Séminaire Équations aux dérivées partielles - École Polytechnique

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Well-posedness and propagation of singularities for initial boundary value problem for second order hyperbolic equation with general boundary condition

Séminaire Équations aux dérivées partielles (Polytechnique) (1979-1980), exp. nº 2, p. 1-14

<http://www.numdam.org/item?id=SEDP_1979-1980____A2_0>

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WELL-POSEDNESS AND PROPAGATION OF SINGULARITIES FOR INITIAL BOUNDARY VALUE PROBLEM FOR SECOND ORDER HYPERBOLIC EQUATION WITH GENERAL BOUNDARY CONDITION.

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Exposé n° II

16 Octobre 1979

§1. STATEMENT OF RESULTS

Let G be a domain in $\mathbb{R}^n$ with a smooth boundary γ and let $A(x,\mathcal{D})$ be a second order differential operator in the cylinder $\Omega = (-\infty,+\infty)\times G \subset \mathbb{R}^{n+1}$, strictly hyperbolic with respect to x_o, where $x = (x_o,x_1,\dots,x_n)$, $x_o \in (-\infty,+\infty)$, $(x_1,x_2,\dots,x_n) \in G$, $\mathcal{D} = (i\frac{\partial}{\partial x_o},\dots,i\frac{\partial}{\partial x_n})$.

Consider the following mixed problem

$$A(x,\mathcal{D})u = 0 \quad \text{in } \Omega , \tag{1.1}$$

$$u = 0 \quad \text{for } x_o < 0, \quad x \in \Omega , \tag{1.2}$$

$$B(x,\mathcal{D})u\big|_\Gamma = h(x') \quad \text{for } x' \in \Gamma = (-\infty,+\infty)\times\gamma , \tag{1.3}$$

where $h \in \mathcal{D}'(\Gamma)$, $h = 0$ for $x_o < 0$, $u \in \mathcal{D}'(\Omega)$ and $B(x,\mathcal{D})$ is a differential operator of order r. We shall find when for a given $B(x,\mathcal{D})$ the problem (1.1), (1.2), (1.3) is well-posed or ill-posed. Denote by $G_\delta \approx \Gamma\times[0,\delta)$ the δ-neighbourhood of Γ in G. We shall use in G_δ the coordinates (x',x_n), where $x' \in \Gamma$ and x_n is the distance to Γ, $x_n \in [0,\delta)$. The dual coordinates in $T^*(G_\delta)$ will be denote by (ξ',ξ_n) where $\xi' \in T^*(\Gamma)$ and $\xi_n \in \mathbb{R}^1$.

The principal symbol of $A(x,\mathcal{D})$ has in this system of coordinates the following form

$$A^{(o)}(x,\xi) = (\xi_n - \lambda(x,\xi'))^2 - \mu(x,\xi'), \tag{1.4}$$

where $\lambda(x,\xi')$, $\mu(x,\xi')$ are real and $\frac{\partial\mu}{\partial\xi_0} \neq 0$ when $\mu = 0$ (cf. [3]). As in [3] denote $N_o = \{(x',\xi') \in T_o^*(\Gamma),\ \mu(x',0,\xi') = 0\}$, $N_\pm = \{(x',\xi') \in T_o^*(\Gamma),\ \mu(x',0,\xi') \gtrless 0\}$.

Set

$$b(x',\zeta') = B^{(o)}(x',0,\zeta',\lambda(x',0,\zeta') - \sqrt{\mu(x',0,\zeta')}), \tag{1.5}$$

where $B^{(o)}(x',x_n,\xi',\xi_n)$ is the principal symbol of the operator $B(x,\mathcal{D})$ written in the system of coordinate introduced above, $\zeta' = (\xi_o + i\tau, \xi_1,\dots,\xi_{n-1})$, $\tau > 0$, and $\sqrt{\mu(x',0,\zeta')}$ is the branch of the square root such that $\operatorname{Im}\sqrt{\mu(x',0,\zeta')} > 0$ for $(x',\xi') \in T_o^*(\Gamma)$, $\tau > 0$.

It is known (see [12]) that the condition

$$(1.6) \qquad b(x', \zeta') \neq 0 \quad \text{for} \quad \forall \tau > 0, \ (x', \xi') \in T_0^*(\Gamma)$$

is necessary for the well-posedness of the initial-boundary problem (1.1), (1.2), (1.3).

We shall see below that one needs additional assumptions on $b(x', \zeta')$ for real $\zeta' = \xi'$. It is well-known (see, for example, [13]) that microlocally on $N_+ \cup N_-$ the problem (1.1), (1.2), (1.3) can be reduced to the solution of a pseudo-differential equation

$$(1.7) \qquad b(x', \partial')u_0 = h ,$$

where $u_0 = u|_\Gamma$.

