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HYPOELLIPTIC OPERATORS WITH CHARACTERISTIC VARIETY OF CODIMENSION TWO AND THE WAVE EQUATION

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INTRODUCTION

We consider a direct Fourier integral method for analysing the simplest type of hypoelliptic operator with double characteristic - where the characteristic variety is a C^∞ symplectic manifold of codimension two. Of course, the structure of these operators is well known (see for example [1]). However the method discussed here, using the Hermite transform, also allows the direct solution of some related problems.

In particular associated wave equations can be solved in a "quasi-classical" manner, giving Poisson relations and optimal estimates on the Weyl expansion for hypoelliptic operators, as well as propagation of singularities results for microlocally hyperbollic operators of analogous type. Moreover, when the condition of hypoellipticity with loss of one derivative is violated, standard results can be applied to an induced operator yielding more subtle conditions for hypoellipticity. A theorem of Weyl's type for the non-zero eigenvalues of Kohn's Laplacian on a compact strictly pseudoconvex hypersurface in $\mathbb{C}^2$ can also be obtained.

Similar results on the propagation of singularities have been found by B. Lascar [12] and for the invariant Laplacian on the Heisenberg group by A. Nachman. A non-homogeneous version of the Hermite transform (not therefore a Fourier integral operator) has also been studied by Guillemin and Sternberg [5] from a more group theoretic point of view.

§ 1. RESULTS

Recall that an odd dimensional submanifold of a symplectic manifold on which the symplectic form has maximal rank carries a natural, "Hamilton", 1-foliation given by the radical of the symplectic form. If a C^∞ function p on a symplectic manifold, E, vanishes to second order at a point $\rho \in E$ then the fundamental matrix of p at ρ is the linear map

$$F_\rho(p) : T_\rho E \ni v \mapsto vH_p \in T_\rho F$$

where H_p is the Hamilton vector field of p.

Theorem 1 : Suppose that P is a classical pseudodifferential operator with real principal symbol, p, on a manifold M. If

$$\Sigma_2(P) = \{dp = 0\} \subset T^*M\setminus 0$$

is a C^∞ manifold of codimension 3 on which the symplectic form has maximal rank, the Hessian of p has rank 3 and negativity 1 and any section of $N^*\Sigma_2$ with Hamilton field non-zero and in the 1-foliation of Σ_2 is in the negative cone of the dual of the Hessian then the fundamental matrix has eigenvalues 0, $\pm$ i tr^+ where $tr^+ \in C^\infty(\Sigma_2)$ is positive. Suppose in addition that

(1) $$\sigma_{sub}(P) + \frac{1}{2} tr^+ : \Sigma_2 \to (0,\infty).$$

Then, any distribution u with $Pu \in C^\infty$ has $WF(u) \cap \Sigma_2$ a complete union of integral curves, maximally extended, of Σ_2.

This theorem is proved by the application of Duhamel's principle after the construction of a microlocal parametrix for a suitable Cauchy problem for P. In particular the result is valid microlocally.

Example : Suppose that

(2) $\left\{ \begin{array}{l} \text{A is a classical pseudodifferential operator with non-negative principal} \\ \text{symbol, a , vanishing to precisely second order on a symplectic manifold,} \\ \Sigma_2, \text{ of codimension 2.} \end{array} \right.$

Then, provided A has order two the hypotheses of Theorem 1 are satisfied by

$$P = D_t^2 - A$$

exactly when the analogue of (1) holds for A :

(3) $$\sigma_{sub}(A) + \frac{1}{2} tr^+ : \Sigma_2 \to (0,\infty).$$

Theorem 2 : Suppose that A is a classical pseudodifferential operator, acting on $\frac{1}{2}$ - densities on a compact manifold M, that A is self-adjoint of order q > 1 and that (2), (3) hold. Then, as an unbounded self-adjoint operator on $L^2(M, \Omega^{1/2})$ A has discrete spectrum :

$$spec(A) = \{\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \to \infty\} ,$$

with

(4) $$N(\lambda) = \inf\{k;\ \lambda_k \geqslant \lambda\}$$

of polynomial growth. The Poisson distribution

$$\hat{\sigma}(t) = \sum_{\mu_j > 0,\ \mu_j^q \in \operatorname{spec}(A)} e^{i\mu_j t}$$

converges in $\mathcal{S}'(\mathbb{R})$, its singularities occur only at 0 and at the set, L , of signed lengths of closed bicharacteristics of a away from Σ_2 :

