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METHODES NUMERIQUES POUR LES EQUATIONS DE NAVIER-STOKES INSTATIONNAIRES DES FLUIDES VISQUEUX INCOMPRESSIBLES

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METHODES NUMERIQUES POUR LES EQUATIONS DE NAVIER-STOKES INSTATIONNAIRES DES FLUIDES VISQUEUX INCOMPRESSIBLES

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On décrit dans ce travail une méthode de résolution des équations de Navier-Stokes pour les fluides visqueux incompressibles lorsque l'écoulement est instationnaire. Cette méthode est basée sur une discrétisation par rapport au temps de type directions alternées, ce qui permet de découpler les difficultées numériques, liées à l'incompressibilité, et aux non linéarités, respectivement. On décrit également des algorithmes de résolution des problèmes découplés ainsi obtenus ainsi qu'une méthode d'approximation par éléments finis mixtes. Des résultats numériques illustrent les possibilités des méthodes décrites dans cet article.

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NUMERICAL METHODS FOR THE TIME DEPENDENT NAVIER-STOKES EQUATIONS FOR INCOMPRESSIBLE VISCOUS FLUIDS

R. GLOWINSKI*

INTRODUCTION.

We describe in this paper some new methods for solving the time dependent Navier-Stokes equations for incompressible viscous fluids ; these methods combine finite elements for the space discretization and alternating directions for the time discretization. The key idea is to use the splitting associated to the alternating direction methods to decouple the two main difficulties of the original problem, namely nonlinearity and incompressibility.

The methods which follow are a natural extension of those described in [1],[2], since least square and conjugate gradient methods are still used to treat the nonlinearity ; however due to the decoupling mentioned above the present methods are in fact more efficient since they require less computer time and lead to more accurate numerical results. They provide in particular quite efficient methods for solving the steady Navier-Stokes equations.

The following paper is closely related to GLOWINSKI [3 , Chapter 7] for which we refer for more details and also for a substantial bibliography concerning the Navier-Stokes equations and their numerical treatment.

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1. - FORMULATION OF THE TIME DEPENDENT NAVIER-STOKES EQUATIONS FOR INCOMPRESSIBLE FLUIDS.

Let us consider a newtonian incompressible viscous fluid. If Ω and Γ denote the region of the flow ($\Omega \subset \mathbb{R}^N$, N=2,3 in practice) and its boundary, respectively, then this flow is governed by the following Navier-Stokes equations

(1) $$\frac{\partial \underset{\sim}{u}}{\partial t} - \nu\Delta\underset{\sim}{u} + (\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\underset{\sim}{u} + \underset{\sim}{\nabla}p = \underset{\sim}{f} \text{ in } \Omega,$$

(2) $$\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0 \text{ in } \Omega \text{ (incompressibility condition)}.$$

In (1),(2)

(a) $\underset{\sim}{\nabla} = \{\frac{\partial}{\partial x_i}\}_{i=1}^{N}$, $\Delta = \underset{\sim}{\nabla}^2 = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$,

(b) $\underset{\sim}{u} = \{u_i\}_{i=1}^{N}$ is the flow velocity,

(c) p is the pressure,

(d) ν is the viscosity of the fluid,

(e) $\underset{\sim}{f}$ is a density of external forces.

In (1), $(\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}$ is a symbolic notation for the nonlinear (vector) term

$$\left\{ \sum_{j=1}^{N} u_j \frac{\partial u_i}{\partial x_j} \right\}_{i=1}^{N} .$$

Boundary conditions have to be added ; for example in the case of the airfoil B of Figure 1, we have (since the fluid is viscous) the following adherence condition

(3) $$\underset{\sim}{u} = \underset{\sim}{0} \text{ on } \partial B = \Gamma_B \; ;$$

typical conditions at infinity are

(4) $$\underset{\sim}{u} = \underset{\sim}{u}_\infty$$

where $\underset{\sim}{u}_\infty$ is a constant vector (with regard to the space variables at least).

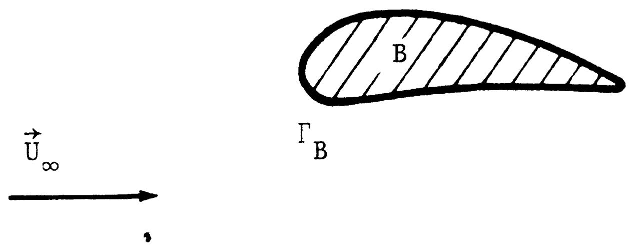


Figure 1

If Ω is a bounded region of $\mathbb{R}^N$ we may prescribe as boundary condition

(5) $$\underset{\sim}{u} = \underset{\sim}{g} \ \underline{\text{on}} \ \Gamma$$

where (from the incompressibility of the fluid) the given function $\underset{\sim}{g}$ has to satisfy

(6) $$\int_\Gamma \underset{\sim}{g} \cdot \underset{\sim}{n} \, d\Gamma = 0,$$

where $\underset{\sim}{n}$ is the outward unit vector normal at Γ.

Finally for the time dependent problem (1),(2) an initial condition such as

(7) $$\underset{\sim}{u}(x,0) = \underset{\sim}{u}_o(x) \ \underline{\text{a.e. on}} \ \Omega,$$

with $\underset{\sim}{u}_o$ given, is usually prescribed.

From the above equations we observe three difficulties (even for flows at low Reynold's numbers in bounded regions Ω) which are

(i) The nonlinear term $(\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}$ in (1),

(ii) The incompressibility condition (2),

(iii) The fact that the solutions of the Navier-Stokes equations are vector-valued functions of x,t, whose components are coupled by the nonlinear term $(\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}$ and by the incompressibility condition $\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0$.

Using convenient alternating direction methods for the time discretization of the Navier-Stokes equations we shall be able to decouple the difficulties due to the nonlinearity and to the incompressibility, respectively.

For simplicity we suppose from now on that Ω is bounded and that we have (5) as boundary condition (with $\underset{\sim}{g}$ satisfying (6) and possibly depending upon t).

2. - TIME DISCRETIZATION BY ALTERNATING DIRECTION METHODS.

Let Δt (> 0) be a time discretization step and θ a parameter such that $0 < \theta < 1$.

2.1. A first alternating direction method.

We consider first the following alternating direction method (of Peaceman-Rachford type) :

(8) $$\underset{\sim}{u}^{o} = \underset{\sim}{u}_{o},$$

then for $n \geq 0$ compute $\{\underset{\sim}{u}^{n+1/2}, p^{n+1/2}\}$ and $\underset{\sim}{u}^{n+1}$, from $\underset{\sim}{u}^{n}$, by solving

(9) $$\begin{cases} \dfrac{\underset{\sim}{u}^{n+1/2}-\underset{\sim}{u}^{n}}{(\Delta t/2)} - \theta\nu\Delta\underset{\sim}{u}^{n+1/2} + \underset{\sim}{\nabla} p^{n+1/2} = \underset{\sim}{f}^{n+1/2} + (1-\theta)\nu\Delta\underset{\sim}{u}^{n} - (\underset{\sim}{u}^{n}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}^{n} \ \underline{\text{in}}\ \Omega, \\ \underset{\sim}{\nabla}\cdot\underset{\sim}{u}^{n+1/2} = 0 \ \underline{\text{in}}\ \Omega, \\ \underset{\sim}{u}^{n+1/2} = \underset{\sim}{g}^{n+1/2} \ \underline{\text{on}}\ \Gamma, \end{cases}$$

and

(10) $$\begin{cases} \dfrac{\underset{\sim}{u}^{n+1}-\underset{\sim}{u}^{n+1/2}}{(\Delta t/2)} - (1-\theta)\nu\Delta\underset{\sim}{u}^{n+1} + (\underset{\sim}{u}^{n+1}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}^{n+1} = \underset{\sim}{f}^{n+1} + \theta\nu\Delta\underset{\sim}{u}^{n+1/2} - \underset{\sim}{\nabla} p^{n+1/2} \ \underline{\text{in}}\ \Omega, \\ \underset{\sim}{u}^{n+1} = \underset{\sim}{g}^{n+1} \ \underline{\text{on}}\ \Gamma, \end{cases}$$

respectively.