We shall assume that $b(x', \partial')$ is an operator of principal type in $N_+ \cup N_-$; more exactly we assume that

$$(1.8) \qquad \frac{\partial b(\hat{x}, \hat{\xi})}{\partial \xi_0} \neq 0 \quad \text{if} \quad (\hat{x}, \hat{\xi}) \in N_+ \cup N_- \ \text{and} \ b(\hat{x}, \hat{\xi}) = 0 .$$

Therefore $b(x', \xi')$ has the following form in the neighbourhood of $(\hat{x}, \hat{\xi})$:

$$(1.9) \qquad b(x', \xi') = (\xi_0 + b_1(x', \xi''))b^{(1)}(x', \xi'),$$

where $b^{(1)}(x', \xi') \neq 0$. The main problem is to find the conditions on $b(x', \xi')$ for $(x', \xi') \in N_0$.

We shall assume additionally that

$$(1.10) \qquad \frac{\partial B^{(0)}(\hat{x}, 0, \hat{\xi}, \lambda(\hat{x}, 0, \hat{\xi}))}{\partial \xi_n} \neq 0 \quad \text{for all} \ (\hat{x}, \hat{\xi}) \in N_0 \ \text{such that} \ b(\hat{x}, \hat{\xi}) = 0 .$$

Then in a small neighbourhood of such point $(\hat{x}, \hat{\xi}) \in N_0$ we have

$$(1.11) \qquad B^{(0)}(x', 0, \xi', \xi_n) = B^{(2)}(x', \xi', \xi_n)(\xi_n + b_2(x', \xi')) ,$$

where $B^{(2)}(x', \xi', \xi_n) \neq 0$. Therefore in this neighbourhood we have

(1.12) $b(x',\zeta) = -B^{(2)}(x',\zeta',\lambda(x',0,\zeta') - \sqrt{\mu(x',0,\zeta')})(\sqrt{\mu(x',0,\zeta')} + \lambda_1(x',\zeta'))$

where

(1.13) $$\lambda_1(x',\zeta') = -(\lambda(x',0,\zeta') + b_2(x',\zeta')).$$

The conditions of the well-posedness of the initial-boundary problem depend on the geometry of the boundary. We shall consider only the cases when the boundary is strictly convex or concave with respect to the null-bicharacteristics of $A(x, \mathcal{D})$. The boundary Γ is called strictly convex for $(x', \xi') \in N_0$ if for $x_n = 0$ and $\mu(x',0,\xi') = 0$ we have (cf. [3]) :

(1.14) $$\{\xi_n - \lambda(x',x_n,\xi'), \mu(x',x_n,\xi')\} < 0,.$$

where $\{\xi_n - \lambda, \mu\}$ is the Poisson bracket.

The boundary Γ is called strictly concave for $(x',\xi') \in N_0$ if (1.14) with the opposite sign holds.

In the case of the strictly concave boundary there are few restrictions for the well-posedness of the initial-boundary problem.

Theorem 1.1 : Let the boundary Γ be strictly concave for all (x',ξ') and let the assumptions (1.6), (1.8) and (1.10) hold. Then for each $h \in H^{\tau}_{s+r-1}(\Gamma)$ there exists a unique solution $u \in H^{\tau}_{s}(\Omega)$ of the problem (1.1), (1.2) (1.3) when τ is sufficiently large.

Here $H^{\tau}_s(\Omega)$ is a Sobolev's space with a finite norm $\|u\|^2_{s,\tau} = \|e^{-\tau x_0} u\|_s = \|\Lambda^{2s}_{\tau} u\|_0$, Λ^{2s}_{τ} is a pseudo-differential operator with a symbol $(|\xi|^2 + \tau^2)^s$, $u = 0$ for $x_0 < 0$, $h = 0$ for $x_0 < 0$. For the convex boundary we shall consider separately three cases.

It follows from (1.12) that if $(\hat{x},\hat{\xi}) \in N_0$ and $b(\hat{x},\hat{\xi}) = 0$ then also $\lambda_1(\hat{x},\hat{\xi}) = 0$. Suppose that if $(\hat{x},\hat{\xi}) \in N_0$ and $b(\hat{x},\hat{\xi}) = 0$ then there exists a conic neighbourhood U_0 of $(\hat{x},\hat{\xi})$ and a constant C_0 such that

(1.15) $$\frac{-\mathrm{Re}\,\lambda_1(x',\xi')}{\ln \dfrac{1}{|\mathrm{Re}\,\lambda_1(x',\xi')|}} \operatorname{sgn} \frac{\partial\mu(\hat{x},\hat{\xi})}{\partial\xi_0} < C_0(\mathrm{Im}\,\lambda_1(x',\xi'))^2$$

for any $(x',\xi') \in U_0 \cap N_0$ and $|\xi'| = 1$.

