent on the equation under consideration. It has been established only for the NLS and NLKG equation with interaction satisfying a suitable repulsivity condition. The proofs are based on a priori estimates on the solutions, derived from conservation laws satisfied by the equations. There are basically two methods available. The first one is based on conformal invariance for the NLKG equation (in which case it works only in the massless case [10]) and its galilean analogue, hereafter called pseudoconformal invariance, for the NLS equation [3]. The second method, which is more complicated, is based on a modified form of dilational invariance and applies to the massive NLKG equation [7] and to the NLS equation in some specific cases [6]. Here, as an illustration, we briefly sketch the pseudo conformal invariance method for the NLS equation.

One first remarks that the free Schrödinger equation, and more remarkably, the NLS equation, in the case of a single power interaction(cf(0.4)) with $p = 1 + 4/n$, is invariant under a projective representation of a group $\mathcal{G}_S$ which is larger than the Galilei group and is generated by the Galilei group, the dilations $(t,x) \longmapsto (t\,e^{2\theta}, xe^{\theta})$ and a one parameter group of transformations, hereagter called pseudoconformal, $(t,x) \to (t(1 + at)^{-1}, x(1 + at)^{-1})$

or, in projective coordinates $(t,x,1) \to (t,x, 1 + at)$. That group $\mathcal{G}_S$ is obtained by enlarging the Galilei group by its transform under the external automorphism generated by the inversion $(t,x) \to (t^{-1}, t^{-1}x)$ (in projective coordinates $(t,x,1) \to (1,x,t)$. Under that automorphism space translations are exchanged with pure Galilei transformations, and time translations with pseudo conformal transformations. $\mathcal{G}_S$ is sometimes called the Schrödinger group. The projective representation under which the free Schrödinger equation is invariant has the pseudoconformal transformations represented by

$$\varphi(t,x) \longmapsto (1-at)^{-n/2} \exp\left[-\frac{1}{2}\, a\, x^2 (1-at)^{-1}\right] \varphi\left(t(1-at)^{-1}, x(1-at)^{-1}\right) \tag{2.18}$$

For a general interaction f_0 (satisfying (1.16), the NLS equation is no longer invariant. However by a standard application of Noether's theorem, one derives easily the following approximate conservation law

$$\frac{1}{2} \| x \tilde{\varphi}(t) \|_2^2 + t^2 \int dx\, V(\varphi(t)) = \text{idem}\ (t \to s) + \int_s^t d\tau\ \tau \int dx\ W(\varphi(\tau)) \tag{2.19}$$

when $W(z) = W(|z|)$ is defined by

$$W(\rho) = (n+2)\, V(\rho) - \frac{n}{2} V'(\rho) \tag{2.20}$$

and $\tilde{\varphi}$ is defined in analogy with (2.2).

Define now Σ at the Hilbert space with norm

$$\| \varphi \|_{\Sigma}^{2} = \| \varphi \|_{2}^{2} + \| \nabla \varphi \|_{2}^{2} + \| x \varphi \|_{2}^{2} \tag{2.21}$$

Let the interaction f_0 satisfy (1.16) and be repulsive in the sense that $V \geq 0$ and $W \leq 0$, and let φ be a solution of (2.5) with initial time $t_0 = 0$ (for simplicity) and with $\tilde{\varphi}_0 (= \varphi_0) \in \Sigma$. It follows then from L^2 norm and energy conservation and from (2.19) that for all t

$$\| \tilde{\varphi}(t) \|_{\Sigma}^{2} \leq \| \varphi_0 \|_{2}^{2} + \| x \cdot \varphi_0 \|_{2}^{2} + 2 E(\varphi_0) \tag{2.22}$$

so that $\tilde{\varphi}$ is bounded in Σ uniformly in time. By a straighforward application of (1.7) and the Sobolev inequalities, this implies that φ itself satisfies the time decay

$$\| \varphi(t) \|_{r} \leq C (1 + |t|)^{-\delta(r)} \tag{2.23}$$

for all r satisying $\frac{1}{2} - \frac{1}{n} < \frac{1}{r} \leq \frac{1}{2}$.

The previous argument, which is partly formal at the moment, has to be combined with the functional framework developed in Section 2.a to study the Cauchy problem at infinity. The result are most satisfactory in case 2, with $X = L^{r_1} \cap L^{r_2}$. In that case $\Sigma \subset X_0$, and the preceding arguments boil down to a proof of asymptotic completeness in Σ for repulsive interactions in the previous sense. In particular (2.22), (2.23) imply that all solutions with initial data $\tilde{\varphi}_0 \in \Sigma$ belong to $X_0(\mathbb{R})$.

Similar results hold for the massless NLKG equation.

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