We use the notation $f^{j}(x) = f(x, j\Delta t)$, $g^{j}(x) = g(x, j\Delta t)$, and $u^{j}(x)$ is an approximation of $u(x, j\Delta t)$.

2.2. A second alternating direction method.

We consider now the following alternating direction method (of Strang type) :

(11) $$\underset{\sim}{u}^{o} = \underset{\sim}{u}_{o} ,$$

then for $n \geq 0$ and starting from $\underset{\sim}{u}^{n}$ we solve

(12) $$\begin{cases} \dfrac{\underset{\sim}{u}^{n+1/4}-\underset{\sim}{u}^{n}}{(\frac{\Delta t}{4})} -\theta\nu\Delta\underset{\sim}{u}^{n+1/4}+ \underset{\sim}{\nabla} p^{n+1/4}=\underset{\sim}{f}^{n+1/4}+(1-\theta)\nu\Delta\underset{\sim}{u}^{n}-(\underset{\sim}{u}^{n}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}^{n} \ \underline{\text{in}}\ \Omega, \\ \underset{\sim}{\nabla}\cdot\underset{\sim}{u}^{n+1/4} = 0 \ \underline{\text{in}}\ \Omega, \\ \underset{\sim}{u}^{n+1/4} = \underset{\sim}{g}^{n+1/4} \ \underline{\text{on}}\ \Gamma, \end{cases}$$

(13) $$\begin{cases} \dfrac{\underset{\sim}{u}^{n+3/4}-\underset{\sim}{u}^{n+1/4}}{(\Delta t/2)} - (1-\theta)\nu\Delta\underset{\sim}{u}^{n+3/4}+(\underset{\sim}{u}^{n+3/4}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}^{n+3/4} = \\ \underset{\sim}{f}^{n+3/4}+\theta\nu\Delta\underset{\sim}{u}^{n+1/4} - \underset{\sim}{\nabla} p^{n+1/4} \ \underline{\text{in}}\ \Omega, \\ \underset{\sim}{u}^{n+3/4} = \underset{\sim}{g}^{n+3/4} \ \underline{\text{on}}\ \Gamma, \end{cases}$$

(14) $$\begin{cases} \dfrac{\underset{\sim}{u}^{n+1}-\underset{\sim}{u}^{n+3/4}}{(\frac{\Delta t}{4})} - \theta\nu\Delta\underset{\sim}{u}^{n+1} + \underset{\sim}{\nabla} p^{n+1} = \underset{\sim}{f}^{n+1}+(1-\theta)\nu\Delta\underset{\sim}{u}^{n+3/4}-(\underset{\sim}{u}^{n+3/4}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}^{n+3/4} \ \underline{\text{in}}\ \Omega, \\ \underset{\sim}{\nabla}\cdot\underset{\sim}{u}^{n+1} = 0 \ \underline{\text{in}}\ \Omega, \\ \underset{\sim}{u}^{n+1} = \underset{\sim}{g}^{n+1} \ \underline{\text{on}}\ \Gamma. \end{cases}$$

2.3 Some comments and remarks concerning the alternating direction schemes (8)-(10) and (11)-(14).

Using the two alternating direction schemes described in Secs. 2.1, 2.2 we have been able to decouple nonlinearity and incompressibility in the Navier-Stokes equations (1),(2). We shall describe in the following sections the specific treatment of the subproblems encountered at each step of (8)-(10) and (11)-(14) ; we shall consider first the case where the subproblems are still continuous in space (since the formalism of the continuous problems is much simpler), and then the discrete case where a finite element method is used to approximate in space the Navier-Stokes equations.

Scheme (8)-(10) has a truncation error in $O(\Delta t)$; due to the symmetrization process involved in it, scheme (11)-(14) has a truncation error in $O(|\Delta t|^2)$.

We observe that $\underset{\sim}{u}^{n+1/2}$ and $\underset{\sim}{u}^{n+1/4}$, $\underset{\sim}{u}^{n+1}$ are obtained from the solution of linear problems ((9) and (12),(14), respectively) very close to the steady Stokes problem. Despite of its greater complexity scheme (11)-(14) is almost as economical to use as scheme (8)-(10) ; this is mainly due to the fact that the "quasi" steady Stokes problems (9) and (12),(14) (in fact convenient finite element approximations of them) can be solved by quite efficient solvers resulting in that most of the computer time used to solve a full alternating direction step ((9),(10) or (12)-(14)) is in fact used to solve the nonlinear subproblem ((10) or (13)).

The good choice for θ is $\theta = 1/2$ (resp. $\theta = 1/3$) if one uses scheme (8)-(10) (resp. (11)-(14)) ; this follows from the fact that with the above choices for θ, many computer subprograms can be used for both the linear and nonlinear subproblems, resulting therefore in quite substantial computer core memory savings.

Remark 2.1 : A variant of scheme (8)-(10) is the following (it corresponds to $\theta=1$) :

(15) $$\underset{\sim}{u}^{o} = \underset{\sim}{u}_{o},$$

then for $n \geq 0$ and starting from $\underset{\sim}{u}^n$,

(16) $$\begin{cases} \dfrac{\underset{\sim}{u}^{n+1/2}-\underset{\sim}{u}^{n}}{(\Delta t/2)} - \nu\Delta\underset{\sim}{u}^{n+1} + \underset{\sim}{\nabla} p^{n+1/2} = \underset{\sim}{f}^{n+1} - (\underset{\sim}{u}^{n}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}^{n} \ \underline{\text{in}}\ \Omega, \\ \underset{\sim}{\nabla}\cdot\underset{\sim}{u}^{n+1/2} = 0 \ \underline{\text{in}}\ \Omega, \\ \underset{\sim}{u}^{n+1/2} = \underset{\sim}{g}^{n+1/2} \ \underline{\text{on}}\ \Gamma, \end{cases}$$

$$(17)\quad \begin{cases} \dfrac{\underset{\sim}{u}^{n+1}-\underset{\sim}{u}^{n+1/2}}{(\Delta t/2)} + (\underset{\sim}{u}^{n+1/2}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}^{n+1} = \underset{\sim}{f}^{n+1} + \nu\Delta\underset{\sim}{u}^{n+1/2} \ \underline{\text{in}}\ \Omega, \\ \underset{\sim}{u}^{n+1} = \underset{\sim}{g}^{n+1} \ \underline{\text{on}}\ \Gamma_{-}^{n+1}, \end{cases}$$

<u>where</u>

$$\Gamma_{-}^{n+1} = \{x \mid x \in \Gamma,\ \underset{\sim}{g}^{n+1}(x)\cdot\underset{\sim}{n}(x) < 0\} .$$

Both subproblems (16) and (17) are <u>linear</u> ; the first one is also a "quasi" steady Stokes problem and the second which is a <u>first order system</u> can be solved by a <u>method of characteristics</u>.

A similar remark holds for scheme (11)-(14).

Such methods have been used by several authors, the space discretization being done by finite element methods very close to those described in Sec. 5 of this paper (see [4],[5] for a discussion of those characteristics - finite element methods for solving Navier-Stokes equations). In our opinion these characteristics-finite element methods are still too dissipative and will not be discussed here any longer (we are presently working at such schemes with very small dissipation).

3. - <u>LEAST SQUARE-CONJUGATE GRADIENT SOLUTION OF THE NONLINEAR SUBPROBLEMS</u>.

3.1. <u>Classical and Variational Formulations</u>. <u>Synopsis</u>.

At each full step of the alternating direction methods (8)-(10) and (11)-(14) we have to solve a <u>nonlinear elliptic system</u> of the following type

$$(18)\quad \begin{cases} \alpha\underset{\sim}{u} - \nu\Delta\underset{\sim}{u} + (\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\underset{\sim}{u} = \underset{\sim}{f} \ \underline{\text{in}}\ \Omega, \\ \underset{\sim}{u} = \underset{\sim}{g} \ \underline{\text{on}}\ \Gamma, \end{cases}$$

where α and ν are two positive parameters and where $\underset{\sim}{f}$ and $\underset{\sim}{g}$ are two <u>given</u> functions defined on Ω and Γ, respectively.