Theorem 1.2 : Let the boundary Γ be strictly convex and the conditions (1.6), (1.8) (1.10) be satisfied. Suppose that for any $(\hat{x},\hat{\xi}) \in N_o$ such that $b(\hat{x},\hat{\xi}) = 0$ the condition (1.15) holds. Then for each $h \in H_{s+r-1}(\Gamma_T)$ there exists a unique solution $u \in H_s(\Omega_T)$ of the problem (1.1), (1.2), (1.3) where $T > 0$ is sufficiently small, $\Omega_T = (-\infty, T) \times G$, $\Gamma_T = (-\infty, T) \times \gamma$ and $H_s(\Omega_T)$ is a Sobolev's space on Ω_T, $u = 0$ for $x_o < 0$, $h = 0$ for $x_o < 0$.

Remark 1.1 : If $\operatorname{Im} \lambda_1 \equiv 0$ then the condition (1.15) means that $\lambda_1(x',\xi') \geq 0$ in $U_o \cap N_o$, if $\frac{\partial\mu(\hat{x},\hat{\xi})}{\partial\xi_o} > 0$ and $\lambda_1(x',\xi') \leq 0$ in $U_o \cap N_o$ if $\frac{\partial\mu(\hat{x},\hat{\xi})}{\partial\xi_o} < 0$. We note that $\frac{\partial\mu}{\partial\xi_o} \neq 0$ when $\mu = 0$ since the symbol $A^{(o)}(x,\xi)$ is hyperbolic with respect to ξ_o.

Now we shall consider the case when the condition (1.15) is not satisfied.

We suppose that there exists a point $(\hat{x},\hat{\xi}) \in N_o$ with the following properties :

(1.16) $\lambda_1(\hat{x},\hat{\xi}) = 0$ and there is a sequence $(\hat{x}_m,\hat{\xi}_m) \in N_o, |\hat{\xi}_m| = 1$ such that $(\hat{x}_m, \hat{\xi}_m) \to (\hat{x},\hat{\xi})$ and $\operatorname{Re} \lambda_1(\hat{x}_m, \hat{\xi}_m) < 0$ if $\frac{\partial\mu(\hat{x},\hat{\xi})}{\partial\xi_o} > 0$ and $\operatorname{Re} \lambda_1(\hat{x}_m, \hat{\xi}_m) > 0$ if $\frac{\partial\mu(\hat{x},\hat{\xi})}{\partial\xi_o} < 0$.

We assume also that

$$\frac{(\operatorname{Im} \lambda_1(\hat{x}_m, \hat{\xi}_m))^2 \ln |\operatorname{Re} \lambda_1(\hat{x}_m, \hat{\xi}_m)|}{\operatorname{Re} \lambda_1(\hat{x}_m, \hat{\xi}_m)} \to 0 \text{ when } m \to \infty \tag{1.17}$$

and

$$|\{\lambda_1(x', \xi'), \mu(x',0, \xi')\}| < C|\operatorname{Re} \lambda_1(x', \xi')|^{\delta_1} \text{ for } (x', \xi') = (\hat{x}_m, \hat{\xi}_m) \tag{1.18}$$

where δ_1 is an arbitrary positive constant and C, δ_1 are independent on m .

Theorem 1. 3 : Suppose that there exists $(\hat{x}, \hat{\xi}) \in N_o$ such that (1.16), (1.17), (1.18) hold and that the boundary Γ is strictly convex at the point $(\hat{x},\hat{\xi})$. Then the initial-boundary problem (1.1) (1.2) (1.3) is ill-posed in the space of distributions. More precisely, there exists $h(x') \in C_o^\infty(\Gamma)$, $h(x') = 0$ for

$x_o < 0$ such that there is no $u(x) \in \mathcal{D}'(\Omega_T)$ which is a solution of the problem (1.1), (1.2), (1.3) for $x_o < T$.
Here $T > \hat{x}_o$ where $\hat{x} = (\hat{x}_o, \hat{x}_1, \ldots, \hat{x}_{n-1})$, $(\hat{x}, \hat{\xi})$ is the point satisfying (1.14), (1.16), (1.17), (1.18) and $\mathcal{D}'(\Omega_T)$ is the space of distributions on Ω_T.

