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INTERACTION OF PROGRESSING WAVES THROUGH A NONLINEAR POTENTIAL

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(including joint work with N.Ritter)

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Consider a wave equation with non-linear potential :

(1) $$Pu = f(t,x,u) \qquad \text{in } \Omega \subset \mathbb{R}_t \times \mathbb{R}_x^n$$

where f is a C^∞-function in all variables and P is a strictly t-hyperbolic linear operator of second order. The linear equation, with $f \equiv 0$, has simple progressing wave solutions which are of considerable direct significance in for example scattering theory. Thus if

(2) $$P_E = D_t^2 - \Delta_x \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}^n$$

is the Minkowski wave operator

(3) $$u = H(t - x\omega) \qquad \omega \in S^{n-1}$$

solves (1), with $f \equiv 0$, and superposition of such solutions gives the fundamental solution. The latter superposition process has no general analogue for (1) but the progressing wave solutions are still of special interest.

Two or more progressing waves, (3), do not interact in the linear case but in the semi-linear case (1) they do and new singularities are produced in the solution-although for two waves the new singularities do not propagate. Some results on interaction of up to three waves were given in [3] and will be outlined here. Related theorems have since been announced by Bony in this seminar [1] so emphasis will be placed on the behaviour of polyhomogeneous, or "classical" singularities. Results on more general propagation theorems for (1) are referenced in [1] , [3].

1. LOCAL EXISTENCE OF PROGRESSING WAVES

There are standard results on the existence of local solutions to the Cauchy problem for (1), typically with initial data $(u,D_t u)|_{t=o} \in H^s(\mathbb{R}^n) \oplus H^{s-1}(\mathbb{R}^n)$, $s > n/2$. Linear solutions such as (3) are only in H^s_{loc} for $s < 1/2$ but part of the significance of the conormal regularity of progressing waves, discussed in [3] , is that there are indeed solutions like (3) to (1). The result below is based on ideas from Ritter [4].

Let P be a second-order differential operator in an open neighborhood $\Omega \subset \mathbb{R}_t \times \mathbb{R}^n_x$ of O, with P stricly hyperbolic with respect to t. Choose $\varepsilon > 0$ so small that the set

$$\Omega_\varepsilon = \{(t,x) \in \Omega;\ -\varepsilon \leqslant t \leqslant \varepsilon \text{ and } (t,x) \text{ is in the past P-dependence domain of } (\varepsilon,0)\}$$

is compact in Ω . If S is a closed embedded characteristic hypersurface for P in Ω then there are conormal (Lagrangian) solutions of $Pu = 0$ associated to S, i.e. $u \in I(\Omega,S)$. This latter space is defined directly below, in § 2, and is just the space denoted $I^*(\Omega,N^*S)$ in Hörmander [2].

<u>Theorem 1</u> : Let P, Ω, Ω_ε be as above and suppose S_1, S_2 and S_3 are closed embedded characteristic hypersurfaces for P, in Ω , each passing through O with the three normals independent. Suppose

$$\text{(5)}\quad \begin{cases} u_o \in [I(\Omega,S_1) + I(\Omega,S_2) + I(\Omega,S_3)] \cap L^\infty(\Omega) \\ Pu_o = 0 \end{cases}$$

is a bounded conormal solution of the linear equation and $\varepsilon > 0$ is chosen small enough. Then to $f \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R})$ with support in $[-\frac{\varepsilon}{2}, \infty) \times \mathbb{R}^n \times \mathbb{R}$ there corresponds $\bar{\delta} > 0$ such that whenever $|\delta| \leqslant \bar{\delta}$ there exists a unique distribution $u \in L^\infty(\Omega_\varepsilon)$ satisfying

$$\text{(6)}\quad \begin{cases} Pu = \delta f(t,x,u) \quad \text{in } t(\Omega_\varepsilon) \\ u = u_o \quad \text{in } t < -\varepsilon/2 \\ \|u\|_\infty \leqslant 2\,\|u_o\|_{L^\infty}. \end{cases}$$

It may seem strange to have such an existence result for the continuation, as a solution of the non-linear wave equation (1), of data with an L^∞ bound. It should be noted that there is considerable regularity hidden in (5), although by no means enough for standard Sobolev estimates to apply. The analogous result with four or more hypersurfaces is probably not correct, more regularity on u_o may be required.