(5) $$\text{sing.supp}\ \hat{\sigma} \subset \{0\} \cup L.$$

L is discrete and near 0 $\hat{\sigma}$ is a Lagrangian distribution. There is a symbol $\tilde{\sigma}(\lambda)$ vanishing in $\lambda \leqslant 0$ with

(6) $$\tilde{\sigma}(\lambda) \sim \sum_j (\alpha_j \lambda^{m_j} + \beta_j \lambda^{m_j'} \log \lambda) \qquad \lambda \to \infty \quad ,$$

where m_j, $m_j' \to \infty$ as $j \to \infty$, such that

(7) $$|N(\lambda^q) - \int_{-\infty}^{\lambda} \tilde{\sigma}(s)ds| \leqslant C(\tilde{\sigma}(\lambda) + 1).$$

<u>Remarks</u> : a) The Poisson relation (5) is obtained by microlocal analysis of the Cauchy problem for

(8) $$P' = D_t - B$$

where B is an essentially positive q^{th}-root of A, constructed operator theoretically. Of course, P' is therefore not a classical pseudodifferential operator, so Theorem 1 does not apply directly. However, the same approach via the Hermite transform allows B to be constructed microlocally and Theorem 1 carries over unchanged. Note that (5) cannot be deduced from Theorem 1, as was done in the elliptic case by Chazarain [3] and Duistermaat and Guillemin [4] since the calculus of (conic) wavefront sets does not exclude the possibility that $\hat{\sigma}$ is everywhere singular.

b) Since the double characteristics play no role in the singularities of $\hat{\sigma}(t)$ away from 0 the formulae obtained for the leading part of these singularities by Duistermaat and Guillemin [4] , under suitable regularity and non-degeneracy conditions on the closed bicharacteristics carry over immediately.

c) The Weyl estimate (7) follows by direct application of Hörmander's simple Tauberian theorem [6] once it is shown that $\hat{\sigma}(t)$ is Lagrangian at 0 with symbol $\tilde{\sigma}(\lambda)$. The exponents in (7) are a mixture of arithmetic progressions. There is the classical progression

$$M_1 = n - 1 - \mathbb{N} \qquad\qquad n = \dim M$$

and a double progression associated to the characteristic variety :

$$M_q = \frac{q(n-1)}{q-1} - 1 - q\mathbb{N} - \mathbb{N}\ .$$

(9) $$m_j \in M_1 \cup M_2\ .$$

The logarithmic terms only occur if q is rational, $q - 1 = a/b$ $a,b \in \mathbb{N}$ relatively prime. Then

(10) $$m_j' \in n'-1-\mathbb{N}$$

where $n' = n$ if $1 \leqslant q-1 \leqslant n-1$ is integral and $n' = n-b$ otherwise. In all cases the m_j' are amongst the m_j in (9). The leading term in (7) was calculated by Menikoff and Sjöstrand [10] using heat equation techniques. The existence of the expansion (6) implies a similar expansion for the trace of the heat kernel. Formulae for certain other coefficients can be obtained in some cases, notably in the case $q = n$ when the leading classical term, α_0, can be obtained as a regularization of the usual volume of Weyl.

d) The remainder estimate in (7) can be further improved by the assumption that the closed bicharacteristics have measure zero by following [4] using b).

e) When (3) is replaced by the known necessary and sufficient condition for hypoellipticity with loss of one derivative

(11) $$\sigma_{sub}(A) + (k + \tfrac{1}{2})\,\mathrm{tr}^+ \neq 0 \quad \text{on } \Sigma_2 \qquad \forall\, k \in \mathbb{N}$$

similar results hold. Without (3) A is not bounded from below, however (4), (5) are valid, although they only involve the positive eigenvalues. Defining analogous objects for the negative eigenvalues