We shall not discuss here the existence and uniqueness of solutions for problem (18).

We introduce now the following functional space of <u>Sobolev</u>'s type (see, e.g., ADAMS [6], NECAS [7], ODEN-REDDY [8] for information on Sobolev spaces) :

(19) $\quad H^1(\Omega) = \{\phi \mid \phi \in L^2(\Omega),\ \frac{\partial\phi}{\partial x_i} \in L^2(\Omega) \quad \forall i=1,\ldots N\}$,

(20) $\quad H^1_o(\Omega) = \{\phi \mid \phi \in H^1(\Omega),\ \phi = 0 \ \underline{\text{on}}\ \Gamma\}$,

(21) $\quad V_o = (H^1_o(\Omega))^N$,

(22) $\quad V_g = \{\underset{\sim}{v} \mid \underset{\sim}{v} \in (H^1(\Omega))^N,\ \underset{\sim}{v} = \underset{\sim}{g}\ \underline{\text{on}}\ \Gamma\}$;

if $\underset{\sim}{g}$ is sufficiently smooth then V_g is <u>nonempty</u>.

We shall use in the sequel the following notation

$$dx = dx_1 \ldots dx_N,$$

and if $\underset{\sim}{u} = \{\underset{\sim}{u}_i\}_{i=1}^N$, $\underset{\sim}{v} = \{\underset{\sim}{v}_i\}_{i=1}^N$

$$\begin{cases} \underset{\sim}{u} \cdot \underset{\sim}{v} = \sum_{i=1}^{N} u_i v_i, \\ \nabla\underset{\sim}{u} \cdot \nabla\underset{\sim}{v} = \sum_{i=1}^{N} \nabla\underset{\sim}{u}_i \cdot \nabla\underset{\sim}{v}_i = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} . \end{cases}$$

Using Green's formula we can prove that for sufficiently smooth functions $\underset{\sim}{u}$ and $\underset{\sim}{v}$, belonging to $(H^1(\Omega))^N$ and V_o, respectively, we have

(23) $$-\int_\Omega \Delta\underset{\sim}{u} \cdot \underset{\sim}{v}\, dx = \int_\Omega \nabla\underset{\sim}{u} \cdot \nabla\underset{\sim}{v}\, dx .$$

It can also be proved that if $\underset{\sim}{u} \in V_g$ is a solution of (18) it is also a solution of the <u>nonlinear variational problem</u>

(24) $$\begin{cases} \underline{\text{Find}}\ \underset{\sim}{u} \in V_g\ \underline{\text{such that}} \\ \alpha\int_\Omega \underset{\sim}{u} \cdot \underset{\sim}{v}\, dx + \nu\int_\Omega \nabla\underset{\sim}{u} \cdot \nabla\underset{\sim}{v}\, dx + \int_\Omega ((\underset{\sim}{u} \cdot \nabla)\underset{\sim}{u}) \cdot \underset{\sim}{v}\, dx = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx \quad \forall \underset{\sim}{v} \in V_o, \end{cases}$$

and <u>conversely</u>.

We observe that (18),(24) <u>is not equivalent</u> to a problem of the <u>Calculus of Variations</u> since there is no functional of $\underset{\sim}{v}$ with $(\underset{\sim}{v} \cdot \nabla)\underset{\sim}{v}$ as differential ; however using a convenient <u>least square formulation</u> we shall be able to solve (18),(24) by efficient methods from <u>Nonlinear Programing</u>, like conjugate gradient, for example.

The finite element approximation of problem (18),(24) will be discussed in Sec. 5.

3.2. Least square formulation of (18),(24).

Let $\underset{\sim}{v} \in V_g$; from $\underset{\sim}{v}$ we define $\underset{\sim}{y}$ (= $\underset{\sim}{y}(\underset{\sim}{v})$) $\in V_o$ as the solution of

$$(25)\quad \begin{cases} \alpha \underset{\sim}{y} - \nu\Delta \underset{\sim}{y} = \alpha \underset{\sim}{v} - \nu\Delta \underset{\sim}{v} + (\underset{\sim}{v}\cdot\underset{\sim}{\nabla})\underset{\sim}{v} - \underset{\sim}{f} \text{ in } \Omega, \\ \underset{\sim}{y} = 0 \text{ on } \Gamma. \end{cases}$$

We observe that $\underset{\sim}{y}$ is obtained from $\underset{\sim}{v}$ via the solution of N uncoupled linear Poisson problems (one for each component of $\underset{\sim}{y}$) ; using (23) it can be shown that problem (25) is actually equivalent to the linear variational problem

$$(26)\quad \begin{cases} \text{Find } \underset{\sim}{y} \in V_o \text{ such that } \forall \underset{\sim}{z} \in V_o \text{ we have} \\ \alpha\int_\Omega \underset{\sim}{y}\cdot\underset{\sim}{z}\, dx + \nu\int_\Omega \underset{\sim}{\nabla}\underset{\sim}{y}\cdot\underset{\sim}{\nabla}\underset{\sim}{z}\, dx = \alpha\int_\Omega \underset{\sim}{v}\cdot\underset{\sim}{z}\, dx + \nu\int_\Omega \underset{\sim}{\nabla}\underset{\sim}{v}\cdot\underset{\sim}{\nabla}\underset{\sim}{z}\, dx + \\ + \int_\Omega ((\underset{\sim}{v}\cdot\underset{\sim}{\nabla})\cdot\underset{\sim}{v})\cdot\underset{\sim}{z}\, dx - \int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{z}\, dx, \end{cases}$$

which has a unique solution.

Suppose now that $\underset{\sim}{v}$ is a solution of the nonlinear problem (18),(24) ; the corresponding $\underset{\sim}{y}$ (obtained through the solution of (25),(26)) is clearly $\underset{\sim}{y} = \underset{\sim}{0}$. From these observations it is quite natural to introduce the following (nonlinear) least square formulation of problem (18),(24) :

$$(27)\quad \begin{cases} \text{Find } \underset{\sim}{u} \in V_g \text{ such that} \\ J(\underset{\sim}{u}) \le J(\underset{\sim}{v}) \quad \forall \underset{\sim}{v} \in V_g , \end{cases}$$

where $J : (H^1(\Omega))^N \to \mathbb{R}$ is that function of $\underset{\sim}{v}$ defined by

$$(28)\quad J(\underset{\sim}{v}) = \frac{1}{2}\int_\Omega \{\alpha|\underset{\sim}{y}|^2 + \nu|\underset{\sim}{\nabla}\underset{\sim}{y}|^2\}\, dx,$$

with $\underset{\sim}{y}$ defined from $\underset{\sim}{v}$ by solving the linear problem (25),(26).

We observe that if $\underset{\sim}{u}$ is solution of (18),(24) it is also a solution of (27) such that $J(\underset{\sim}{u}) = 0$; conversely if $\underset{\sim}{u}$ is a solution of (27) such that $J(\underset{\sim}{u}) = 0$ it is also a solution of (18),(24).

3.3. Conjugate gradient solution of the least square problem (27).

3.3.1. Description of the algorithm.

We use the Polak-Ribière version (see POLAK [9]) of the conjugate gradient method to solve the minimization problem (27) ; we have then (with $J'(\underset{\sim}{v})$ the differential of J at $\underset{\sim}{v}$)

Step 0 : Initialization

(29) $\underset{\sim}{u}^o \in V_g$, given,

we define then $\underset{\sim}{g}^o, \underset{\sim}{w}^o \in V_o$ by

(30) $$\begin{cases} \alpha\int_\Omega \underset{\sim}{g}^o \cdot \underset{\sim}{z}\, dx + \nu\int_\Omega \nabla\underset{\sim}{g}^o \cdot \nabla\underset{\sim}{z}\, dx = \langle J'(\underset{\sim}{u}^o), \underset{\sim}{z}\rangle \quad \forall \underset{\sim}{z} \in V_o, \\ \underset{\sim}{g}^o \in V_o, \end{cases}$$

(31) $\underset{\sim}{w}^o = \underset{\sim}{g}^o$,

respectively. ■

Then for $n \geq 0$, assuming that $\underset{\sim}{u}^n, \underset{\sim}{g}^n, \underset{\sim}{w}^n$ are known we obtain $\underset{\sim}{u}^{n+1}, \underset{\sim}{g}^{n+1}, \underset{\sim}{u}^{n+1}$ by

Step 1 : Descent

(32) $$\begin{cases} \text{Find } \lambda^n \in \mathbb{R} \text{ such that} \\ J(\underset{\sim}{u}^n - \lambda^n \underset{\sim}{w}^n) \leq J(\underset{\sim}{u}^n - \lambda \underset{\sim}{w}^n) \quad \forall \lambda \in \mathbb{R}, \end{cases}$$

(33) $\underset{\sim}{u}^{n+1} = \underset{\sim}{u}^n - \lambda^n \underset{\sim}{w}^n$. ■

Step 2 : Calculation of the new descent direction.