2. ONE WAVE

The simplest case of Theorem 1 is when there is no interaction, $u_o \in I(\Omega,S_1) \cap L^\infty$, in (5). The regularity properties of the solution are then readily determined, and more specific results are easily obtained. Recall the subspace of polyhomogeneous, or classical, distributions $I_h(\Omega,s) \subset I(\Omega,s)$, for a hypersurface S. Here if $u \in I(\Omega,S)$ then $u \in I_h(\Omega,S)$ if near each $s \in S$ in local coordinates $x_1,\dots,x_p$ in which S is given by $x_1 = 0$ for any $M \in \mathbb{N}$

(7) $$u = \sum_{\text{finite}} u_j + u_{(M)} \ , \ u_{(M)} \in C^M \ , \quad \text{near } S$$

with the u_j quasi-homogeneous

(8) $$u_j(tx_1,x_2,\dots,x_p) = t^{n(j)}\, U_j(t\ ,x_1,\dots,x_p) \qquad t > 0$$

and U_j a polynomial in $\ell n(t)$.

<u>Theorem 2</u> : Let P, Ω, Ω_ε, S_1 be as in Theorem 1 and suppose $u \in L^\infty(\Omega_\varepsilon)$ satisfies

$$Pu = f(t,x,u) \qquad \text{in int } \Omega_\varepsilon$$

for some $f \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R})$. If $\varepsilon > 0$ is small enough, and $u \in I(\Omega'_\varepsilon,S_1)$ in $t < -\varepsilon/2$ it follows that $u \in I(\Omega'_\varepsilon, S_1)$, $\Omega'_\varepsilon = \text{int}\ \Omega_\varepsilon$; if $u \in I_h(\Omega'_\varepsilon,S_1)$ in $t < -\varepsilon/2$ then $u \in I_h(\Omega'_\varepsilon,S_1)$.

To illustrate the proof of this elementary result, and Theorem 1, consider the special case $P = P_E = D_t^2 - \Delta$, the constant coefficient wave operator. One can take ε arbitrary in Theorem 2 if $S_1 = \{t = \mathbf{x}.\omega\}$ for some $\omega \in S^{n-1}$, so suppose

(9) $$f \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}) \ , \ \text{supp}(f) \subset \{t^2 + |x|^2 \leqslant 1\} \times \mathbb{R} \ .$$

Returning to (2) look for u satisfying

$$(10)\qquad \begin{cases} P_E u = \delta f(t,x,u) & t < 2 \\ u = H(t - x.\ \omega) & t < -1 \\ \|u\|^\infty \leqslant 2 & \end{cases}$$

Thus, $u_o = H(t - x.\omega)$, set $u_1 = u - u_o$ with u the putative solution to (10), which becomes

$$(11)\qquad u_1 = \delta E_+(f(t,x,u_o + u_1)), \quad t < 2\ ,\ u_1 \in L^\infty\ ,$$

where E_+ is the forward fundamental solution of P_E. Given a solution $u_1 \in L^\infty$ it follows from (11) that $u_1 \in H^1_c((-\infty,2] \times \mathbb{R}^n)$. For simplicity setting $\omega = (1,0,\ldots,0)$ u_o satisfies the conormal estimates

$$(12)\qquad (t - x_1)^k\ (D_t - D_{x_1})^k (D_t + D_{x_1})^p\ D^{\alpha'}_{x'}\ u_o \in L^\infty \qquad \forall\ p,k,\ \alpha'$$