(12) $$\hat{\sigma}_-(t) = \sum_{\mu_j > 0,\, -\mu_j^q \in \operatorname{spec}(A)} e^{i\mu_j t}$$

the Poisson relation simplifies to

$$\hat{\sigma}_- \in C^\infty(\mathbb{R}\setminus\{0\})$$

with σ_- Lagrangian at 0 with classical symbol :

(13) $$\tilde{\sigma}_-(\lambda) \sim \sum_j \alpha_j^- \lambda^{m_j}$$

with exponents as in (9). The leading coefficient was found in [10].

f) In the hypoelliptic case e) there are only finitely many finite eigenvalues. If on the contrary (11) is identically violated on one or more components $\Sigma_2^{(j)}$ of Σ_2 :

(14) $$\sigma_{sub}(A) \equiv -(k_j + \tfrac{1}{2})\,\mathrm{tr}^+ \qquad \text{on } \Sigma_2^{(j)}$$

but still holds on the remaining components and A is of order two then the conclusions of e) remain valid provided $N_\pm$, $\hat{\sigma}_\pm$ are defined using only the eigenvalues of appropriate sign outside a sufficiently large interval around 0. These results apply in particular to Kohn's Laplacian $\square_b$ on functions on a strictly pseudo convex compact hypersurface in $\mathbb{C}^2$.

Theorems 1 and 2 are proved by the same basic method. First the operator concerned is reduced, microlocally, to normal form. Then the Hermite transform discussed below is used to reduce the problem to one on the circle. This must be analysed globally in S^1 (but only microlocally in the other variables) and this is readily done using product-type pseudodifferential and Fourier integral series operators. As a byproduct of such analysis one obtains : (cf. [13], see also [14])

<u>Theorem 3</u> : Suppose A is a classical pseudodifferential operator on a manifold M and (2) holds near $\rho \in \Sigma_2$. If (11) is false at ρ then there is a germ of canonical transformation

$$\chi : \Sigma_2, \rho \to T^*\mathbb{R}^{n-1}, \rho'$$

and a pseudodifferential operator $\hat{A}$ of order q-1, on $\mathbb{R}^{n-1}$, such that near ρ

$$\sigma_{sub}(A) + (k + \tfrac{1}{2})\ \mathrm{tr}^+(A) = \chi^*(\sigma(\tilde{\hat{A}}))$$

(= 0 at ρ) and A is hypoelliptic at ρ if, and only if, $\tilde{A}$ is hypoelliptic at ρ'.

The usual condition (11) corresponds to the ellipticity of $\tilde{\tilde{A}}$.

§ 2. HERMITE TRANSFORM

The first step in the proof of Theorems 1-3 is the reduction to an operator associated to the homogeneous harmonic oscillator.

Proposition 1 : If A is of order $q \neq 1$ and satisfies (2) near $\rho \in \Sigma_2$ then there is a germ of canonical diffeomorphism

$$\chi : T^*M,\rho \longrightarrow T^*(\mathbb{R}_x \times \mathbb{R}_y^{n-1}),\ \rho'$$

$\rho' = (0,0;0,\tilde{\eta})$, $\tilde{\eta} = (0,0,\ldots,0,\tilde{\eta}_{n-1})$, $\tilde{\eta}_{n-1} > 0$ in canonical coordinates $(x,y;\ \xi,\eta)$ and associated Fourier integral operators of order zero

$$F : \mathcal{E}'(M) \to \mathcal{E}'(\mathbb{R}^n),\ G : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(M)$$

which are microlocal inverses at ρ :

(15) $$GF \equiv Id \text{ at } \rho \text{ , } FG \equiv Id \text{ at } \rho'$$

such that

(16) $$\begin{cases} A \equiv G'.A'.F \quad \text{at } \rho \\ A' \equiv D_{y_{n-1}}^{q-2}\ [D_x^2 + x^2 D_{y_{n-1}}^2 + S(y_1,\ldots,y_{n-2} + \dfrac{xD_x}{2D_{y_{n-1}}},D_y)] \end{cases}$$

where S is an operator of order 1 with classical symbol.

