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HYPERBOLIC EQUATIONS AND IRREGULARITY

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1. GEVREY CLASSES OF ULTRADIFFERENTIABLE FUNCTIONS AND ULTRADISTRIBUTIONS

In many of his works Hörmander discussed infinitely differentiable solutions and distribution solutions of linear partial differential equations with simple characteristics. On the other hand, in the works of Sato-Kawai-Kashiwara and Bony-Schapira they assume very little about multiplicity of characteristics and discussed real analytic solutions and hyperfunction solutions.

We are interested in inbetween. There are infinitely many classes of functions between the infinitely differentiable functions and the real analytic functions and also infinitely many classes of generalized functions between the distributions and the hyperfunctions. The most important among them are Gevrey classes of ultradifferentiable functions and corresponding classes of ultradistributions.

Let $s > 1$. An infinitely differentiable function ϕ on an open set Ω in $\mathbb{R}^n$ is said to be an ultradifferentiable function of class (s) (resp. $\{s\}$) if for any compact set K in Ω and any $h > 0$ there is a constant C (resp. there are constants h and C) such that

$$(1) \qquad \sup_{x \in K} |\partial^\alpha \phi(x)| \leqq C h^{|\alpha|} |\alpha|!^s.$$

We denote by $*$ either (s) or $\{s\}$ and by $\mathcal{E}^*(\Omega)$ the space of all ultradifferentiable functions of class $*$ on Ω. The subspace $\mathcal{D}^*(\Omega)$ of all elements in $\mathcal{E}^*(\Omega)$ with compact support has a natural locally convex topology and an ultradistribution of class $*$ on Ω is by definition a continuous linear functional on $\mathcal{D}^*(\Omega)$.

We denote by $\mathcal{D}^{*\prime}(\Omega)$ the space of all ultradistributions of class $*$ on Ω and endow it with the strong topology as the dual of the locally convex space $\mathcal{D}^*(\Omega)$. Since there are partitions of unity composed of ultra-differentiable functions of class $*$, the notion of support is defined for ultradistributions in the same way as for distributions ([9], [10], [11]).

It is often convenient to regard Schwartz' $\mathcal{E}$ as $\mathcal{E}^{(\infty)}$, the distributions $\mathcal{D}'$ as $\mathcal{D}^{(\infty)}{}'$, the real analytic functions $\mathcal{A}$ as $\mathcal{E}^{\{1\}}$ and the hyperfunctions $\mathcal{B}$ as $\mathcal{D}^{\{1\}}{}'$.

2. IRREGULARITY OF CHARACTERISTIC ELEMENTS

Let

$$P(x, \partial) = \sum_{|\alpha| \leqq m} a_\alpha(x) \partial^\alpha \tag{2}$$

be a single linear partial differential operator defined on an open set Ω in $\mathbb{R}^n$ and let

$$p(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \tag{3}$$

be its characteristic polynomial. We assume that $P(x, \partial)$ is non-degenerate or that for every $x \in \Omega$ there is an α with $|\alpha| = m$ such that $a_\alpha(x) \neq 0$. Then the multiplicity d of $P(x, \partial)$ is defined to be the supremum of the multiplicities of algebraic varieties $\{\xi \in \mathbb{C}^n; p(\mathring{x}, \xi)\} = 0$ as $\mathring{x}$ ranges over Ω. It is determined only by the principal part. As Hörmander shows, multiplicity is a good index. However, there are also many results in which lower order terms play an essential role. Taking this into account we introduced another invariant named irregularity in [13].

We assume that the coefficients $a_\alpha(x)$ are real analytic so that they are continued to holomorphic functions $a_\alpha(z)$ on a complex neighborhood $\Omega^{\mathbb{C}}$ of Ω. Then the characteristic variety $V = \{(x, \xi) \in \Omega^{\mathbb{C}} \times \mathbb{C}^n; p(x, \xi) = 0\}$ is an analytic-algebraic set. A non-singular characteristic element $(\mathring{x}, \mathring{\xi})$ is by definition a point in the non-singular part $\mathring{V}$ of V.