$x' = (x_2,\ldots,x_n)$. Using the Lie algebra of vector fields $\mathcal{V}$ generated (as a C^∞-module) by $V_1 = (t - x_1)(D_t - D_{x_1})$, $V_2 = D_t + D_{x_1}$, $w_j = D_{x_j}$, $j \geqslant 2$, and the commutation relation

$$(13)\qquad [P,(t - x_1)(D_t - D_{x_1})] = -2i(P + \sum_{j=2}^{n} D^2_{x_j})$$

it follows directly from (11) and (12) that $\forall\ k,p,\alpha'$

$$(14)\qquad V_1^k\ V_2^p\ W^{\alpha'}\ u_1 \in H^1_c\ ((-\infty,2] \times \mathbb{R}^n).$$

Fixing $r > \frac{n+1}{2}$, for simplicity an integer, let

$$H_{(r)} = \{u_1 \in H^1_c((-\infty,2] \times \mathbb{R}^n)\ ;\ u_1 = 0 \quad \text{if } |x| \geqslant 3 + t \text{ or } t \leqslant -2\ ,$$
$$\text{and (14) holds for } k + p + |\alpha'| \leqslant r\}\ ,$$

which is a Hilbert space with the obvious norms based on the estimates (14). Moreover

$$(15)\qquad H_r \hookrightarrow L^\infty((-\infty,2] \times \mathbb{R}^n)\ .$$

The multiplicative properties discussed in § 3 below show that

(16) $$H_{(r)} \ni u_1 \longmapsto E_+(f(.,.,u_o + u_1)) \in H_{(r)}$$

is a bounded map and (11) is a contraction on the unit ball in $H_{(r)}$ for $\bar{\delta}$ small enough.

It is clear that the solution obtained in this way to (10) is conormal since the estimates (14) hold. That it is polyhomogeneous follows from the direct construction of a formal solution ; thus the solution of (10) has an expansion of the form

(17) $$u \sim H(t - x.\omega) + \sum_{j \geqslant 1} c_j(x,\omega)(t - x.\omega)_+^j$$

with $c_j \in C^\infty(\mathbb{R}^n \times S^{n-1})$ well-defined in $x.\omega \leqslant 2$ (always for $|\delta| < \bar{\delta}$) and satisfying non linear transport equations .

$$\omega . D_x c_j = g_j(t,x,c_1,\ldots,c_{j-1}), \ c_j = 0 \quad \text{in } x.\omega < -1 \ .$$

Comparison with the linear case $f(t,x,u) = V(t,x)u$ makes the status of (1) as a non linear potential problem more evident.

§ 3. CONORMAL RINGS

On a C^∞-manifold X let $\mathcal{V} \subset C^\infty(X,TX)$ be a linear space of real vector fields with the properties :

(18) $$C^\infty(X) . \mathcal{V} \subset \mathcal{V}$$

(19) $$[\mathcal{V},\mathcal{V}] \subset \mathcal{V}$$

(20) $\mathcal{V}$ is locally finitely generated as a C^∞-module.

More explicitly (20) requires that each point $x \in X$ have a neighbourhood Ω such that for some finite set $\{V_1,\ldots,V_N\} \subset \mathcal{V}$

$$C_c^\infty(\Omega) . \mathcal{V} = \sum_{j=1}^{N} C_c^\infty(\Omega) . V_j \ .$$

Given such a Lie algebra of vector fields one can define spaces of distributions regular in the directions of $\mathcal{V}$

(21) $$I_{(k)}H^s_{loc}(X,\mathcal{V}) = \{u \in H^s_{loc}(x) \ ; \mathcal{V}^p u \subset H^s_{loc}(x) \quad \forall\, p \leqslant k\}$$

for $s \in \mathbb{R}$. In particular the spaces of $\mathcal{V}$-regular distributions are given by

(22) $$IH^s_{loc}(X,\mathcal{V}) = \bigcap_k I_{(k)} H^s_{loc}(X,\mathcal{V}), \quad I(X,\mathcal{V}) = \bigcup_s I\, H^s_{loc}(X,\mathcal{V}).$$

For nonlinear differential equations the important property of these spaces is their behaviour under pointwise multiplication. Set

(23) $$L^\infty I_{(k)}H^s_{loc}(X,\mathcal{V}) = L^\infty_{loc}(X) \cap I_{(k)}H^s_{loc}(X,\mathcal{V}).$$