This proposition represents, in this special case, an improvement over the normal form result of Boutet de Monvel only in so far as the simplified operator A' on $\mathbb{R}^n$ is microlocally conjugate to A, without the occurrence of an elliptic factor. This is important in a discussion of eigenvalues. It is equally possible to obtain, in (16), the lower order terms in the apparently simpler form $S(y,D_y)$. The reason for (16) is clarified by (17) below. A result similar to Proposition 1 has been obtained by M. E. Taylor (private communication).

The principal symbol of the transformed operator A' is $(x^2\eta_{n-1}^2 + \xi^2)\ \eta_{n-1}^{2-q}$. A standard method from celestial mechanics suggests the introduction of symplectic polar coordinates, $\nu(x,y;\ \xi,\eta) = (\theta\ ,y';\ r,\ \eta')$

$$(17)\quad \begin{cases} x = \sqrt{2}\, r^{1/2}\, \eta_{n-1}^{-1/2} \cos\theta & y_j = y'_j \qquad j < n-1 \\ \xi = -\sqrt{2}\, r^{1/2}\, \eta_{n-1}^{1/2} \sin\theta & \eta = \eta' \\ y_{n-1} = y'_{n-1} + \frac{r}{2}\sin 2\theta\ . \end{cases}$$

The square-root in (17) which gives

$$r = \frac{1}{2}(x^2 + \xi^2/\,\eta_{n-1}^2)$$

makes ν symplectic and also ensures that

$$(18)\qquad \operatorname{graph}(\nu) \subset T^*(S^1 \times \mathbb{R}^{n-1}_{y'}) \times T^*(\mathbb{R}^n)$$

is a C^∞ canonical relation, at least in $|(\xi,\eta)| < C\eta_{n-1}$ the region of interest.

Using the classical method of Caratheodory [2] one can construct a generating functions for ν. This allows one to implement, or "quantize", ν to a Fourier integral operator :

$$(19)\quad Tu(s,y) = (2\pi)^{-n}(1+s^2)^{1/4} \int e^{ix^2 s\eta_{n-1}/2 + i(y-y')\cdot\eta}\, u(x,y')\,dy'\,dx\,\lambda\eta,$$

which we call the Hermite transform. Here, S^1 is realized as the real projective line $\mathbb{R}\mathbb{P}^1$ in which s is the affine variable along the line (s,1) in $\mathbb{R}^2$. It is clear from (19) that

$$T\,\mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}_s \times \mathbb{R}^{n-1}_y).$$

Applying Plancherel's formula to the x variable in (19) shows that

$$(20)\quad \begin{aligned} Tu(\tfrac{1}{t},y) = {} & (2\pi)^{-\frac{1}{2}-\eta}(1+t^2)^{2/4}\, e^{i\frac{\pi}{4}\operatorname{sgn} t} \\ & \int e^{-i\xi^2 t/2\,\eta_{n-1} + i(y-y')\cdot\eta}\,\hat{u}(\xi,y')\,dy'\,d\xi\,\,d\eta \end{aligned}$$

where $\hat{u}$ is the partial Fourier transform of u in x. Since $t = 1/s$ is another affine variable, covering S^1 together with s, (20) shows that Tu is C^∞ on S^1 apart from the factor $\exp(\frac{i\pi}{4}\operatorname{sgn} t)$. This is just the transition function for the Arnold-Keller-Maslov bundle, L, over $\mathbb{R}\mathbb{P}^1$ thought of as the Grassmannian of Lagrangian planes (lines) in $\mathbb{R}^2$. Thus,

$$T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(S^1 \times \mathbb{R}^{n-1}; L)$$

In $|(\xi,\eta)| < C\eta_{n-1}$, T is a Fourier integral operator associated to the canonical relation (18). Clearly T annihilates odd functions of x. This is its entire kernel in view of the inversion formula :

(21) $$T^* \alpha(\mathcal{L}_{D_\theta}) D_{y_{n-1}}^{1/2} T = Id \quad \text{on} \quad \mathcal{S}_E(\mathbb{R}^n)$$

where T^* is the adjoint with respect to the invariant measures and α is a translation invariant classical pseudodifferential on S^1, of order 1/2 and elliptic in $r \geq 0$. The range of T is the Hardy space

$$\mathcal{S}_H(S^1 \times \mathbb{R}^{n-1}, L)$$

of sections extending holomorphically into the interior of the unit disc with boundary S^1.