(34) $$\begin{cases} \underline{\text{Find}}\ \underset{\sim}{g}^{n+1} \in V_o\ \underline{\text{such that}} \\ \alpha\int_\Omega \underset{\sim}{g}^{n+1}\cdot\underset{\sim}{z}\,dx + \nu\int_\Omega \nabla\underset{\sim}{g}^{n+1}\cdot\nabla\underset{\sim}{z}\,dx = \langle J'(\underset{\sim}{u}^{n+1}),\underset{\sim}{z}\rangle \quad \forall \underset{\sim}{z}\in V_o, \end{cases}$$

(35) $$\gamma_n = \frac{\alpha\int_\Omega \underset{\sim}{g}^{n+1}\cdot(\underset{\sim}{g}^{n+1}-\underset{\sim}{g}^{n})dx + \nu\int_\Omega \nabla\underset{\sim}{g}^{n+1}\cdot\nabla(\underset{\sim}{g}^{n+1}-\underset{\sim}{g}^{n})dx}{\alpha\int_\Omega |\underset{\sim}{g}^n|^2\,dx + \nu\int_\Omega |\nabla\underset{\sim}{g}^n|^2\,dx},$$

(36) $$\underset{\sim}{w}^{n+1} = \underset{\sim}{g}^{n+1} + \gamma_n\underset{\sim}{w}^n,$$

$n=n+1$, go to (32).

As we shall see in Secs. 3.3.2, 3.3.3, applying algorithm (29)-(36) to solve the least square problem (27) requires the solution at each iteration of several Dirichlet problems associated to the elliptic operator $\alpha I-\nu\Delta$.

3.3.2. Calculation of J'.

A most important step, when making use of algorithm (29)-(36) to solve the least square problem (27), is the calculation of $\langle J'(\underset{\sim}{u}^{n+1}),\underset{\sim}{z}\rangle$ at each iteration ; owing to the importance of this calculation we shall give it in detail.

Let $\underset{\sim}{v}\in V_g$ and let $\delta\underset{\sim}{v}$ be a perturbation of $\underset{\sim}{v}$ such that $\delta\underset{\sim}{v}\in V_o$ (i.e. $\delta\underset{\sim}{v}=\underset{\sim}{0}$ on Γ) ; we have for the corresponding variation of $J(\underset{\sim}{v})$

(37) $$\delta J(\underset{\sim}{v}) = \langle J'(\underset{\sim}{v}),\delta\underset{\sim}{v}\rangle .$$

Using (26),(28) we also have that

(38) $$J(\underset{\sim}{v}) = \int_\Omega \{\alpha\underset{\sim}{y}\cdot\delta\underset{\sim}{y} + \nu\nabla\underset{\sim}{y}\cdot\nabla\delta\underset{\sim}{y}\}\,dx ,$$

where $\delta\underset{\sim}{y}$ is the solution of the linear problem

$$(39)\quad \begin{cases} \delta y \in V_o, \\ \alpha\int_\Omega \delta y\cdot z\,dx + \nu\int_\Omega \nabla\delta y\cdot\nabla z\,dx = \alpha\int_\Omega \delta v\cdot z\,dx + \nu\int_\Omega \nabla\delta v\cdot\nabla z\,dx + \\ + \int_\Omega ((\delta v\cdot\nabla)v)\cdot z\,dx + \int_\Omega ((v\cdot\nabla)\delta v)\cdot z\,dx \quad \forall z \in V_o. \end{cases}$$

Taking $z=y$ in (39) we obtain from (37),(38) that

$$\begin{cases} \langle J'(v),\delta v\rangle = \alpha\int_\Omega y\cdot\delta v\,dx + \nu\int_\Omega \nabla y\cdot\nabla\delta v\,dx + \int_\Omega ((\delta v\cdot\nabla)v)\cdot y\,dx + \\ + \int_\Omega ((v\cdot\nabla)\cdot\delta v)\cdot y\,dx. \end{cases}$$

Thus $J'(v)$ can be identified with the linear functional from V_o to $\mathbb{R}$ defined by

$$(40)\quad \begin{cases} \langle J'(v),z\rangle = \alpha\int_\Omega y\cdot z\,dx + \nu\int_\Omega \nabla y\cdot\nabla z\,dx + \int_\Omega y\cdot(z\cdot\nabla)v\,dx + \\ + \int_\Omega y\cdot(v\cdot\nabla)z\,dx \quad \forall z \in V_o\ ; \end{cases}$$

it has therefore a purely integral representation, which is of major importance in view of finite element implementations of algorithm (29)-(36).

From the above results, to obtain $\langle J'(u^{n+1}),z\rangle$ we proceed as follows :

(i) We compute y^{n+1} from u^{n+1} through the solution of (25) with $v = u^{n+1}$, i.e. we solve the Dirichlet system

$$(41)\quad \begin{cases} \alpha y^{n+1}-\nu\Delta y^{n+1} = \alpha u^{n+1}-\nu\Delta u^{n+1}+(u^{n+1}\cdot\nabla)u^{n+1}-f \ \text{in}\ \Omega, \\ y^{n+1} = 0 \ \text{on}\ \Gamma. \end{cases}$$

(ii) We finally obtain $\langle J'(u^{n+1}),z\rangle$ by taking in (40) $v = u^{n+1}$ and $y = y^{n+1}$.

3.3.3. Further comments on algorithm (29)-(36).

Each step of algorithm (29)-(36) requires the solution of several Dirichlet systems for the operator $\alpha I-\nu\Delta$; more precisely we have to solve the following such systems :

(i) System (41) to obtain y^{n+1} from u^{n+1} ,

(ii) System (34) to obtain $\tilde{g}^{n+1}$ from $\tilde{u}^{n+1}, y^{n+1}$,

(iii) Two systems to obtain the coefficients of the quartic polynomial

$$\lambda \to J(u^n - \lambda w^n).$$

Thus we have to solve 4 Dirichlet systems for $\alpha I - \nu\Delta$ at each iteration (or equivalently 4N scalar Dirichlet problems for $\nu I - \nu\Delta$ at each iteration).

From the above observations it appears clearly that the practical implementation of algorithm (29)-(36) will require an efficient (direct or iterative) elliptic solver.

The solution of the one-dimensional problem (32) can be done very efficiently since it is equivalent to finding the roots of a single variable cubic polynomial whose coefficients are known.

As a last comment we would like to mention that algorithm (29)-(36) (in fact its finite element variants) is quite efficient ; when used in combination with the alternating direction methods of Sec. 2 to solve the test problems of Sec. 6, three iterations suffice to reduce the value of the cost function J by a factor of 10^4 to 10^6.

4. - SOLUTION OF THE "QUASI" STOKES LINEAR SUBPROBLEMS.

4.1. Formulation. Synopsis.

At each full step of the alternating direction methods (8)-(10) and (11)-(14) we have to solve a linear problem of the following type

$$(42) \quad \begin{cases} \alpha u - \nu\Delta u + \nabla p = f \ \text{in} \ \Omega, \\ \nabla \cdot u = 0 \ \text{in} \ \Omega, \\ u = g \ \text{on} \ \Gamma \ \left(\text{with} \int_\Gamma g \cdot n \, d\Gamma = 0\right), \end{cases}$$

where α and ν are two positive parameters and where f and g are two given functions defined on Ω and Γ, respectively.