<u>Proposition 1</u> : For any $k,s \geqslant 0$ and any $\mathcal{V}$ satisfying (18), (19) and (20) $L^\infty I_{(k)} H^s_{loc}(X,\mathcal{V})$ is a ring and if $f \in C^\infty(X,\mathbb{R}^m)$ then

(24) $$f(x,u_1,\dots,u_m) \in L^\infty I_{(k)} H^s_{loc}(X,\mathcal{V})$$

$$\text{whenever } u_1,\dots,u_m \in L^\infty I_{(k)} H^s_{loc}(X,\mathcal{V}).$$

These spaces are all local and they are all C^∞-modules. For each $x \in X$ set

(25) $$\mathcal{V}_x = \{v \in T_xX \ ; \ \exists\, V \in \mathcal{V} \text{ with } V(x) = \mathcal{V}\}.$$

The Sobolev embedding theorem implies :

<u>Proposition 2</u> : If $\mathcal{V}$ satisfies (18), (19) and (20) then each $\bar{x} \in X$ has a neighbourhood Ω such that provided $k > \dim \mathcal{V}_{\bar{x}}/2$, $s > (\dim X - \dim \mathcal{V}_{\bar{x}})/2$

$$L^\infty I_{(k)}H^s_{loc}(\Omega,\mathcal{V}) = I_{(k)}H^s_{loc}(\Omega,\mathcal{V})$$

i.e. the L^∞ bound is automatic.

The support of $\mathcal{V}$ is the closed set

(26) $$S(\mathcal{V}) = \{x \in X \ ; \ \mathcal{V}_x \neq T_xX\} .$$

The most interesting cases here arise from tangency to C^∞-varieties. Thus, suppose :

(27) $$\mathcal{S} = \{S_1,\dots,S_N\}$$

is a finite union of embedded (but not necessarily closed) submanifolds $S_i \hookrightarrow X$. Set

(28) $$\mathcal{V}(\mathcal{S}) = \{V \in C^\infty(X,TX)\ ;\ V \text{ is tangent to each } S_i \in \mathcal{S}\}.$$

$\mathcal{S}$ is said to be of finite type if $\mathcal{V}(\mathcal{S})$ satisfies (20), (18) and (19) being automatic, and provided in terms of (26)

$$S(\mathcal{V}(\mathcal{S})) = S_1 \cup \dots \cup S_N .$$

For such an $\mathcal{S}$ of finite type set

$$N^*\mathcal{S} = \bigcup_{x \in S} (\mathcal{V}_x)^\circ \subset T^*_S X$$

(the set of common zeroes of all the vector fields in $\mathcal{V}$, as linear functions on the cotangent fibres). Put $M = T^*X \setminus 0$ and set

$$\mathcal{M}(\mathcal{S}) = \{f \in C^\infty(M)\ ;\ f = 0 \text{ on } M \cap N^*\mathcal{S}\}.$$

Then $\mathcal{S}$ is said to be <u>microlocally complete</u> if, locally,

(29) $$\mathcal{M}(\mathcal{S}) = C^\infty(M).\ \sigma_1(\mathcal{V}(\mathcal{S})),$$

a type of irreducibility condition. For $\mathcal{S}$ of finite type let $\mathcal{N}(\mathcal{S}) \subset \Psi^1_p(X)$ be the space of properly supported pseudodifferential operators characteristic on $N^*\mathcal{S}$, that is

$$A \in \mathcal{N}(\mathcal{S}) \text{ iff } \exists\, a' \in S^0(T^*X) \text{ such that } \sigma_1(A) - a' = 0 \text{ on } N^*\mathcal{S} .$$

<u>Proposition 3</u> : If $\mathcal{S}$ is of finite type then $\mathcal{S}$ is microlocally complete if and only if

$$I_{(k)}H^s_{loc}(X,\mathcal{S}) = I_{(k)}H^s_{loc}(X,\mathcal{V}(\mathcal{S}))$$
$$= \{u \in H^s_{loc}(X)\ ;\ \mathcal{N}(\mathcal{S})^p u \subset H^s_{loc}(x)\quad \forall\, p \leqslant k\} .$$

Example : If $H_1,\ldots,H_k \hookrightarrow X$ are closed immersed hypersurfaces meeting and self-intersecting independently, i.e. with all normals independent at each point of intersection or self-intersection, then the variety $\mathfrak{J}$ consisting of all H_k and all intersection and self-intersection submanifolds is a microlocally complete C^∞ -variety of finite type.