For each non-singular characteristic element $(\mathring{x}, \mathring{\xi})$ we can find an irreducible homogeneous polynomial $q(x, \xi) \in \mathcal{O}_{\mathring{x}}[\xi]$ with coefficients in the ring $\mathcal{O}_{\mathring{x}}$ of germs of holomorphic functions at the point $\mathring{x}$ such that V coincides with the zeros of q near $(\mathring{x}, \mathring{\xi})$ and $(\mathring{x}, \mathring{\xi})$ is a simple zero of

q. We have

<u>Lemma 1</u> : Let $q(x, \xi)$ and $b(x, \xi)$ be homogeneous polynomials in $\mathcal{O}_{\mathring{x}}[\xi]$. If $q(x, \xi)$ is irreducible in $\mathcal{O}_{\mathring{x}}[\xi]$ and if $b(x, \xi)$ vanishes identically on the zeros of $q(x, \xi)$ on a neighborhood of $(\mathring{x}, \mathring{\xi})$ in $\Omega^{\mathbb{C}} \times \mathbb{C}^n$, then there is a homogeneous polynomial $\ell(x, \xi) \in \mathcal{O}_{\mathring{x}}[\xi]$ such that

$$b(x, \xi) = \ell(x, \xi)q(x, \xi).$$

Hence if d is the multiplicity of V at $(\mathring{x}, \mathring{\xi})$, then we can find a polynomial $\ell_m(x, \xi) \in \mathcal{O}_{\mathring{x}}[\xi]$ such that

$$p(x, \xi) = \ell_m(x, \xi)q(x, \xi)^d$$

and $\ell_m(\mathring{x}, \mathring{\xi}) \not\equiv 0$. Let $Q(x, \partial)$ and $L_m(x, \partial)$ be differential operators with coefficients in $\mathcal{O}_{\mathring{x}}$ whose characteristic polynomials are equal to $q(x, \xi)$ and $\ell_m(x, \xi)$ respectively. Then $P(x, \partial) - L_m(x, \partial)Q(x, \partial)^d$ is an operator of order at most $m - 1$. Applying Lemma 1 to the characteristic polynomial of $P - L_m Q^d$ and so on, we have the following.

<u>Proposition 1</u> : There are $d_i \in \{0,1,2,\ldots,\infty\}$ and differential operators $L_i(x, \partial)$ with coefficients in $\mathcal{O}_{\mathring{x}}$ such that

$$(4) \quad P(x, \partial) = L_m(x, \partial)Q(x, \partial)^{d_m} + L_{m-1}(x, \partial)Q(x, \partial)^{d_{m-1}} + \cdots + L_0(x, \partial),$$

where either $d_i = \infty$ and $L_i = 0$ or else $\mathrm{ord}(L_i Q^{d_i}) = i$ and $\ell_i(x, \xi) \not\equiv 0$ on V near $(\mathring{x}, \mathring{\xi})$.

(4) is called the De Paris decomposition of P relative to Q after De Paris [4].

<u>Definition</u> : The irregualrity σ of the non-singular characteristic element

$(\mathring{x}, \mathring{\xi})$ is defined by

(5) $$\sigma = \max\{1, \max_{0\leq i\leq m} \frac{d_m - d_i}{m - i}\}.$$

The De Paris decomposition is not unique but the irregularity is uniquely determined by $(\mathring{x}, \mathring{\xi})$. It actually depends only on the operator P and the connected component of $\mathring{V}$.

J. Leray did not like the author's proof of invariance and proposed to start with a decomposition of $P(x, \partial)$ of the form

(6) $$P(x, \partial) = \sum_{j\in J} L_j(x, \partial)Q(x, \partial)^j,$$

where

(i) J is a finite subset of $\mathbb{N}$;

(ii) $\ell_j(x, \xi) \not\equiv 0$ on $\{q(x, \xi) = 0\}$;

(iii) $\omega(j) = \mathrm{ord}(L_j Q^j)$ increases strictly with $j \in J$.

Then

(7) $$\sigma = \max_{j\in J\setminus\{d\}} \{1, \frac{d - j}{\omega(d) - \omega(j)}\}.$$

Two definitions are clearly equivalent. Leray's proof of invariance of irregularity is longer than the author's but more natural. His approach has another advantage, i.e. it applies to pseudodifferential operators as well.

Employing the Weierstrass type division theorem of Sato-Kawai-Kashiwara [22], Aoki [1] has proved the existence of the strict De Paris decomposition (6) for pairs of pseudodifferential operators (P, Q) such that $d_{(x,\xi)}q(\mathring{x}, \mathring{\xi}) \neq 0$. Thus he considers also a class of degenerate operators including ordinary differential operators at their singular points as discussed in [12].