Remark 1.2 : It is obvious that (1.17) is satisfied, for example, when $\mathrm{Im}\,\lambda_1(x', \xi') \equiv 0$. We note that (1.18) implies

$$(1.18') \qquad \{\lambda_1(x', \xi'), \mu(x', 0, \xi')\} = 0 \quad \text{for} \quad (x', \xi') = (\hat{x}, \hat{\xi}).$$

One can show that when $\mathrm{Im}\,\lambda_1(x', \xi') \equiv 0$ and (1.16), (1.18') hold then there exists a sequence $(\tilde{x}_m, \tilde{\xi}_m) \in N_o$ such that $(\tilde{x}_m, \tilde{\xi}_m) \to (\hat{x}, \hat{\xi})$, $|\tilde{\xi}_m| = 1$, $\lambda_1(\tilde{x}_m, \tilde{\xi}_m).\mathrm{sgn}\,\dfrac{\partial\mu(\hat{x}, \hat{\xi})}{\partial\xi_o} < 0$ and (1.18) holds for $(x', \xi') = (\tilde{x}_m, \tilde{\xi}_m)$ with $\delta_1 = \frac{1}{2}$.

Theorem 1.4 : Let the boundary Γ be strictly convex and the assumptions (1.6), (1.8) and (1.10) hold. Suppose that for each $(\hat{x}, \hat{\xi}) \in N_o$ such that $b(\hat{x}, \hat{\xi}) = 0$ and (1.15) is not satisfied we have

$$(1.19) \qquad \{\lambda_1(x', \xi'), \mu(x', 0, \xi')\} \neq 0 \quad \text{for} \quad (x', \xi') = (\hat{x}, \hat{\xi})$$

Then for each $h \in H_{s+r-1+m}(\Gamma_T)$, $h = 0$ for $x_o < 0$, there exists a unique solution $u \in H_s(\Omega_T)$ of the problem (1.1), (1.2), (1.3) where T is sufficiently small and

$$(1.20) \qquad m > \max \frac{1}{3} \; \frac{-\{\xi_n - \lambda_1, \mu\}}{|\{\lambda_1, \mu\}|}$$

where the maximum in (1.20) is taken over all $(\hat{x}, \hat{\xi}) \in N_o$ such that $b(\hat{x}, \hat{\xi}) = 0$ and (1.15) is not satisfied .

Remark 1.3 : Part of the results containing in the Theorems 1.1 and 1.3 was proved by M. Ikawa (see [7] , [8] , [9]). The Theorem 1.2 has an intersection with results of M. Miyatake [11] , L. Gårding [5] and R. Melrose and J. Sjöstrand [10] .

Remark 1.4 : We have described also the singularities of the solution of the problem (1.1), (1.2), (1.3) under the assumptions of the Theorems 1.1 and 1.4 and additional assumptions that $b_1(x', \xi'')$ and $b_2(x', \xi')$ are real (see (1.9) and (1.11)).

Remark 1.5 : Roughly speaking the phenomen of ill-posedness of the problem (1.1) (1.2), (1.3) under the assumptions of the Theorem 1.3 can be explained in a following way :
When (1.16) holds and $\lambda_1(x', \xi')$ is real there is a propagation of singularities along the boundary (the boundary waves). These singularities are overgoing multiple reflections the number of which tends to the infinity for any time interval containing $\hat{x}_o$ when the reflecting points approach $(\hat{x}, \hat{\xi})$. Therefore under the assumptions of the Theorem 1.3 the Green function of the problem (1.1), (1.2), (1.3) has a singularity of an infinite order, i.e. it is not a distribution. Indeed it can be shown for the model initial boundary problem (see the section 3) that under the assumptions of the Theorem 1.3 the problem (1.1), (1.2), (1.3) is well-posed in the space of ultra-distributions, i.e. functionals over C^∞ -functions belonging to some Gevrey classes. More precisely such ultradistributions have a finite order of singularities outside of N_o which increases when we approach N_o and which becomes infinite only on N_o.

It follows from the Theorem 1.4 that when (1.18') don't take place there is still a well-posedness of the problem (1.1), (1.2), (1.3). We note that when $\{\mu, \lambda_1\} \to 0$ then $m \to \infty$ (see (1.20)) in accordance with the Theorem 1.3.

§2. EXAMPLES

We give now two simple examples which illustrate the Theorems 1.1, 1.2, 1.3, 1.4.