§ 4. P-COMPLETENESS

If P is a linear differential operator of order m, with principal symbol p then a submanifold S is characteristic for P if

$$(30) \qquad p = 0 \quad \text{on} \quad N^*S .$$

For C^∞-varieties a more general notion is useful. If $\mathfrak{J}$ is of finite type it will be said to be characteristic for P if

$$(31) \qquad N^*\mathfrak{J} \cap \{p = 0\} = \Lambda_1 \cup \ldots \cup \Lambda_N$$

is a finite union of Lagrangian submanifolds. With $\Sigma(p) = \{p = 0\} \subset M = T^*X \setminus 0$ set

$$\mathfrak{I}_p(\mathfrak{J}) = \{f \in C^\infty(\Sigma(p));\ f = 0 \quad \text{on} \quad \Lambda_j\}$$

where it is assumed that P is of real principal type so that $\Sigma(p)$ is a manifold, and (31) holds. Then $\mathfrak{J}$ is said to be P-complete (characteristically complete in [3]) if it is characteristic for P and

$$(32) \qquad \mathfrak{I}_p(\mathfrak{J}) = C^\infty(\Sigma(p)).[\sigma_1(\mathfrak{V}(\mathfrak{J}))\,|\,\Sigma(p)]$$

locally in $\Sigma(p)$.

Proposition 4 : If P is a linear differential operator of order m and real principal type and $\mathfrak{J}$ is a C^∞ variety of finite type which is P-complete,

$$(33) \qquad [\mathfrak{V}(\mathfrak{J}),P] \subset \Psi_h^o(x).P + \Psi_h^{m-1}(x).\ \mathfrak{V}(\mathfrak{J}) + \Psi_h^{m-1}(x).$$

Examples : Suppose P is a strictly hyperbolic operator of second order then

(34) $\mathcal{S} = \{S\}$, S a characteristic hypersurface is P-complete.

If S_1, S_2 are two characteristic hypersurfaces meeting transversally (independently) then

(35) $\mathcal{S} = \{S_1 \setminus S_2, S_2 \setminus S_1, S_1 \cap S_2\}$ is P-complete .

If S is characteristic for P and $\bar{x} \in S$ let K be the characteristic cone of P with pole $\bar{x}$, then near $\bar{x}$

(36) $\mathcal{S} = \{S \setminus K, K \setminus \{\bar{x}\}, (S \cap K) \setminus \{\bar{x}\}, \{\bar{x}\}\}$ is P-complete.

Some important examples of varieties which are not P-complete include for S_1, S_2 as in (35), $\bar{x} \in S_1 \cap S_2$

(37) $\mathcal{S} = \{S_1 \setminus S_2, S_2 \setminus S_1, S_1 \cap S_2 \setminus \{\bar{x}\}, \{\bar{x}\}\}$ is not P-complete

and similarly

(38) $\mathcal{S} = \{S_1 \setminus (S_2 \cup S_3), S_2 \setminus (S_1 \cup S_3), S_3 \setminus (S_1 \cup S_2), (S_1 \cap S_2) \setminus S_3 ,$
$(S_2 \cap S_3) \setminus S_1, (S_1 \cap S_2) \setminus S_3, S_1 \cap S_2 \cap S_3\}$ is not P-complete

when S_1, S_2 and S_3 are characteristic hypersurfaces meeting independently. In fact (37), (38) are not even characteristic, it is necessary in the case (38) to add the characteristic surface emanating from $S_1 \cap S_2 \cap S_3$, a conic surface S_4. However, even the resulting C^∞ variety with eleven component manifolds :

(39) $\mathcal{S} = \{S_1 \setminus (S_2 \cup S_3 \cup S_4), (S_2 \cap S_3) \setminus S_1, (S_4 \cap S_1) \setminus (S_1 \cap S_2 \cap S_3)$
and cyclic permutations in 1, 2, 3, $S_4 \setminus (S_1 \cup S_2 \cup S_3)$,
$S_1 \cap S_2 \cap S_3\}$

whilst characteristic is not P-complete.