Clearly we have $1 \leqq \sigma \leqq d$. An operator P is said to satisfy Levi's condition at $(\mathring{x}, \mathring{\xi})$ if $\sigma = 1$ there. De Paris [4] calls such an operator "bien décomposable".

Let $\phi(z)$ be a holomorphic solution of

$$q(z, \operatorname{grad} \phi(z)) = 0 \tag{8}$$

and let $W_j(t)$ be a sequence of wave forms satisfying

$$\frac{dW_j(t)}{dt} = W_{j-1}(t). \tag{9}$$

When the operator is of simple characteristics J. Beudon, J. Hadamard, P. D. Lax, D. Ludwig, Mizohata [20], Wagschal [23] and many others constructed a solution $u(z)$ of

$$P(z, \partial)u(z) = 0 \tag{10}$$

of the form

$$u(z) = \sum_{j=0}^{\infty} A_j(z) W_j(\phi(z)), \tag{11}$$

where the amplitudes $A_j(z)$ are holomorphic functions defined on a neighborhood U of $\mathring{x}$ and satisfying the estimates

$$|A_j(z)| \leqq C^{j+1} j!, \quad j \geqq 0, \tag{12}$$

for a constant C. De Paris [5] proved the same for operators with Levi's condition.

When the multiplicity d is greater than 1, Hamada [7], [8] constructed a solution $u(z)$ of (10) of the form

(13) $$u(z) = \sum_{j=-\infty}^{\infty} A_j(z) W_j(\phi(z)).$$

We have shown in [13] that the amplitudes $A_j(z)$ satisfy (12) and

(14) $$|A_j(z)| \leqq C^{-j+1}\left(\frac{|x_n|^{-j}}{(-j)!}\right)^{\frac{\sigma}{\sigma-1}}, \quad j < 0.$$

By the method of Ōuchi [21] we can show that (14) is the best possible.

Since $A_j(z)$ are independent of the wave forms $W_j(t)$, we can construct many solutions of (10). We proved in particular the existence of null-solutions for analytic characteristic surfaces of constant multiplicity [13].

3. HYPERBOLIC EQUATIONS

We consider the Cauchy problem

(15) $$\begin{cases} P(x, \partial)u(x) = f(x), \\ \partial_1^{j-1} u(\mathring{x}_1, x') = g_j(x'), \quad j = 1,\dots,m, \end{cases}$$

on the domain

(16) $$\Omega_T = (-T, T) \times \mathbb{R}^{n-1}.$$

Let $\mathcal{F}$ and $\mathcal{F}_{n-1}$ be linear spaces of (generalized) functions on Ω_T and $\mathbb{R}^{n-1}$ respectively. The operator $P(x, \partial)$ is said to be $(\mathcal{F}, \mathcal{F}_{n-1})$-hyperbolic if for any $\mathring{x}_1 \in (-T, T)$ and any $f \in \mathcal{F}$ and $g_1,\dots,g_m \in \mathcal{F}_{m-1}$ there is a unique solution $u \in \mathcal{F}$ of (15).

Usually hyperbolicity condition consists of three parts:

(i) $P(x, \partial)$ is non-characteristic with respect to the initial surface $x_1 = \mathring{x}_1$, i.e.

(17) $$a_{(m,0,\ldots,0)}(x) \neq 0, \quad x \in \Omega_T;$$

(ii) The characteristic roots are real, i.e.

(18) $$p(x, \tau, \xi') = 0, \; x \in \Omega_T, \; \xi' \in \mathbb{R}^{n-1} \implies \tau \text{ real};$$

(iii) Conditions for lower order terms depending on $(\mathcal{F}, \mathcal{F}_{n-1})$.

Operators satisfying the first two conditions are called formally hyperbolic. The necessity in general of those conditions is proved by the behavior of solutions (13) in case the coefficients are real analytic ([13], [14]). Conversely Bony and Schapira [2] have proved that formally hyperbolic operators with real analytic coefficients are $\mathcal{A}$-hyperbolic and $\mathcal{B}$-hyperbolic.

