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VARIATIONAL METHODS IN NONLINEAR PROBLEMS

by L. NIRENBERG

This lecture presents several methods for finding stationary points of functions defined on finite or (often) infinite dimensional spaces in case the functions are unbounded from above and below -- so that one seeks stationary points which are, in general, saddle points. Applications to the problem of finding nontrivial time-periodic solutions of a nonlinear string equation

$$(1) \quad u_{tt} - u_{xx} + g(u) = 0$$

are presented. Here u is defined for $0 \leq x \leq \pi$, $t \in \mathbb{R}$, and is to satisfy the boundary conditions $u(0,t) = u(\pi,t) = 0$ and to be periodic in t with given period T .

The author has recently published an expository article [10] devoted to variational and topological methods, and this lecture takes up some of the material of [10].

In treating variational problems on infinite dimensional spaces it is now rather standard to make use of a kind of compactness condition that was introduced by R. Palais and S. Smale often called the (P.S.) condition. For a

real C^1 function f defined on a Banach space X it takes the form:

(P.S.): Any sequence $\{x_j\} \in X$ for which

(i) $|f(x_j)|$ is bounded,

and

(ii) $f'(x_j)$, as an element of the dual space X^* , tends strongly to zero,

has a strongly convergent subsequence $\{x_{j_i}\}$.

This is a rather strong condition. For instance if f satisfies (P.S.) and is bounded from below then f achieves its minimum.

The variational methods described here are all extensions of the Mountain Pass Lemma:

(MPL) Let f be a C^1 real function on X satisfying (PS).

Assume that for some open neighborhood U of the origin in X we have

$$f(0) < c_1 \leq f(u) \quad \forall u \in \partial U,$$

and assume that for some $u_0 \notin \bar{U}$,

$$f(u_0) < c_1.$$

Then the following number c is a stationary value of f :

$$c = \inf_P \max_{u \in P} f(u) \geq c_1,$$

where P is any continuous path in X going from u_0 to 0 , and we take the infimum with respect to all paths.

If $X = \mathbb{R}^2$ and $f(x)$ represents the height of land above the point $u \in X$, then o lies in a valley U . The number c then represents the height of the lowest mountain pass crossing the mountain range ∂U from u_0 to 0 .

This intuitively obvious lemma appeared for the first time in this form only in 1973 in a paper of Ambrosetti and Rabinowitz [1]. Together with its various extensions it has proved to be very useful. Here is an extension due to Rabinowitz [11] which we present in a slightly more general form of Ni [9].

(Gen'd MPL): Let f be a real C^1 function on X satisfying (PS).
Let ϕ be a continuous map of the k -sphere S^k into X and
assume

$$(2) \quad \max_{x \in S^k} f(\phi(x)) < c_1.$$

Assume furthermore that for every continuous extension
 $h: B^{k+1} \rightarrow X$ of the map ϕ inside the unit ball B^{k+1} in \mathbb{R}^{k+1}
into the space X ,

$$(3) \quad \max_{x \in B^{k+1}} f(h(x)) \geq c_1.$$

Then

$$c = \inf_h \max_{x \in B^{k+1}} f(h(x)) \geq c_1$$

is a stationary value of f .

The conditions (2), (3) are difficult to verify in practice, and in [11] and [9] useful sufficient conditions are given. References to further extensions and related variational principles may be found in [10]. See in particular the papers there: [54], [29] and [8-10].

Turning to equation (1) we consider g monotone and satisfying $g(0) = 0$ and $|g|$ growing faster than linearly as $|u| \rightarrow \infty$ -- for example $g = u^3$. We suppose also that T/π is rational. In [11] Rabinowitz applied the Gen'd MPL to obtain nontrivial solutions for (1). Adapting a dual variational method of Ekeland [8], Brézis, Coron and Nirenberg [6] have given a different proof and extension of the results of Rabinowitz -- using the original (MPL). Further references may be found there and in [10]. Here are some more recent papers by Bahri and Brézis [5] and by Coron [7]. Further variational techniques have been devised by Bahri and Berestycki, see [2-4].

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