Theorem 3 : Let P be a stricIty t-hyperbolic operator of second order in a neighbourhood Ω of $0 \in \mathbb{R}_t \times \mathbb{R}^n_x$. Suppose $\mathcal{S}$ is a P-complete C^∞-variety of finite type and $0 \in S$. If $u \in L^\infty(\Omega)$ satisfies

(40)
$$Pu = f(t,x,u) \quad \text{in} \quad \Omega \qquad f \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R})$$

$$u \in L^\infty I_{(k)} H^s_{loc}(\Omega, \mathcal{S}) \quad \text{in} \quad t < 0 \ , \quad s \geqslant 0 \ , \quad k \geqslant 0$$

then $u \in L^\infty I_{(k)} H^s_{loc}(\Omega', \mathcal{S})$ for some neighbourhood Ω' of O.

The proof of this general result follows that of Theorem 2, outlined above in a special case, with due allowance for the replacement of (13) by (33).

§ 5. TWO WAVES

Theorem 3 applies to the variety (35) and hence to the circumstances of Theorem 1, when one of the hypersurfaces , S_3, carries no singularities in (5). Thus in that case the solution to (6) is conormal with respect to $\mathcal{S}$ generated by S_1 and S_2. The existence part of Theorem 1 follows by judicious use of the same estimates. Consider more refined questions along the line of Theorem 2. Thus, taking $\mathcal{S}$ in (35), what are the polyhomogeneous elements of $IH^s_{loc}(X, \mathcal{S})$?

(41) If $u \in IH^s_{loc}(X, \mathcal{S})$, $\mathcal{S} = \{S_1 \setminus S_2, S_2 \setminus S_1, S_1 \cap S_2\}$ for S_1, S_2 hypersurfaces meeting transversally then $u \in I_h H^s_{loc}(X, \mathcal{S})$ if for any s' there is a decomposition

$$u = \sum_{\text{finite}} u_j v_k + \sum_{\text{finite}} u'_p v'_q f_{pq}, \ u_j, u'_p \in I_h(X, S_1),$$

$$v_k, v'_q \in I_h(X, S_2), \ f_{pq} \in IH^{s'}_{loc}(X, \mathcal{S})$$

Theorem 4 : Under the hypothesis of Theorem 3, suppose $\mathcal{S}$ is given by two transversal closed characteristic hypersurfaces through 0. If (40) holds for $k = \infty$ and $u \in L^\infty I_h H^s_{loc}(\Omega,)$ in $t < 0$, $u \in I_h H^s_{loc}(\Omega', \mathcal{S})$ for some neighbourhood Ω' of O .

More specifically still consider the generalization of (10) :

$$(42) \qquad \begin{cases} P_E u = \delta f(t,x,u) & t < 2 \quad u \in L^\infty \\ u = aH(t - x.\omega_1) + bH(t - x\omega_2) & t < -1 \end{cases}$$

Then, as before, there is a unique solution for $|\delta| < \bar{\delta}$, $\bar{\delta} > 0$ depending on f, a and b.

<u>Proposition 5</u> : If $\bar{\delta} > 0$ is small enough and $|\delta| < \bar{\delta}$ the solution to (42) is piecewise C^∞ , with all derivatives bounded, in each of the four regions (assuming $\omega_1 \neq \omega_2$)

$$t - x.\, \omega_1 \gtrless 0 \qquad t - x.\, \omega_2 \gtrless 0 .$$

In fact in $t - x.\omega_1 < 0$, $t - x.\ \omega_2 < 0$ u vanishes by finite propagation speed. When the arguments are of opposite sign one of the waves has not arrived so the analysis for one wave can be applied. In the region where both signs are positive one can use the conormal estimates above to show that only the two characteristic variables are important, the others being essentially parameters. This reduces the analysis to that of a wave equation in two variables where the traditional iteration methods leading to the construction of the Riemann function yield the regularity of u.