For each $x \in \Omega_T$ and $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ the characteristic polynomial $p(x, \xi)$ of a formally hyperbolic operator $P(x, \partial)$ is decomposed as

$$p(x, \tau, \xi') = a_{(m,0,\ldots,0)}(x) \prod_{i=1}^{k} (\tau - \tau_i(x; \xi'))^{\nu_i},$$

where

$$\tau_1(x; \xi') < \tau_2(x; \xi') < \cdots < \tau_k(x; \xi').$$

$P(x, \partial)$ is said to be simply hyperbolic if $\nu_1 = \cdots = \nu_k = 1$ for all $x \in \Omega_T$ and $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, and of constant multiplicity if ν_i are constant on each connected component of $\Omega_T \times (\mathbb{R}^{n-1} \setminus \{0\})$. The latter holds if and only if every characteristic element is non-singular.

We claim that the third condition for hyperbolicity with respect to

Gevrey classes of ultradifferentiable functions and ultradistributions is given in terms of irregularity if the operator is of constant multiplicity.

First we note that if $P(x, \partial)$ is a formally hyperbolic operator of constant multiplicity, we can define the irregularity also in the case where the coefficients of $P(x, \partial)$ are not real analytic.

Matsuura [18] has shown that the characteristic polynomial $p(x, \xi)$ of a formally hyperbolic operator of constant multiplicity and with coefficients in $\mathcal{E}^*(\Omega_T)$ can be decomposed as

$$(19) \qquad p(x, \xi) = q_1(x, \xi)^{d_1} \cdots q_r(x, \xi)^{d_r}$$

by the characteristic polynomials $q_j(x, \xi)$ of simply hyperbolic operators $Q_j(x, \partial)$ with coefficients in $\mathcal{E}^*(\Omega_T)$ such that the product $Q_1(x, \partial) \cdots \times Q_r(x, \partial)$ is also simply hyperbolic.

Then we use the following.

Lemma 2 : Let $q(x, \xi)$ be the characteristic polynomial of a simply hyperbolic operator with coefficients in $\mathcal{E}^*(\Omega_T)$. If a homogeneous polynomial $b(x, \xi) \in \mathcal{E}^*(\Omega_T)[\xi]$ vanishes identically on the zeros of $q(x, \xi)$ in $\Omega_T \times \mathbb{R}^n$, then there is a homogeneous polynomial $\ell(x, \xi) \in \mathcal{E}^*(\Omega_T)[\xi]$ such that

$$b(x, \xi) = \ell(x, \xi) q(x, \xi).$$

In fact, let μ be the order of $q(x, \xi)$ and let

$$b(x, \xi) = \quad (x, \xi) q(x, \xi) + r(x, \xi)$$

be the division as a polynomial in ξ_1. Then the polynomial $r(x, \xi)$ of order less than μ has μ distinct roots for every x and ξ' so that we have $r(x, \xi) = 0$.

If $Q(x, \partial)$ is one of $Q_j(x, \partial)$ of the Matsuura decomposition (19), then we can prove by Lemma 2 that $P(x, \partial)$ has a De Paris decomposition (4) by operators $L_i(x, \partial)$ with coefficients in $\mathcal{E}^*(\Omega_T)$. Thus we can define the irregularity of $P(x, \partial)$ relative to $Q(x, \partial)$ by (5). When $P(x, \partial)$ has real analytic coefficients, it is equal to the supremum of the irregularities of characteristic elements which are zeros of $q(x, \xi)$.

We define the irregularity σ of the operator $P(x, \partial)$ to be the maximum of the irregularities of $P(x, \partial)$ relative to all factors $Q_j(x, \partial)$ of the Matsuura decomposition.

Theorem : Let $P(x, \partial)$ be a formally hyperbolic operator of constant multiplicity and of irregularity σ such that the characteristic roots $\tau_i(x; \xi')$ are uniformly bounded on $\Omega_T \times \{\xi' \in \mathbb{R}^n; |\xi'| = 1\}$.

(i) If $\sigma = 1$ and the coefficients of $P(x, \partial)$ are in $\mathcal{E}(\Omega_T)$, then $P(x, \partial)$ is $\mathcal{E}$-hyperbolic and $\mathcal{D}'$-hyperbolic;

(ii) If $\sigma \leqq s/(s - 1)$ and the coefficients are in $\mathcal{E}^{(s)}(\Omega_T)$, then $P(x, \partial)$ is $\mathcal{E}^{(s)}$-hyperbolic and $\mathcal{D}^{(s)}{}'$-hyperbolic;

(iii) If $\sigma < s/(s - 1)$ and the coefficients are in $\mathcal{E}^{\{s\}}(\Omega_T)$, then $P(x, \partial)$ is $\mathcal{E}^{\{s\}}$-hyperbolic and $\mathcal{D}^{\{s\}}{}'$-hyperbolic.