Example 2.1 : Let G be a strictly convex bounded domain in $\mathbb{R}^2$ and $A(x, \partial) \equiv \square = \frac{\partial^2}{\partial x_o^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ is the wave operator. Consider the following initial boundary problem

(2.1) $$\square u = 0 \quad \text{for} \quad x \in \Omega$$

(2.2) $$u = 0 \quad \text{for } x_o < 0, \quad x \in \Omega$$

(2.3) $$\left. \frac{\partial u}{\partial n} - a(x_o) \frac{\partial u}{\partial x_o} \right|_\Gamma = h ,$$

where $\frac{\partial u}{\partial h}$ is the normal derivative to ∂G and $a(x_o)$ is a real C^∞ function.

The condition (1.6) gives

$$a(x_o) > -1. \tag{2.4}$$

If $a(x_o) \geq 0$ then all assumptions of the Theorem 1.2 are satisfied.
If all zeros of $a(x_o)$ are of the first order and (2.4) holds then according to the Theorem 1.4 the problem (2.1), (2.2), (2.3) is well-posed.
If $x_o^{(o)}$ is a zero of $a(x_o)$ and $a'(x_o)$ such that there exists a sequence $\{x_o^{(n)}\}$ where $a(x_o^{(n)}) < 0$ and $x_o^{(n)} \to x_o^{(o)}$ then it follows from the Theorem 1.3 that the problem (2.1), (2.2), (2.3) is ill-posed.

We note that if $\Omega = (-\infty, +\infty) \times \complement G$ where $\complement G$ is the exterior of the domain G then according to the Theorem 1.1 the problem (2.1), (2.2), (2.3) is well-posed under the only assumption (2.4).

<u>Example 2.2</u> : Let G be the same domain as in the example 2.1. Consider the following boundary operator

$$\frac{\partial u}{\partial h} + b(s)\frac{\partial u}{\partial s}\Big|_{\Gamma} = h , \tag{2.5}$$

where $\frac{\partial}{\partial n}$ is the normal derivative to ∂G, $\frac{\partial}{\partial s}$ is the tangential derivative and $b(s)$ is a real C^∞ -function.

It follows from the Theorem 1.4 that the problem (2.1), (2.2), (2.5) is well-posed if $b(s)$ has no roots on ∂G of the multiplicity greater than one . If there is a point s_o such that $b(s_o) = b'(s_o) = 0$ and $b(s) \not\equiv 0$ then the Theorem 1.3 gives that the problem (2.1), (2.2), (2.5) is ill-posed.

The exterior problem (2.1), (2.2) (2.5) in the domain $\Omega = (-\infty, +\infty) \times \complement G$ is well-posed for any real $b(s) \in C^\infty$ (see the Theorem 1.1) and [7]).

§3. A MODEL FOR THE THEOREM 1.3

To clarify the results of the section 1 we shall consider some model equations with model boundary conditions. A good model for the initial-boundary problem in the case of convex domain is the following problem in $\mathbb{R}^{n+1}_+$:

$$a_-(x_n, \mathcal{D})u = 0, \quad x \in \mathbb{R}^{n+1}_+ , \tag{3.1}$$

$$u = 0 \qquad \text{for } x_o < 0 , \ x \in \mathbb{R}^{n+1}_+ , \tag{3.2}$$

$$B(\mathcal{D})u\Big|_{x_n=0} = h(x') , \tag{3.3}$$

where $h(x')$ is a tempered distribution, $h = 0$ for $x_o < 0$,
$a_-(x_n, \xi) = \xi_n^2 - (\xi_o - x_n|\xi''|)\,|\xi''|$,
$B(\xi) = \xi_n - B_1(\xi'')$, $B_1(\xi'') \in C^\infty$ for $\xi'' \neq 0$ and $B_1(\xi'')$ is a homogenicus function of degree 1.

The initial-boundary Dirichlet problem for the equation (3.1) in $\mathbb{R}^{n+1}_+$ was considered in [4].

As it was stated in [4] the unique solution of (3.1) in the class of tempered distributions which satisfies the initial data (3.2) and the Dirichlet condition $u|_{x_n=o} = v$, $v = 0$ for $x_o < 0$ is given by the following formula

$$(3.4)\quad h(x',x_n) = \frac{1}{(2\pi)^n}\int_\infty^\infty \frac{A_o\left(\frac{\xi_o+i\tau_o}{|\xi''|^{1/3}} - x_n|\xi''|^{\frac{2}{3}}\right)}{A_o\left(\frac{\xi_o+i\tau_o}{|\xi''|^{1/3}}\right)} e^{-i(x'',\xi'')-i(\xi_o+i\tau_o)x_o}\tilde{v}(\xi_o+i\tau_o,\xi'')\,d\xi_o d\xi''$$

where $\tau_o > 0$ is arbitrary and $A_o(Z)$ is the Airy function with the following properties

$$(3.5)\quad \begin{aligned} A_o(t) &\approx \frac{C}{|t|^{1/4}} e^{-\frac{2}{3}|t|^{3/2}} \quad \text{for } t \to -\infty \\ A_o(t) &\approx \frac{C}{t^{1/4}} \sin\left(\frac{2}{3}t^{3/2} + \frac{\pi}{4}\right) \quad \text{for } t \to +\infty . \end{aligned}$$