§ 6. <u>THREE WAVES</u>

As noted above the full characteristic variety, (39), corresponding to three independent characteristic hypersurfaces is not P-complete. The commutator methods used above do not therefore apply directly. One can view the difficulty as being that $I(X, \mathfrak{Z}_{3^q})$ is too large. To obtain a more appropriate space the components of S_{3^q} need to be separated to some degree. This is most readily done through the introduction of polar coordinates around $S_1 \cap S_2 \cap S_3$, i.e. by blowing-up this submanifold. In this way $X = \Omega$ in Theorem 1 is replaced by a manifold with boundary

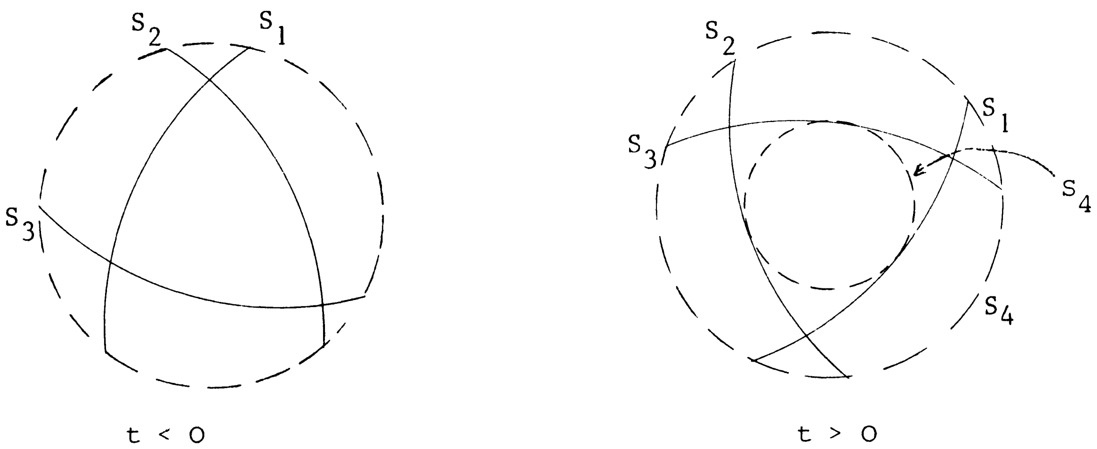
$$X' \to X$$

which is a diffeomorphism of $X' \setminus \partial X'$ onto $X \setminus (S_1 \cap S_2 \cap S_3)$ and covers $S_1 \cap S_2 \cap S_3$ by $\partial X'$, as a 2-sphere bundle. All the hypersurfaces and other components of $\mathcal{J}_{3q}$ lift to form a C^∞-variety $\mathcal{S}'$ in X'. Let $I_{(k)}L^2(X,\mathcal{S}')$ be the conormal spaces on X of elements lifting into $I_{(k)}L^2(X', \mathcal{S}')$, where L^2 is computed with respect to the measure in polar coordinates.

<u>Theorem 5</u> : Under the hypotheses of Theorem 1 the solution, u, to (6) lies in the space $L^\infty \cap L^2(X, \mathcal{S}')$.

Again it is the estimation of this type which enables one to prove the existence result in Theorem 1.

The proof of Theorem 1 proceeds by radial decomposition of the solution. Consider the structure of $\mathcal{S}'$, in each fibre above $S_1 \cap S_2 \cap S_3$:



The region outside S_4 has not been influenced by the most degenerate surface $S_1 \cap S_2 \cap S_3$. The estimates, obtained by commutation on the solution can therefore be applied in weighted L^2-spaces, outside any conic neighbourhood of S_4. Using a radial cut off in such a conic neighbourhood localizes the solution near S_4, excluding intersections of the hypersurfaces S_1, S_2 and S_3. By further radial decomposition this allows the commutation method to be applied to the P-complete variety (36), consisting of the cone S_4 and of one of the hypersurfaces S_1, S_2, S_3 .

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