We recall that if f and g are sufficiently smooth, then problem (42) has a unique solution in $V_g \times (L^2(\Omega)/\mathbb{R})$ (with V_g still defined by (22) ; $p \in L^2(\Omega)/\mathbb{R}$ means that p is defined only to within an arbitrary constant).

We shall discuss in Secs. 4.2, 4.3 several iterative methods for solving (42), quite easy to implement using finite element methods (other methods are discussed in [3, Chap. 7]).

4.2. Gradient and conjugate gradient methods for solving (42).

This method which is quite classical is defined as follows :

(43) $p^o \in L^2(\Omega)$, given ,

then for $n \geq 0$, define $\underset{\sim}{u}^n$ and p^{n+1} from p^n by

(44) $$\begin{cases} \alpha \underset{\sim}{u}^n - \nu\Delta \underset{\sim}{u}^n = \underset{\sim}{f} - \underset{\sim}{\nabla} p^n \text{ in } \Omega, \\ \underset{\sim}{u}^n = \underset{\sim}{g} \text{ on } \Gamma, \end{cases}$$

(45) $$p^{n+1} = p^n - \rho \underset{\sim}{\nabla} \cdot \underset{\sim}{u}^n .$$

Concerning the convergence of algorithm (43)-(45) we have the following

Proposition 4.1 : Suppose that

(46) $$0 < \rho < 2\frac{\nu}{N} ;$$

we have then

(47) $$\lim_{n\to+\infty} \{\underset{\sim}{u}^n, p^n\} = \{\underset{\sim}{u}, p_o\} \text{ strongly in } (H^1(\Omega))^N \times L^2(\Omega),$$

where $\{\underset{\sim}{u}, p_o\}$ is that solution of (42) such that

(48) $$\int_\Omega p_o \, dx = \int_\Omega p^o \, dx.$$

Proof : We shall only prove the convergence of $\{\underset{\sim}{u}^n\}$; for a proof of the convergence of $\{p^n\}_{n\geq 0}$ (which is more complicated) see, e.g., [3, Chap. 7]. Let $\{\underset{\sim}{u}, p\}$ be a solution of (42) ; we clearly have

(49) $$p = p - \rho \underset{\sim}{\nabla} \cdot \underset{\sim}{u} \quad \forall \rho > 0.$$

We define then $\bar{\underset{\sim}{u}}^n, \bar{p}^n$ by $\bar{\underset{\sim}{u}}^n = \underset{\sim}{u}^n - \underset{\sim}{u}$, $\bar{p}^n = p^n - p$, respectively ; by substraction between (44) and (42) (resp. (49) and (45)) we obtain

$$(50)\qquad \begin{cases} \alpha \bar{\underset{\sim}{u}}^n - \nu\Delta\bar{\underset{\sim}{u}}^n = -\underset{\sim}{\nabla}\bar{p}^n \ \underline{\text{in}}\ \Omega, \\ \bar{\underset{\sim}{u}}^n = 0 \ \underline{\text{on}}\ \Gamma \end{cases}$$

(resp.

$$(51)\qquad \bar{p}^{n+1} = \bar{p}^n - \rho\underset{\sim}{\nabla}\cdot\bar{\underset{\sim}{u}}^n).$$

Using the notation $|q|_o = ||q||_{L^2(\Omega)}$, we obtain from (51) that

$$(52)\qquad |\bar{p}^n|_o^2 - |\bar{p}^{n+1}|_o^2 = 2\rho\int_\Omega \bar{p}^n\underset{\sim}{\nabla}\cdot\bar{\underset{\sim}{u}}^n - \rho^2|\underset{\sim}{\nabla}\cdot\bar{\underset{\sim}{u}}^n|_o^2 \ ;$$

on the other hand multiplying (50) by $\bar{\underset{\sim}{u}}^n$ and <u>integrating by parts</u> we obtain

$$(53)\qquad \alpha\int_\Omega |\bar{\underset{\sim}{u}}^n|^2\,dx + \nu\int_\Omega |\underset{\sim}{\nabla}\bar{\underset{\sim}{u}}^n|^2\,dx = \int_\Omega \bar{p}^n\underset{\sim}{\nabla}\cdot\bar{\underset{\sim}{u}}^n\,dx.$$

Combining (52) and (53) we finally have

$$(54)\qquad |\bar{p}^n|_o^2 - |\bar{p}^{n+1}|_o^2 = 2\rho\int_\Omega\{\alpha|\bar{\underset{\sim}{u}}^n|^2 + \nu|\underset{\sim}{\nabla}\bar{\underset{\sim}{u}}^n|^2\}dx - \rho^2|\underset{\sim}{\nabla}\cdot\bar{\underset{\sim}{u}}^n|_o^2 .$$

Let $\underset{\sim}{v} \in (H^1(\Omega))^N$; we have

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \sum_{i=1}^N \frac{\partial v_i}{\partial x_i},$$

which implies

$$(55)\qquad |\underset{\sim}{\nabla}\cdot\underset{\sim}{v}|^2 = \sum_{i=1}^N \left(\frac{\partial v_i}{\partial x_i}\right)^2 + 2\sum_{\substack{1\le i,j\le N\\ i\ne j}} \frac{\partial v_i}{\partial x_i}\,\frac{\partial v_j}{\partial x_j} .$$

We also have

$$(56)\qquad 2\,\frac{\partial v_i}{\partial x_i}\,\frac{\partial v_j}{\partial x_j} \le \left(\frac{\partial v_i}{\partial x_i}\right)^2 + \left(\frac{\partial v_j}{\partial x_j}\right)^2 .$$

Combining (55),(56) we clearly obtain

(57) $$|\nabla\cdot v|^2 \leq N \sum_{i=1}^{N} \left(\frac{\partial v_i}{\partial x_i}\right)^2 \leq N\,|\nabla v|^2 \leq \frac{N}{\nu}\{\alpha|v|^2 + \nu|\nabla v|^2\} .$$

It follows then from (54),(57) that

(58) $$|\bar{p}^n|_o^2 - |\bar{p}^{n+1}|_o^2 \geq \rho(2-\rho\frac{N}{\nu}) \int_\Omega \{\alpha|\bar{u}^n|^2 + \nu|\nabla\bar{u}^n|^2\}\,dx.$$

Suppose that (46) holds (i.e. $0<\rho<2\frac{\nu}{N}$) ; it follows then from (58) that the sequence $\{|\bar{p}^n|_o^2\}_{n\geq 0}$ is decreasing ; since it is bounded from below by 0 it converges, implying

(59) $$\lim_{n\to+\infty} (|\bar{p}^n|_o^2 - |\bar{p}^{n+1}|_o^2) = 0.$$

Since (46) implies $\rho(2-\rho\frac{N}{\nu})>0$ we have from (58),(59) that

$$\lim_{n\to+\infty} \int_\Omega \{\alpha|\bar{u}^n|^2 + \nu|\nabla\bar{u}|^2\}\,dx = 0.$$

Since $\bar{u}^n = u^n - u$, we have thus proved that

$$\lim_{n\to+\infty} u^n = u \text{ strongly in } (H^1(\Omega))^N.$$

Remark 4.1 : It can be proved (see [3, Chap. 7]) that $\{\{u^n,p^n\}\}_{n\geq 0}$ converges to $\{u,p_o\}$ linearly (i.e. the sequences $\{\|u^n-u\|_{(H^1(\Omega))^N}\}_{n\geq 0}$ and $\{\|p^n-p_o\|_{L^2(\Omega)}\}_{n\geq 0}$ converges to zero as fast, at least, as a geometric sequence).

Remark 4.2 : When using algorithm (43)-(45) to solve the "quasi" Stokes problem (42), we have to solve at each iteration N uncoupled scalar Dirichlet problems for $\alpha I-\nu\Delta$, to obtain u^n from p^n. We see again (as in Sec. 3.3.3) the importance to have efficient Dirichlet solvers for $\alpha I-\nu\Delta$.

