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BEHAVIOUR AT THE BOUNDARY OF THE COMPLEX MONGE-AMPERE EQUATION

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Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded strictly pseudo convex domain. Then, Ω always has a strictly plurisubharmonic defining function $r \in C^\infty(\overline{\Omega})$ such that

$$(1) \quad \begin{cases} r = 0 & \text{on } \partial\Omega \\ r < 0 & \text{on } \overset{\circ}{\Omega} \\ r_{i\bar{j}} = \dfrac{\partial}{\partial z^i} \dfrac{\partial}{\partial \overline{z^j}} r > 0 & \text{in } \overline{\Omega} \end{cases} .$$

If one takes

$$g = -\log(-r)$$

then the tensor

$$\sum_{i,j=1}^{n} g_{i\bar{j}}\, dz^i . d\overline{z^j} \quad , \quad g_{i\bar{j}} = \frac{\partial}{\partial z^i} \frac{\partial}{\partial \overline{z^j}} g$$

is a complete Kähler metric on $\overset{\circ}{\Omega}$, and it is always equivalent to the Bergman metric. For a Kähler metric the (Hermitian) Ricci tensor is

$$R_{i\bar{j}} = -\partial z^i \, \partial \overline{z^j} \log \det g_{i\bar{j}}$$

and Einstein's equation :

$$(2) \qquad R_{i\bar{j}} = c\, g_{i\bar{j}}$$

takes a particularly simple form. In fact, to ensure (2) for the metric derived from g one demands

$$(3) \qquad \det (g_{i\bar{j}}) = e^{(n+1)g} .$$

This is the complex Monge-Ampère equation discussed below. The choice of constant, $n+1$, is essentially arbitrary up to sign but is especially convenient (see [1]). If one sets

$$g = G + u$$

where $G = -\log(-R)$ corresponds to some particular choice of plurisubharmonic defining function then (3) can be rewritten

$$M(u) = \det(G_{i\bar{j}})^{-1} \cdot \det(G_{i\bar{j}} + u_{i\bar{j}}) \cdot e^{-(n+1)n}$$

(4)

$$= e^{F}$$

where $F \in C^{\infty}(\bar{\Omega})$ is given by

$$e^{F} = e^{(n+1)G} (\det G_{i\bar{j}})^{-1} .$$

The form (4) is due to Cheng and Yau [1] who showed that it has a unique solution $u \in \overset{o}{C}{}^{2}(\Omega)$ such that $g_{i\bar{j}}$ is again equivalent to the Bergman metric :

(5) $$\frac{1}{C} G_{i\bar{j}} \leq g_{i\bar{j}} \leq C\, G_{i\bar{j}} \qquad C \text{ constant.}$$

Theorem : The solution g to (3), (5) is a graded conormal distribution associated to the boundary $\partial\Omega$. More exactly, there are functions $\psi_j \in C^{\infty}(\bar{\Omega})$ such that

(6) $$u \sim \sum_{j=o}^{\infty} \psi_j (\log(-R))^j \qquad R \to 0$$

where $\psi_j = O(R^{(n+1)j})$ so that (6) completely determines the singularity of u (and hence g) at $\partial\Omega$.

The Kähler-Einstein metric $g_{i\bar{j}}$ is an important biholomorphic invariant of the domain Ω and the Taylor series of the ψ_j at $\partial\Omega$ are related to invariants of the CR geometry of the boundary.

The proof of (6), which was in essence conjectured by Fefferman [2], is carried out in [3]. It can be divided into four steps.

I. The explicit description of the degeneracy of $M(u)$ at the boundary.

II. Continuity properties of M, and its linear part $\Delta_G + (n+1)$ on natural degenerate Hölder spaces.

III. Tangential regularity of solutions, obtained by commutator methods, leading to the conormal property.

IV. The extraction of the "classical" expansion (6), by symbolic methods.

(I) One can arrange that u in (4) vanishes at the boundary, so it is natural to examine the linearization of $M(u)$ about $u = 0$:

$$M(u) = 1 + \sum_{i,\bar{j}} G^{i\bar{j}} u_{i\bar{j}} - (u+1)\, u + \text{quadratic terms} .$$

Here $G^{i\bar{j}}$ is the inverse matrix to $G_{i\bar{j}}$ and since this is a Kähler metric one has the simple formula for the Laplace-Beltrami operator :

$$\Delta_G u = -\sum_{i,\bar{j}} G^{i\bar{j}} u_{i\bar{j}}$$

and the linear part of $M(u)$ is just

$$-(\Delta+(n+1)) .$$

If one introduces $x = -R$ as a first coordinate near a boundary point and then takes local coordinates $y_1,\dots,y_{2n-1}$ in $\partial\Omega$ there is a natural way, using the metric $G_{i\bar{j}}$, to extend the y_j's to give <u>normal coordinates</u> near the boundary, $x, y_1,\dots,y_{2n-1}$. With respect to these coordinates