Part (i) has been proved by Chazarain [3] but our proof is much simpler.

Let $Q_j(x, \partial)$ be the simply hyperbolic operators of the Matsuura decomposition (19). Define simply hyperbolic operators $R_i(x, \partial)$, $i = 1,\dots,d$, by

$$R_i(x, \partial) = \prod_{d_j \geqq i} Q_i(x, \partial). \tag{20}$$

For example, if $p = q_1^2 q_2^3 q_3$, then $R_1 = Q_1Q_2Q_3$, $R_2 = Q_1Q_2$ and $R_3 = Q_2$.

After a few computation we derive the following global decomposition of $P(x, \partial)$ from the De Paris decomposition (4) for each factor $Q_j(x, \partial)$.

<u>Proposition 2</u> : There are differential operators $L_i(x, \partial)$, $i = 1,\ldots,d$, with coefficients in $\mathcal{E}^*(\Omega_T)$ such that

$$
\begin{aligned}
P(x, \partial) &= R_1(x, \partial) \cdots R_d(x, \partial) \\
&\quad + L_1(x, \partial)R_2(x, \partial) \cdots R_d(x, \partial) + \cdots \\
&\quad + L_i(x, \partial)R_{i+1}(x, \partial) \cdots R_d(x, \partial) + \cdots \\
&\quad + L_d(x, \partial)
\end{aligned} \tag{22}
$$

and

$$
\operatorname{ord} L_i \leqq \operatorname{ord}(R_1 \cdots R_d) - i/\sigma. \tag{23}
$$

Then the theorem is proved by the classical result of regularly hyperbolic equations due to I. G. Petrowoky, J. Leray, L. Gårding and Mizohata [19]. Let $R(x, \partial)$ be a regularly hyperbolic operator of order μ on $[-T_1, T_1] \times \mathbb{R}^{n-1}$. Then it asserts that for any data

$$
\left\{
\begin{aligned}
& f \in \bigcap_{j=0}^{p} C^j([-T_1, T_1], H^{p+q-j}(\mathbb{R}^{n-1})), \\
& g_j \in H^{p+q-j+\mu}(\mathbb{R}^n), \qquad j = 1,\ldots,\mu,
\end{aligned}
\right. \tag{24}
$$

there is a unique solution

$$
u \in \bigcap_{j=0}^{p+\mu-1} C^j([-T_1, T_1], H^{p+q-j+\mu-1}(\mathbb{R}^{n-1})) \tag{25}
$$

of the Cauchy problem

$$
\left\{
\begin{aligned}
& R(x, \partial)u(x) = f(x), \\
& \partial_1^{j-1}u(0, x') = g_j(x'), \qquad j = 1,\ldots,\mu,
\end{aligned}
\right. \tag{26}
$$

where $H^p(\mathbb{R}^{n-1})$ is the Sobolev space of order p. Moreover, there is a constant C such that

$$(27) \quad \|\nabla^{p+\mu-1,q} u(t)\| \leqq C\{\int_0^t \|\nabla^{p,q} f(s)\| |ds| + \|\nabla^{p+\mu-1,q} u(0)\|\},$$

where

$$\|\nabla^{p,q} v(t)\| = \sup_{\substack{|\alpha| \leqq p+q \\ \alpha_1 \leqq p}} \|\partial^\alpha v(t, \cdot)\|_{L^2(\mathbb{R}^{n-1})}.$$

To solve the Cauchy problem (15) with $\mathring{x}_1 = 0$ we solve

$$(28) \quad \begin{cases} R_1 \cdots R_d u_0 = f, \\ \partial_1^{j-1} u_0(0, x') = g_j, \quad j = 1,\dots,m, \end{cases}$$

and solve

$$(29) \quad \begin{cases} R_1 \cdots R_d u_{k+1} = -(L_1 R_2 \cdots R_d + \cdots + L_d) u_k, \\ \partial_1^{j-1} u_{k+1}(0, x') = 0, \quad j = 1,\dots,m, \end{cases}$$

for $k = 0,1,\dots$. Then a solution u is obtained as the sum $\sum_{k=0}^{\infty} u_k$.