Remark 4.3 : Algorithm (43)-(45) is related to the so-called method of artificial compressibility of Chorin-Yanenko ; indeed we can view (45), (49) as obtained by a time discretization process from the equation

$$\frac{\partial p}{\partial t} + \nabla\cdot u = 0$$

(ρ being the size of the time discretization step).

Remark 4.4 : In practice we should use instead of algorithm (43)-(45) the following conjugate gradient variant of it, whose convergence is much faster in most cases, and which is, in addition, no more costly to implement :

Description of the conjugate gradient algorithm :

Step 0 : Initialization

(60) $p^o \in L^2(\Omega)$, given arbitrarily,

solve then

(61) $$\begin{cases} \alpha \underset{\sim}{u}^o - \nu \Delta \underset{\sim}{u}^o = \underset{\sim}{f} - \underset{\sim}{\nabla} p^o \text{ in } \Omega, \\ \underset{\sim}{u}^o = \underset{\sim}{g} \text{ on } \Gamma \end{cases}$$

and set

(62) $g^o = \underset{\sim}{\nabla} \cdot \underset{\sim}{u}^o,$

(63) $w^o = g^o.$

Then for $n \geq 0$, we obtain p^{n+1}, g^{n+1}, w^{n+1} from p^n, g^n, w^n by

Step 1 : Descent

Compute first $\underset{\sim}{\chi}^n \in (H_o^1(\Omega))^N$ as the solution of

(64) $$\begin{cases} \alpha \underset{\sim}{\chi}^n - \nu \Delta \underset{\sim}{\chi}^n = - \underset{\sim}{\nabla} w^n \text{ in } \Omega, \\ \underset{\sim}{\chi}^n = \underset{\sim}{0} \text{ on } \Gamma, \end{cases}$$

then

(65) $$\rho_n = \frac{\int_\Omega w^n g^n \, dx}{\int_\Omega \underset{\sim}{\nabla} \cdot \underset{\sim}{\chi}^n w_n \, dx} = \frac{\int_\Omega |g^n|^2 \, dx}{\int_\Omega \underset{\sim}{\nabla} \cdot \underset{\sim}{\chi}^n w^n \, dx},$$

and finally

(66) $p^{n+1} = p^n - \rho_n w^n.$

Step 2 : Calculation of the new direction of descent

(67) $$g^{n+1} = g^n - \rho_n \nabla\cdot\chi^n ,$$

(68) $$\gamma_n = \frac{\|g^{n+1}\|^2_{L^2(\Omega)}}{\|g^n\|^2_{L^2(\Omega)}} ,$$

(69) $$w^{n+1} = g^{n+1} + \gamma_n w^n .$$

Do then n=n+1 and go to (64).

Once the convergence of (60)-(69) to p_o (that pressure solution such that $\int_\Omega p_o\,dx = \int_\Omega p^o\,dx$) has been obtained, we compute u from p_o by the solution of the Dirichlet system

$$\begin{cases} \alpha u - \nu\Delta u = f - \nabla p_o \text{ in } \Omega, \\ u = g \text{ on } \Gamma. \end{cases}$$

4.3. Another iterative method for solving (42).

This second method is in fact a generalization of algorithm (43)-(45), defined as follows (with r a positive parameter) :

(70) $$p^o \in L^2(\Omega) \text{ given,}$$

then for $n \geq 0$ define u^n and p^{n+1} from p^n by

(71) $$\begin{cases} \alpha u^n - \nu\Delta u^n - r\nabla(\nabla\cdot u^n) = f - \nabla p^n \text{ in } \Omega, \\ u^n = g \text{ on } \Gamma, \end{cases}$$

(72) $$p^{n+1} = p^n - \rho\nabla\cdot u^n .$$

Concerning the convergence of algorithm (70)-(72) we have the following

Proposition 4.2 : Suppose that

(73) $$0 < \rho < 2(r + \frac{\nu}{N}) ;$$

then the convergence result (47) still holds for $\{u^n, p^n\}$.

The proof of Proposition 4.2 is quite similar to that of Proposition 4.1 ; moreover the convergence of $\{\{\underset{\sim}{u}^n, p^n\}\}_{n\geq 0}$ is also linear (as shown in [3, Chap. 7]).

Remark 4.5 (About the choice of ρ and r) : In practice we should use ρ=r, since it can be proved that in that case the convergence ratio of algorithm (70)-(72) is $O(\frac{1}{r})$, for large values of r. In many applications, taking $r=10^4\nu$ we have a practical convergence of algorithm (70)-(72) in 3 to 4 iterations. There is however a practical upper bound for r ; this follows from the fact that for too large values of r, problem (71) will be ill-conditioned and its practical solution sensitive to round off errors.

Remark 4.6 : Problem (71) is more complicated to solve in practice than problem (44), since the components of $\underset{\sim}{u}^n$ are coupled by the linear term $\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}^n)$. Actually the partial differential elliptic operator in the left hand side of (71) is very close to the linear elasticity operator, and close variants of it occur naturally in compressible and/or turbulent viscous flow problems.

Remark 4.7 : Other techniques for solving the "quasi" Stokes problem (42) are discussed in references [1],[2],[3].

5. - FINITE ELEMENT APPROXIMATION OF THE TIME DEPENDENT NAVIER-STOKES EQUATIONS.

5.1. Generalities. Synopsis.

We shall describe in this section a specific finite element approximation fo the time dependent Navier-Stokes equations. Actually this method which leads to continuous approximations for both pressure and velocity is fairly simple and has been known for years; it has been advocated for example by HOOD-TAYLOR [10], among other people. Other finite element approximations of the incompressible Navier-Stokes equations can be found in [1],[2],[3], and also in GIRAULT-RAVIART [11] and TEMAM [12] (see also the references therein).

5.2. Basic hypotheses. Fundamental discrete spaces.

We suppose that Ω is a bounded polygonal domain of $\mathbb{R}^2$. With $\mathcal{T}_h$ a standard finite element triangulation of Ω, and h the maximal length of the edges of the triangles of $\mathcal{T}_h$, we introduce the following discrete spaces (with P_k = space of the polynomials in two variables of degree ≤k)

(74) $$H_h^1 = \{q_h | q_h \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega}) \ , \ q_h|_T \in P_1 \quad \forall T \in \mathcal{T}_h\},$$

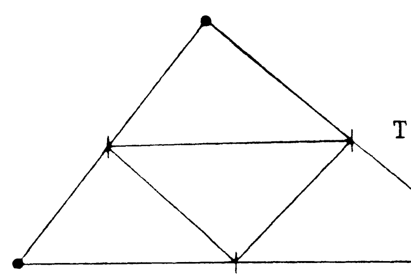
(75) $\quad V_h = \{v_h | v_h \in C^o(\bar{\Omega}) \times C^o(\bar{\Omega}),\ v_h|_T \in P_2 \times P_2 \quad \forall T \in \mathcal{T}_h\}$,

(76) $\quad V_{oh} = V_o \cap V_h = \{\underset{\sim}{v}_h | \underset{\sim}{v}_h \in V_h\ ,\ \underset{\sim}{v}_h = \underset{\sim}{0}\ \underline{\text{on}}\ \Gamma\}$.

A useful variant of V_h (and V_{oh}) is obtained as follows

(77) $\quad V_h = \{v_h | v_h \in C^o(\bar{\Omega}) \times C^o(\bar{\Omega}),\ v_h|_T \in P_1 \times P_1 \quad \forall T \in \tilde{\mathcal{T}}_h\ \}$

where, in (77), $\tilde{\mathcal{T}}_h$ is that triangulation of Ω obtained from $\mathcal{T}_h$ by joining the midpoints of the edges of $T \in \mathcal{T}_h$ as indicated on Figure 2.



<u>Figure 2</u>

We have the same global number of unknowns if we use V_h defined by either (75) or (77) ; however the matrices encountered in the second case are more compact.

As usual the functions of H_h^1 will be defined from the values they take at the vertices of $\mathcal{T}_h$; in the same fashion the functions of V_h will be defined by the values they take at the vertices of $\tilde{\mathcal{T}}_h$ (resp. the vertices and the mid-points of $\mathcal{T}_h$ if V_h is defined by (77) (resp. (75)).