Although T is a Fourier integral operator, ν is not a canonical diffeomorphism so Egorov's theorem is not immediately available. Direct computation shows that A' is intertwined by T

(22) $$TA' \equiv A''T \quad \text{on} \quad \mathcal{S}_E$$

with the operator

(23) $$A'' = D_{y_{n-1}}^{q-2} \left(4 \mathcal{L}_{D_\theta} D_{y_{n-1}} + S'(y, D_y)\right)$$

where it is important to observe that (22) is not quite microlocal (see (17)). Singularities at $(0,y,0,\eta) \in \Sigma_2$ are related to the whole circle $\{(\theta,y,0,\eta), \quad \theta \in S^1\}$.

Thus, it is essential to analyse (23) globally on S^1 and since it is translation-invariant this is conveniently done in terms of Fourier series. If L is invariantly trivialized by lifting it to the four-fold cover of $S^1 \hookrightarrow \mathbb{C}$ then the Hardy space of sections of L is spanned by

$$z^{4k+1} \otimes \mathcal{S}(\mathbb{R}^{n-1}) .$$

On each of these spaces A" acts as a pseudodifferential operator of order $q-1$:

(24) $$A''_k = D^{q-2}_{y_{n-1}} \left((4k+1)D_{y_{n-1}} + S'(y,D_y)\right) .$$

It is this operator, for the appropriate k, which appears in Theorem 3.

Note that the operator A' is even under reflection in x, so to discuss it completely it is only necessary to produce a transformation similar to (22) for its action on odd functions. Using T itself a microlocal inverse for the operator $D^{-1}_{y_{n-1}} D_x + ix$, which transforms odd to even functions, can be readily produced. Then one obtains in place of (23)

$$A''_o = D^{q-2}_{y_{n-1}} \left(4 \mathcal{L}_{D_\theta} D_{y_{n-1}} + S'_o(y,D_y)\right)$$

where $\sigma_1(S'_o) = \sigma_1(S') + 2\,\eta_{n-1}$.

§ 3. INDICATION OF PROOFS

The proof of Theorem 2 is based on the fact that $\hat{\sigma}$ can be realized, modulo C^∞, as the trace of the solution operator

$$u_o \longmapsto E(t)u_o = u(t)$$

of the Cauchy problem for P' in (8). Away from Σ_2 B is microlocally a pseudodifferential operator and Fourier integral operator methods apply. Thus it suffices to assume that $WF(u_o)$ is concentrated near Σ_2 in

$$P'u = 0 \qquad u(0) = u_o .$$

A suitable pseudodifferential partition of unity can be constructed so that $WF(u_o)$ can be assumed to lie in a small conic neighbourhood of some $\rho \in \Sigma_2$.

Using the results above one is reduced to solving the Cauchy problem

(25) $$P''v = (D_t - B'(y,\mathcal{L}_{D_\theta},D_y))v = 0, \qquad v(0) = v_o$$

where $v_o \in \mathcal{E}'_H(S^1 \times \mathbb{R}^{n-1}, L)$ has its wavefront set in a conic neighbourhood of $S^1 \times \{(0;0,\tilde{\eta})\}$, and B' is an essentially positive q^{th} root of A" in (23).

The solution to (25) can be obtained via Lax's method [9] for the Cauchy problem. Thus, look for v in the form

$$v \equiv (2\pi)^{-n} \int \sum_{k\geqslant 0} e^{i(\theta-\theta')(4k+1)+(y-y')\eta + i\varphi(t,y,k,\eta)} \tag{26}$$

$$\times\, a(t,y,k,\eta) b(\theta',y',k,\eta)\, v_o(\theta',y') d\theta' dy' d\eta$$

where b is a classical symbol in k,η coming from the partition of unity.