To solve the problem (3.1), (3.2), (3.3) it is enough to find $v(x')$ such that

$$(3.6)\quad b_{\tau_o}(\mathcal{D})v = h,$$

where $b_{\tau_o}(\mathcal{D})$ is a pseudodifferential operator with a symbol

$$(3.7)\quad b_{\tau_o}(\xi_o,\xi'') = b(\xi_o+i\tau_o,\xi'') = -i|\xi''|^{2/3}\frac{A_o'((\xi_o+i\tau_o)|\xi''|^{-1/3})}{A_o((\xi_o+i\tau_o)|\xi''|^{-1/3})} - B_1(\xi'').$$

The necessary and sufficient condition for the existence of the solution $v(x')$ of the equation (3.6) which is a tempered distribution in $(-\infty,T)\times\mathbb{R}^{n-1}$ for any finite T is the following : there exist C_1, C_2, C_3, N such that

$$(3.8)\quad |b(\xi_o + i\tau, \xi'')| \geq C_1(1 + |\xi'| + \tau)^{-N}$$

for any $\tau > C_2 \ln |\xi'| + C_3$.

Suppose that there exists ω''_o , $|\omega''_o| = 1$ such that $B_1(\omega''_o) = 0$ and suppose that there is a sequence ω''_m , $|\omega''_m| = 1$ such that

(3.9) $$\operatorname{Re} B_1(\omega''_m) < 0 , \quad \omega''_m \to \omega''_o$$

and

(3.10) $$(\operatorname{Im} B_1, (\omega''_m))^2 < \frac{1}{N} \frac{|\operatorname{Re} B_1(\omega_m)|}{|\ln|\operatorname{Re} B_1(\omega_m)||} \quad \text{for } m > m_N ,$$

where $N > 0$ is arbitrary, m_N is depending on N.

Let K_o be an arbitrary zero of the Airy function $A_o(Z)$, i.e.

(3.11) $$A_o(K_o) = 0$$

Take a sequence $|\xi''_m| \to \infty$ such that

(3.12) $$\frac{1}{|\xi''_m|^{1/3}} \frac{1}{B_1(\omega''_m)} \to 0 \quad \text{for } m \to \infty \ .$$

We note that (3.10) implies $\operatorname{Re} \frac{1}{B_1(\omega''_m)} \to -\infty$ when $m \to \infty$.

For example, take $|\xi''_m|^{1/3} = |\operatorname{Re} B_1(\omega''_m)|^{1+\varepsilon}$, $\varepsilon > 0$. It is easy to show that there exists a root $\xi_o^{(m)} + i\tau_m$ of the equation

$$\frac{A_o((\xi_o + i\tau)|\xi''_m|^{-1/3})}{A'_o((\xi_o + i\tau)|\xi''_m|^{-1/3})} = - \frac{i}{|\xi''_m|^{1/3} B_1(\omega''_m)}$$

which has the following asymptotic behaviour

$$\xi_o^{(m)} + i\tau_m = K_o|\xi''|^{-1/3} - \frac{i}{B_1(\omega''_m)} + \ldots$$

Therefore we have

(3.13) $$b(\xi_o^{(m)} + i\tau_m, |\xi''_m|\omega''_m) = 0$$

where

(3.14) $$\tau_m \approx -\mathrm{Re}\,\frac{1}{B_1(\omega_m)} \geq CN\ \ln|\xi''_m| \ ,$$

N is arbitrary, $m > m_N$.

Since the condition (3.8) is not satisfied we obtain that the problem (3.1), (3.2), (3.3) is ill-posed.