$$(7) \qquad \Delta_G = I(x\,D_x) - xr.(x\,D_x)^2 + x\,\Box_b + \tfrac{i}{2}(n-1)\,x\,T + x^2 R_2(x,y,x\,D_x,D_y)$$

where

$$I\,(x\,D_x) = (x\,D_x)^2 + i\,n\,x\,D_x$$

is the indicial operator, $r \in C^{\infty}$, $\Box_b$ is Kohn's Laplacian for $\bar{\partial}_b$ with respect to the Levi form induced by $G_{i\bar{j}}$ and T is a C^{∞} vector field. The remainder terms in R_2 have the important property that at the boundary R_2 is elliptic, in the totally characteristic sense of [4], where $x\,D_x$ and $\Box_B$ are both characteristic.

The structure of (7) is of paramount importance in the analysis. If one recalls that, at least as far as the principal part is concerned, $\Box_b$ is made up of vector fields in the maximal complex subspace of $T\partial\Omega$:

$$H(\partial\Omega) = T\partial\Omega \cap i\, T\partial\Omega \subset T\partial\Omega \ .$$

Intuitively one has :

$$\Delta = (x\,D_x)^2 + (x^{1/2}\,V)^2 + (x\,W)^2$$

where the middle terms are in $H(\partial\Omega)$ over the boundary. Not only is this decomposition meaningful but the whole of the non-linear operator $M(u)$ has a similar structure.

(II) The degenerate Hölder spaces that are used in the estimation of M and Δ are based on this decomposition. For each integer k and ε with $0<\varepsilon<1$ one can define spaces $\Lambda^{k,\varepsilon}(\Omega) \subset L^{\infty}(\Omega)$ such that

$$x\,D_x\ ,\ x^{1/2}\,V\ ,\ x\,W : \Lambda^{k+1,\varepsilon}(\Omega) \to \Lambda^{k,\varepsilon}(\Omega)\ ,$$

if $V \in C^{\infty}(\bar{\Omega}, T\bar{\Omega})$ has $V\big|_{\partial\Omega} \in C^{\infty}(\partial\Omega\ ;\ H(\partial\Omega))$. Following the estimates of Cheng and Yau one then has

$$(8) \qquad \Delta + (n+1) : x^t\,\Lambda^{k+2,\varepsilon}(\Omega) \to x^t\,\Lambda^{k,\varepsilon}(\Omega)$$

an isomorphism whenever $0 \le t < n+1$.

The upper restriction on the range is optimal since the indicial operator satisfies :

$$(9) \qquad [I(x\,D_x) + (n+1)]\,(x^{-1}\ ,\ x^{n+1}) = 0\ .$$

By the method of continuity (see [1]) these estimates can be applied to the non-linear equation too.

(III) As a result of (8) one finds that a distribution u satisfying

(10) $$(\Delta + n+1)\, u = f \in C^{\infty}(\overline{\Omega}),\ u \in L^{\infty}(\Omega)$$

actually lies in the space

(11) $$u \in \Lambda^{\infty} = \bigcap_{k} \Lambda^{k,\varepsilon}(\Omega)\ .$$

These estimates are not very strong however; for example the best isotropic estimates on tangential derivatives implied by (11) are

$$|D_y^{\alpha}\, u| \le C|x|^{-|\alpha|}\ .$$

However, a more systematic study of the filtration associated to Δ - for example it is easy to see that if $V \in C^{\infty}(\overline{\Omega},T\overline{\Omega})$ has $V\big|_{\partial\Omega} \in C^{\infty}(\partial\Omega,H\partial\Omega)$ then

$$[\Delta,V] : x^t\, \Lambda^{k+2,\varepsilon}\,(\Omega) \to x^t\, \Lambda^{k,\varepsilon}(\Omega) \qquad \forall k,t \quad -$$

allows one to improve (11) to

(12) $$(x\, D_x)^k\, D_y^{\alpha}\, u \in L^{\infty}(\overline{\Omega}) \qquad \forall k,\alpha\ .$$

(IV) In [4] it is shown that the estimates (12) are essentially equivalent, apart from questions of order, to the statement that u is a Lagrangian (Fourier integral) distribution associated to the conormal bundle to the boundary, i.e. a conormal distribution. Thus, one can apply symbolic methods, using (7). In the recursive description obtained in this way u is found, modulo lower order terms, by solving the equation

$$[I(x\, D_x) + (n+1)]\, u' = f'$$

where f' is C^∞ in x and y, except for terms which arise from earlier approximations to u. The only non smooth terms which can arise in this way come from the kennel x^{n+1}. Thus

Proposition : The solution u to (10) is of the form

$$u = u_1 + u_2 \log(-R)$$

where $u_1, u_2 \in C^\infty(\overline{\Omega})$ and $u_2 = O(R^{n+1})$.

Once again similar considerations apply to the non-linear problem, yielding the theorem as announced.

References

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