Applying P" to v in (26) Fourier decomposes it and allows the construction of B' as the q^{th} root of

$$D^{q-2}_{y_{n-1}} [(4k+1) D_{y_{n-1}} + S'(y, D_y)]$$

componentwize, uniformly in k. Note that condition (3) says precisely that all the A''_k in (24) are elliptic with positive symbols.

For (26) to solve (25) one needs φ to satisfy the characteristic equation

$$\partial_t \varphi = \gamma^{1/q}(y,k,d_y\varphi) , \qquad \varphi = 0 \quad \text{at} \quad t = 0$$

where $\gamma = [\eta^{2-q}_{n-1}(4k+1)\ \eta_n + S_1(y,\eta))]$ with S_1 the principal symbol of S' in (23). This can be done by Hamilton-Jacobi theory and yields a real symbol which satisfies product-type estimates :

$$|\partial^{\alpha}_{y}\, \partial^{\ell}_{t}\, \partial^{j}_{k}\partial^{\beta}_{\eta}\varphi| \leqslant C(1+k)^{1/q-j}(1+|\eta|)^{1-\frac{1}{q}-|\beta|} \tag{27}$$

in $0 \leqslant k \leqslant \varepsilon\eta_n$, $|\eta| \leqslant C\eta_n$. The transport equations for the symbol a can be solved with a satisfying similar estimates (27), of order 0.

The Poisson relation (5) follows readily from (26). To obtain the asymptotic expansion (6) one needs to further examine φ and a. It turns out these are "product classical". That is, as $\eta_n \to \infty$

$$\varphi \sim \Sigma\varphi_j(t,y,k,\omega)\ \eta_n^{r_j} \qquad\qquad (0 \leqslant k \leqslant \varepsilon\eta_n)$$

where $r_j \to -\infty$ as $j \to \infty$, $\omega = (\eta_1,\ldots,\eta_{n-2})/\eta_{n-1}$ and the φ_j are classical in k :

$$\varphi_j \sim \Sigma \ \varphi_{ij}(t,y,\omega)k^{r'_i} \quad .$$

Then (6) follows by computation and use of

<u>Proposition 2</u> : If $b \in C_o^\infty(\mathbb{R})$ the series

$$\mathcal{N}(a) = \sum_{k \geqslant o} a(k)\ b(k/r\ )$$

defines a continuous linear map $\mathcal{N}$: $S^m_{cl}(\mathbb{R}) \rightarrow S^{n'}_{cll}(\mathbb{R})$ where $m' = \max(0,m+1)$ and $S^{m'}_{cll}(\mathbb{R})$ is the space of classical symbols with logarithmic and constant terms.

<u>CONCLUDING REMARKS</u>

a) It would be interesting to extend these constructions at least to the case where Σ_2 is symplectic of higher codimension; this is not quite straightforward however since, in particular, there is no simple normal form.

b) For operators of the type considered in Theorem 1 there is an analogue of Theorem 3. Namely, Ivrii and Petkov [8] and Hörmander [7] obtain necessary conditions for the solvability of the Cauchy problem for a differential operator of second order. These are readily microlocalized as necessary conditions for the solvability of the microlocal Cauchy problem and in the present case (the simplest examples of non effectively hyperbolic operators) take the form

$$(28) \qquad \sigma_{sub}(P) + \frac{1}{2}\,tr^+(P) \in [0,\infty),$$

weakening condition (1). The methods above reduce the study of sufficiency to the microlocal solvability of the Cauchy problem for an induced operator

$$(29) \qquad D_t^2 - B'(t,y,D_y)$$

under the assumption that B' is of order 1 with non negative principal symbol. When the principal symbol of B' is independent of t in (29) this Cauchy problem is solvable. More generally solvability can presumably be studied using the methods of [7] , [8] .

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