<u>Remark 3.1</u> : <u>Solution of the problem (3.1), (3.2), (3.3) in the space of ultradistributions.</u>

We shall show that the solution of the ill-posed problem (3.1), (3.2) (3.3) exists in some class of ultradistributions. The general initial boundary problems for hyperbolic equations in the spaces of ultradistributions was studied by J. Chazarain [2] and R. Beals [1] , but their results are not precise. We note that the Cauchy problem for partial differential equations with constant coefficients in the spaces of ultradistributions was studied at first by G. E. Shilov and his students (see [6]). Denote by $\tau_0(\xi')$ the following function :

$$\tau_0(\xi') = \frac{C}{\sqrt{\alpha}} \ln(1+|\xi''|) \quad \text{for} \quad \alpha = \frac{\xi_0}{|\xi''|} > \frac{C_1}{|\xi''|^{2/3}}$$

$$\tau_0(\xi') = C_2|\xi''|^{1/3} \quad \text{for} \quad |\alpha| < \frac{C_1}{|\xi''|^{2/3}}$$

$$\tau_0(\xi') = 0 \quad \text{for} \quad \alpha < -\frac{C_1}{|\xi''|^{2/3}} \ .$$

One can show assuming that the condition (1.6) holds that

(3.15) $$|b(\xi_0 + i\tau \ , \xi'')| \geq C \quad \text{for } \tau \geq \tau_0\ (\xi') \ .$$

Let $\varphi(x_0,x'') \in C_0^\infty(\mathbb{R}^n_+)$ and $\varphi(x_0,x'')$ belongs to the Gevrey class $G_{3-\varepsilon}$ of order $3-\varepsilon$ in x". Then

(3.16) $$|\tilde{\varphi}\ (\xi_0+i\tau \ , \xi'')| \leq C_N(1 + |\xi'|+\tau\ )^{-N} e^{-N|\xi''|^{1/3}} \quad \text{for any N.}$$

Let $h(x_0,x'')$ be an arbitrary tempered distribution, h = 0 for $x_0 < 0$. We define $v(x')$ by the following formula

$$v = \frac{1}{(2\pi)^n} \int_L \frac{\tilde{h}(z, \xi'')}{b(z, \xi'')}\, e^{-ix_0 z - i(x'', \xi'')}\, dz\, d\xi'' \ ,$$

where L is a curve $z(\xi_0) = \xi_0 + i\tau\ (\xi_0)$ in the complex plane $\mathbb{C}$ such that $C\ |\xi''|^{1/3} \geq \tau(\xi_0) \geq \tau_0(\xi')$.

One can see that $v(x')$ is a ultradistribution belonging to $G'_{3-\varepsilon}$ and $v(x')$ is a solution of the equation (3.6). We note that $v(x)$ is a ultradistribution microlocally only for $|\xi_0| < \varepsilon\ |\xi''|$. Let χ be a pseudodifferential operator with a symbol $\chi(\frac{\xi_o}{|\xi''|})$ where $\chi(t) \in C^\infty(\mathbb{R}^1)$, $\chi(t) = 0$ for $|t| < \varepsilon$. Since

$$e^{x_o \tau_o(\xi')} < e^{\frac{x_o}{\sqrt{\varepsilon}} \ln(1+|\xi''|)} = (1+|\xi''|)^{\frac{x_o}{\sqrt{\varepsilon}}} \quad \text{for } \alpha = \frac{\xi_o}{|\xi''|} > \varepsilon$$

we have that $\chi(\mathcal{Z})v$ belongs to a Sobolev's space of the order $\frac{T}{\sqrt{\varepsilon}}$ when $x_o < T$.

§4. A MODEL FOR THE THEOREM 1.4

In this section we consider the equation (3.1) in $\mathbb{R}^{n+1}_+$ with the initial data (3.2) and with the following boundary condition

$$(4.1) \qquad i\frac{\partial u}{\partial x_n} + \gamma(x_o - a)\Lambda''u\Big|_{x_n=0} = h(x'),$$

where h is a tempered distribution, $h = 0$ for $x_o < 0$, Λ'' is a pseudodifferential operator with the symbol $|\xi''|$, $a > 0$ and $\gamma \neq 0$. We note that the condition (1.6) gives that $\operatorname{Re}\gamma.\operatorname{Im}\gamma < 0$.

As in §3 (see (3.6)) we reduce the solution of the problem (3.1), (3.2), (4.1) to the solution of the following equation in $\mathbb{R}^n$:

$$(4.2) \qquad N^\tau v_\tau + \gamma(x_o - a)\Lambda''v_\tau = h_\tau$$

where $v_\tau = e^{-x_o\tau} v$, $h_\tau = e^{-x_o\tau} h$, $v(x') = u(x',0)$ and N^τ is the Neumann operator, i.e.

$$(4.3) \qquad N^\tau v_\tau = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} e^{-i(x',\xi')}(-i|\xi''|^{2/3})\frac{A_o'((\xi_o+i\tau)|\xi''|^{-1/3})}{A_o((\xi_o+i\tau)|\xi''|^{-1/3})}\tilde{v}_\tau(\xi')d\xi' \quad .$$