5.3. <u>Approximation of the boundary conditions</u>.

Suppose that the boundary conditions are still defined by

(78) $\quad \underset{\sim}{u} = \underset{\sim}{g}\ \underline{\text{on}}\ \Gamma,\ \underline{\text{with}} \int_\Gamma \underset{\sim}{g} \cdot \underset{\sim}{n}\, d\Gamma = 0$;

for simplicity we suppose that $\underset{\sim}{g}$ is <u>continuous</u> over Γ. We define now the space γV_h as

(79) $\quad \gamma V_h = \{\mu_h | \mu_h = v_h|_\Gamma\ ,\ v_h \in V_h\}$,

i.e. γV_h is the space of the traces on Γ of those functions v_h belonging to V_h.

Actually if V_h is defined by (75) (resp. (77)), V_h is also the space of those functions defined over Γ, taking their values in $\mathbb{R}^2$, continuous over Γ and piecewise quadratic (resp. linear) over the edges of $\mathcal{T}_h$ (resp. $\tilde{\mathcal{T}}_h$) contained in Γ.

Our problem is to construct an approximation g_h of g such that

$$g_h \in \gamma V_h, \int_\Gamma g_h \cdot n \, d\Gamma = 0. \tag{80}$$

If $\pi_h g$ is the unique element of γV_h obtained from the values taken by g at those nodes of $\mathcal{T}_h$ (or $\tilde{\mathcal{T}}_h$) belonging to Γ, we usually have

$$\int_\Gamma \pi_h g \cdot n \, d\Gamma \neq 0.$$

To overcome the above difficulty we may proceed as follows :

(i) We define an approximation n_h of n as the solution of the following <u>linear variational problem</u> in γV_h

$$\begin{cases} n_h \in \gamma V_h , \\ \int_\Gamma n_h \cdot \mu_h \, d\Gamma = \int_\Gamma n \cdot \mu_h \, d\Gamma \quad \forall \mu_h \in \gamma V_h \ ; \end{cases} \tag{81}$$

problem (81) is in fact equivalent to a linear system whose matrix is sparse, symmetric, positive definite, and quite easy to compute.

(ii) Define then g_h by

$$g_h = \pi_h g - \left(\frac{\int_\Gamma \pi_h g \cdot n \, d\Gamma}{\int_\Gamma n \cdot n_h d\Gamma} \right) n_h \ . \tag{82}$$

It is quite easy to check that (81),(82) imply (80).

5.4. <u>Space discretization of the time dependent Navier-Stokes equations</u>.

Using spaces H_h^1, V_h and V_{oh} we approximate the time dependent Navier-Stokes equations as follows :

$$\begin{cases} \underline{\text{Find}} \ \{u_h(t), p_h(t)\} \in V_h \times H_h^1 \quad \forall t \geq 0 \ \underline{\text{such that}} \\ \int_\Omega \frac{\partial u_h}{\partial t} \cdot v_h \, dx + \nu \int_\Omega \nabla u_h \cdot \nabla v_h \, dx + \int_\Omega (u_h \cdot \nabla) u_h \cdot v_h \, dx + \int_\Omega \nabla p_h \cdot v_h \, dx = \\ = \int_\Omega f_h \cdot v_h \, dx \quad \forall v_h \in V_{oh} , \end{cases} \tag{83}$$

(84) $$\int_\Omega \nabla \cdot u_h q_h \, dx = 0 \quad \forall q_h \in H_h^1 ,$$

(85) $$u_h = g_h \text{ on } \Gamma,$$

(86) $$u_h(x,0) = u_{oh}(x) \quad (\text{with } u_{oh} \in V_h);$$

in (83)-(86), f_h and u_{oh} are convenient approximations of f and u_o, respectively, and g_h has been defined in Sec. 5.3.

We have thus reduced to solution of the time dependent Navier-Stokes equations to that of a nonlinear system of algebraic and ordinary differential equations.

We observe that the incompressibility condition is approximately satisfied only. The time discretization of system (83)-(86) is discussed in the following Sec. 5.5.

5.5. Time discretization of (83)-(86) by alternating direction methods.

We consider now a fully discrete version of the scheme (8)-(10) discussed in Sec. 2.1 ; it is defined as follows (with Δt and θ as in Sec. 2) :

(87) $$u_h^o = u_{oh} ,$$

then for $n \geq 0$, compute (from u_h^n) $\{u_h^{n+1/2}, p_h^{n+1/2}\} \in V_h \times H_h^1$, and then $u_h^{n+1} \in V_h$, by solving

(88) $$\begin{cases} \int_\Omega \frac{u_h^{n+1/2} - u_h^n}{(\Delta t/2)} \cdot v_h \, dx + \theta\nu \int_\Omega \nabla u_h^{n+1/2} \cdot \nabla v_h \, dx + \int_\Omega \nabla p_h^{n+1/2} \cdot v_h \, dx = \\ = \int_\Omega f_h^{n+1/2} v_h dx - (1-\theta)\, \nu \int_\Omega \nabla u_h^n \cdot \nabla v_h \, dx - \int_\Omega (u_h^n \cdot \nabla) u_h^n \cdot v_h \, dx \quad \forall v_h \in V_{oh} , \end{cases}$$

(89) $$\int_\Omega \nabla \cdot u_h^{n+1/2} \, q_h \, dx = 0 \quad \forall q_h \in H_h^1 ,$$

(90) $$u_h^{n+1/2} \in V_h , \; p_h^{n+1/2} \in H_h^1 , \; u_h^{n+1/2} = g_h^{n+1/2} \text{ on } \Gamma,$$

and then

(91) $$\begin{cases} \int_\Omega \frac{u_h^{n+1} - u_h^{n+1/2}}{(\Delta t/2)} \cdot v_h \, dx + (1-\theta)\nu \int_\Omega \nabla u_h^{n+1} \cdot \nabla v_h \, dx + \int_\Omega (u_h^{n+1} \cdot \nabla) u_h^{n+1} \cdot v_h \, dx = \\ = \int_\Omega f_h^{n+1} \cdot v_h \, dx - \theta\nu \int_\Omega \nabla u_h^{n+1/2} \cdot \nabla v_h \, dx - \int_\Omega \nabla p_h^{n+1/2} \cdot v_h \, dx \quad \forall v_h \in V_{oh} , \end{cases}$$

(92) $\underset{\sim}{u}_h^{n+1} \in V_h$, $\underset{\sim}{u}_h^{n+1} = \underset{\sim}{g}_h^{n+1}$ on Γ.

Obtaining the fully discrete analogue of the scheme (11)-(14) described in Sec. 2.2 is left as an exercise to the reader.

5.6. Some brief comments on the solution of the linear and nonlinear discrete subproblems.

The linear and nonlinear subproblems which have to be solved at each full step of scheme (87)-(92) are the discrete analogues (in space) of those continuous subproblems whose solution has been discussed in Secs. 3 and 4 ; actually the methods described in these sections apply with almost no modification to the solution of problems (88)-(90) and (91), (92). For this reason they will not be discussed here (they are however discussed in details in [3, Chap. 7]).

6. - NUMERICAL EXPERIMENTS.

We illustrate the numerical techniques described in the previous sections by presenting the results of numerical experiments where these techniques have been used to simulate several flows modelled by the Navier-Stokes equations for incompressible viscous fluids.

6.1. Flow in a channel with a step.

The first numerical experiment that we have done concerns a Navier-Stokes flow in a channel with a step, at Re = 191 ; the characteristic length used to compute the Reynold's number is the height of the step. Poiseuille profiles of velocity have been prescribed upstream and quite far downstream.

The alternating direction schemes of Sec. 5.5 have been used to integrate the time dependent Navier-Stokes equations until a steady state has been reached. The corresponding stream-lines are shown on Figure 3.

We clearly see on Figure 3 a thin separation layer starting slightly below the upper corner of the step, and separating a recirculation zone from a zone where the flow is quasi-potential.