Since we have a priori bounds of dependence domains by assumption, we may deform the coefficients of $P(x, \partial)$ so that R_i in (22) are regularly hyperbolic and the coefficients of L_i satisfy (1) with a constant C (resp. constants h and C) independent of x. We may also assume that $g_j = 0$ by subtracting a suitable function from u_0. Then (28) and (29) are solved by d regularly hyperbolic equations

$$(30)\qquad \begin{cases} R_i v_i = v_{i-1}, \\ \partial_1^{j-1} v_i(0, x') = 0, \quad j = 1,\ldots,m_i. \end{cases}$$

If

$$v_0 \in \bigcap_{j=0}^{p} C^j([-T_1, T_1], H^{p+q-j}(\mathbb{R}^{n-1}))$$

and $\|\nabla^{p-1,q} v_0(0)\| = 0$, then (30) has a unique solution

$$v_i \in \bigcap_{j=0}^{p+m_1+\cdots+m_i-i} C^j\left([-T_1, T_1], H^{p+q-j+m_1+\cdots+m_i-i}(\mathbb{R}^{n-1})\right)$$

for which we have

$$\|\nabla^{p+m_1+\cdots+m_i-i,q} v_i(t)\| \leqq C^i \int_0^t \frac{(t-s)^{i-1}}{(i-1)!} \|\nabla^{p,q} v(s)\| \, ds, \qquad 0 \leqq t \leqq T_1.$$

In case (i) we have

$$\text{ord } L_i \leqq m_1 + \cdots + m_i - i.$$

Hence

$$\|\nabla^{p,q}(L_1 R_2 \cdots R_d + \cdots + L_d) v_d(t)\| \leq \sum_{i=1}^{d} B_i C^i \int_0^t \frac{(t-s)^{i-1}}{(i-1)!} \|\nabla^{p,q} v_0(s)\| ds.$$

Thus we have

$$\|\nabla^{p,q}(L_1 \cdots R_d + \cdots + L_d) u_k(t)\| \leqq C_1^{k+1} \int_0^t \frac{(t-s)^k}{k!} \|\nabla^{p,q} f(s)\| \, ds,$$

proving the convergence of $\sum u_k$. The uniqueness is proved similarly.

Since p and q are arbitrary, the solution is infinitely differentiable if the data are so.

Considering the adjoint problem, we have the $\mathcal{D}'$-hyperbolicity. It is easily proved that the adjoint of a formally hyperbolic operator of constant multiplicity has the same irregularity as the original operator.

The proofs in cases (ii) and (iii) are essentially the same as above. To prove the ultradifferentiability of solutions in the space variables $(x_2, \ldots, x_n)$ we need only to estimate the commutators of the operators R_i with differentiations in the space variables as was done by Leray and Ohya [17] in case (iii) for operators of the form

$$P(x, \partial) = R_1(x, \partial) \cdots R_d(x, \partial) + L(x, \partial).$$

Then the ultradifferentiability in all variables follows from an analogue of the Cauchy-Kowalevsky theorem [15]. The $\mathcal{D}^{*\prime}$-hyperbolicity is again proved by the duality. Details are given in [16].

When $P(x, \partial)$ has real analytic coefficients another proof has been given by De Paris and Wagschal [6].

We employed the assumption that $P(x, \partial)$ is of constant multiplicity and of irregularity σ only in the form that it admits a global decomposition (22) with (23). Therefore the conclusion of the theorem holds also for a type of formally hyperbolic operators of varying multiplicity.

Lastly we note that the condition $\sigma = 1$ (resp. $\sigma \leqq s/(s - 1)$, resp. $\sigma < s/(s - 1)$) are necessary at each non-singular characteristic element in order that a formally hyperbolic operator with real analytic coefficients be $\mathcal{E}$-hyperbolic (resp. $\mathcal{E}^{(s)}$-hyperbolic, resp. $\mathcal{E}^{\{s\}}$-hyperbolic). This is proved by the behavior of solutions (13) and a characterization ([10], [11]) of local operators $\mathcal{E}^* \to \mathcal{E}^*$.

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