If we take the Fourier transform in (4.2) we obtain an ordinary differential equation of the first order :

$$(4.4) \qquad N^\tau\tilde{v}(\xi_o+i\tau) - i\gamma|\xi''|\frac{d\tilde{v}(\xi_o+i\tau, \xi'')}{d\xi_o} - \gamma a|\xi''|\tilde{v} = \tilde{h}(\xi_o+i\tau, \xi''),$$

where

$$(4.5) \qquad N^\tau = (-i|\xi''|^{2/3})\frac{A_o'(Z)}{A_o(Z)}, \quad z = (\xi_o + i\tau)|\xi''|^{-1/3}.$$

The general solution of the homogeneous equation (4.4), i.e. with h = 0, is given by the formula

$$(4.6)\qquad w(\xi_o+i\tau,\xi") = C(\xi")A_o^{-\frac{1}{\gamma}}((\xi_o+i\tau)|\xi"|^{-1/3})e^{ia(\xi_o+i\tau)},$$

where we take a branch of $Z^{-\frac{1}{\gamma}}$ which is positive for the positive Z.

By using (4.6) one can show that the equation (4.4) has a unique solution which is a tempered distribution and so that the problem (3.1), (3.2), (4.1) is well-posed. The following indirect way to solve the equation (4.2) is more useful : in the region $\xi_o < 0$ the symbol N^τ is a symbol of class $S_{1/3,0}$ and so that there is no difficulties to solve the equation (4.2) microlocally in this region. Let now $\xi_o > 0$. Denote by $\chi_1(t)$ a C^∞-function such that $\chi_1(t) = 1$ for $t > 1$, $\chi_1(t) = 0$ for $t < 0$. Let $\chi_1(\xi_o|\xi"|^{-1/3})$ be a pseudodifferential operator with a symbol $\chi_1(\xi_o|\xi"|^{-1/3})$.

Let $B\chi_1$ be the following operator

$$(4.7)\qquad B\chi_1 V = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} B(Z)\,\chi_1(\xi_o|\xi"|^{-1/3})\tilde{V}(\xi')e^{-i(x',\xi')}d\xi' ,$$

where $Z = (\xi_o+i\tau)|\xi"|^{-1/3}$, $B(z) = \frac{A_1(Z)}{A(Z)}$, $A_1(Z) = \overline{A(\bar{Z})}$ and A(Z) is the Airy function with the following asymptotic behaviour for $Z \to +\infty$ (see, for example [4])

$$(4.8)\qquad A(Z) \approx C\, Z^{-1/4} e^{-2/3\; iZ^{3/2}}$$

We note that $A_o(Z) = A(Z) - A_1(Z)$.

The operator $B\chi_1$ is a Fourier integral operator with the phase function $\varphi(X',\zeta') = (X',\zeta') - \frac{4}{3}\zeta_o^{3/2}|\zeta"|^{-1/2}$. It was indicated in [4] that the phase function $\varphi(X',\zeta')$ for $\zeta_o > 0$ is the generating function for the canonical transformation $(y',\zeta') \to (x',\xi')$ such that the image (X_o',ξ_o') of a point $(y_o',\zeta_o'),\zeta_o^{(o)} > 0$ is the endpoint of an outgoing bicharacteristics in $\mathbb{R}_+^{n+1}$ of the operator (3.1) which starts at the point (y_o',ζ_o').

It is easy to verify that the following identity holds

$$(4.9)\qquad N^\tau\chi_1 V + \gamma\,(X_o-a)\,\Lambda"\chi_1 V = (I-B)^{-1/\gamma}\,L_o(I-B)^{1/\gamma}\chi_1 V$$

where L_o is a pseudodifferential operator with a symbol $-i|\xi"|^{2/3}\frac{A'(Z)}{A(Z)} + \gamma(X_o-a)\Lambda"$, $Z = (\xi_o+i\tau)|\xi"|^{-1/3}$.

We note that an operator of the form L_o arises also when one consider the case of the concave boundary. It is not difficult to prove the existence of the inverse of L_o and therefore to find the inverse of (4.9).

It was shown in [4] that

$$(4.10) \qquad 1 - |B(Z)| > \frac{C\tau}{|\xi''|^{1/3}}(1 + \sqrt{\alpha}|\xi''|^{1/3}) \quad \text{for } \alpha = \frac{\xi_o}{|\xi''|} > 0$$

One can obtain from (4.10) an a priori estimate for the solution of the equation (4.2) with a loss of $\frac{1}{3}\ \frac{1}{\gamma}+\frac{2}{3}$ derivatives.

The identity (4.9) allows also to describe the wave front set of the solution $v(x')$.

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