The results obtained for this test are in very good agreement with those obtained by several authors, using different methods (see in particular [1] and HUTTON [13]).

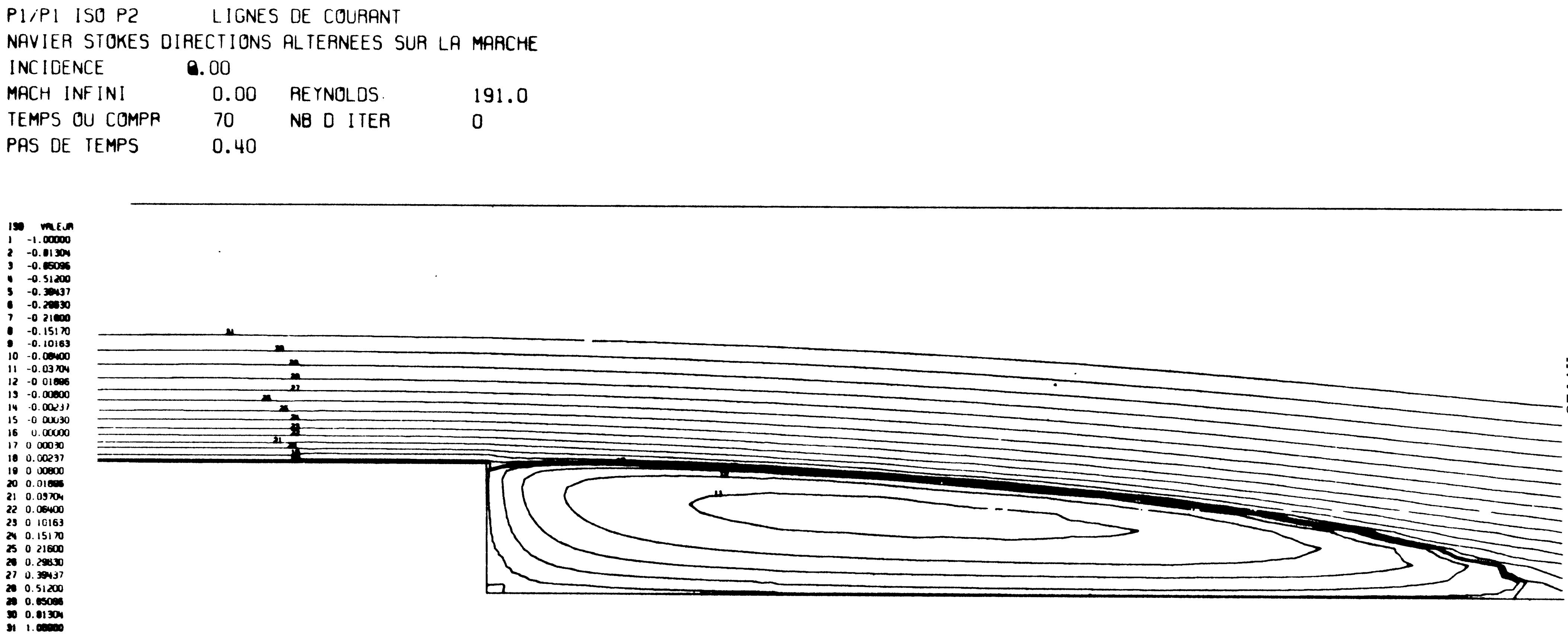
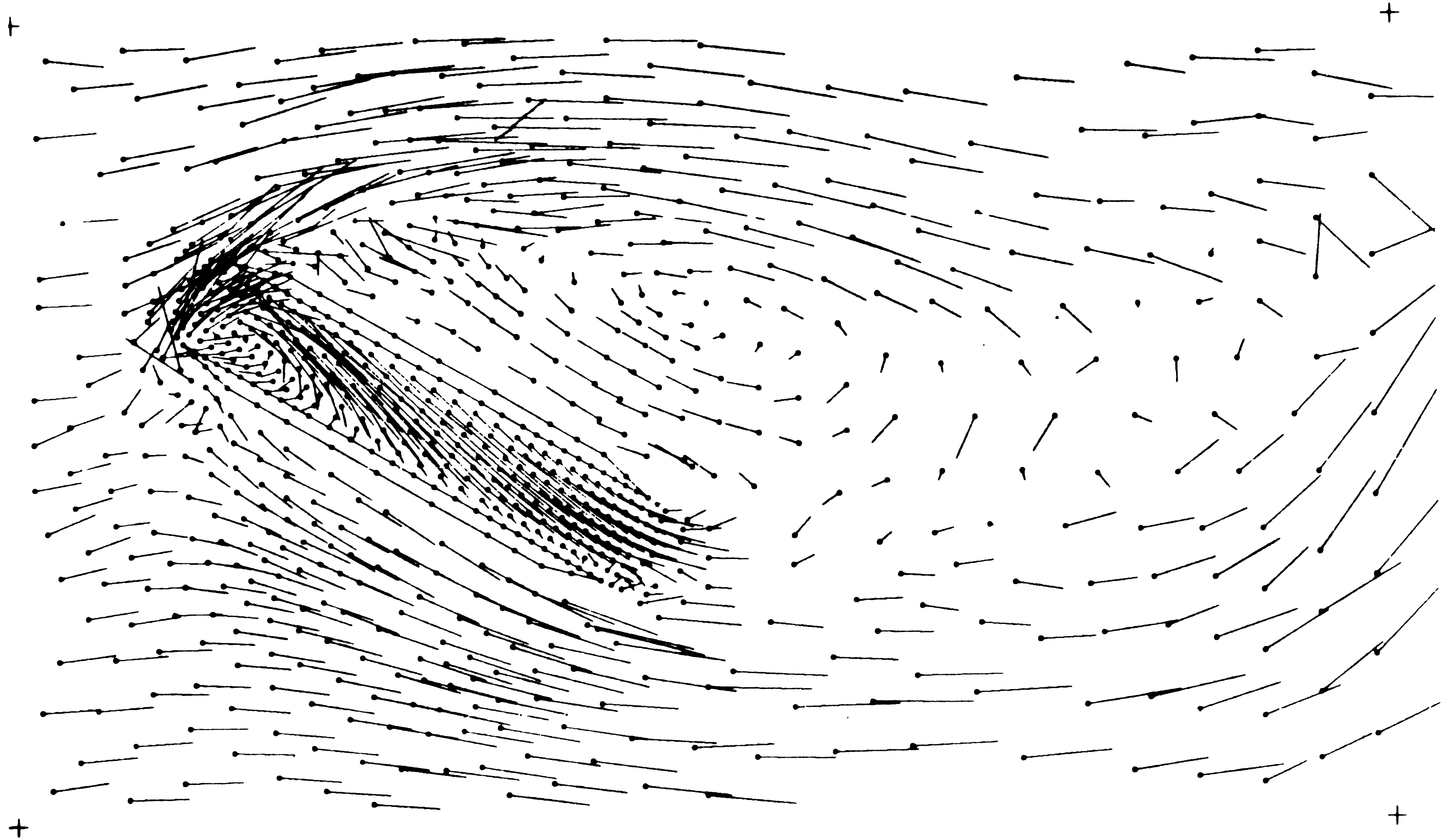


Figure 3

Stream lines for a flow in a channel with a step at Re = 191.

Figure 4

Velocity distribution for a flow around and inside a nozzle at Re = 100



6.2. Flow around and inside a nozzle.

This experiment concerns an unsteady flow around and inside a nozzle at high incidence, at Re = 100 (the characteristic length being the distance between the nozzle walls).

The velocity distribution has been visualized on Figure 4, showing clearly the creation and the motion of eddies inside and behind the nozzle.

7. - CONCLUSION.

The methods discussed in this paper combine a time discretization by alternating direction schemes and a space approximation by finite elements. Compared to the methods discussed in [1] they produce more accurate results for less computational efforts. The main cause of that improvement is the decoupling between nonlinearity and incompressibility obtained through the application of alternating direction schemes.

We are still looking at further improvements and we have the feeling that methods making use of characteristic methods have a good future in order to simulate viscous flows governed by the incompressible or compressible Navier-Stokes equations.

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