

SÉMINAIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES – ÉCOLE POLYTECHNIQUE

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Diffraction by convex bodies

Séminaire Équations aux dérivées partielles (Polytechnique) (1978-1979), exp. n° 23,
p. 1-9

<http://www.numdam.org/item?id=SEDP_1978-1979___A23_0>

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DIFFRACTION BY CONVEX BODIES

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Exposé n° XXIII

22 Mai 1979

In the spectral theory of the laplacian in exterior domains "distorted plane waves" are fundamental. For the exterior domain $\mathbf{R}^n \setminus K$, where K is a compact set with smooth boundary ∂K , one defines the distorted plane wave $\phi(x, \omega, k)$ for the Dirichlet problem as follows :

- i) $(\Delta + k^2)\phi = 0$ on $\mathbf{R}^n \setminus K$,
- ii) $\phi = 0$ on ∂K (Dirichlet condition),
- iii) $\phi = e^{-ikx \cdot \omega} + v$, where as $|x| \rightarrow \infty$
 $v = |x|^{\frac{1-n}{2}} e^{-ik|x|} (f(\frac{x}{|x|}) + O(\frac{1}{|x|}))$ (Sommerfeld condition).

For a proof of the existence and uniqueness of ϕ satisfying i) - iii) one may consult [11].

This seminar deals with an approximate construction of $\phi(x, \omega, k)$ in the case that K is strictly convex - in the sense that the normal curvatures of ∂K are everywhere strictly positive. The construction is asymptotic to order k^{-N} for any given N as k tends to ∞ , and it permits the explicit asymptotic expansion of two quantities of interest in scattering theory, the scattering phase $s(k)$ and the forward diffraction peak $a(\theta, \theta, k)$. These can be expressed in terms of $\phi(x, \omega, k)$ as follows :

$$\frac{ds}{dk} = \frac{1}{8\pi^2} \left(\frac{k}{2\pi}\right)^{n-3} \int_{|\omega|=1} d\omega \int_{\partial K} \left| \frac{\partial \phi}{\partial \nu} \right|^2 (x \cdot \nu) dS$$

$$a(\theta, \omega, k) = \frac{i}{4\pi} \left(\frac{k}{2\pi}\right)^{n-2} \int_{\partial K} e^{ik\theta \cdot x} \frac{\partial \phi}{\partial \nu} dS ,$$

where ν is the unit normal to ∂K pointing into $\mathbf{R}^n \setminus K$. By substituting the approximations for ϕ into these formulas one can conclude that ds/dk and $a(\theta, \theta, k)$ have complete asymptotic expansions of the form

$$\sim \sum k^{-n_i} (a_i + b_i \log k), \quad n_i \searrow -\infty ,$$

and compute the first few terms :

$$(1) \quad \frac{1}{2\pi} \frac{ds}{dk} = \frac{n(4\pi)}{\Gamma(1+\frac{n}{2})} k^{n-1} V(K) - \frac{(n-1)(4\pi)}{\Gamma(1+\frac{n-1}{2})} k^{n-2} A(\partial K) + O(k^{n-3})$$

$$(2) \quad a(\theta, \theta, k) = \left(\frac{k}{2\pi}\right)^{n-1} A(\theta) + c_n k^{n-\frac{5}{3}} \int_{\Gamma} K^{-\frac{1}{3}}(\theta) dS + \dots$$

Here $V(K)$ is the volume of K , $A(\partial K)$ is volume of ∂K , $A(\theta)$ is the volume of the projection of K onto $x \cdot \theta = 0$, Γ is the boundary of this projection, dS is the volume form on Γ , and $K(\theta)$ is the normal curvature in direction θ on the pre-image of Γ in K . The constant c_n is the finite part of a definite integral of Airy functions and depends only on n .

The constructions given here follows those of Ludwig [3] very closely but make use of improvements made possible by Melrose's proof of the symplectic equivalence of glancing hypersurfaces [7]. For a discussion of (1) one may see [5]. The expansion (2) was derived when K is a sphere by Rubinow and Wu [10], and conjectured for convex bodies by Keller and Rubinow [2]. The leading term was derived rigorously by Majda and Taylor [6]. The complete asymptotic expansion is due to R. Melrose [9]. The method of [9] is different from that used here and appears to be more powerful as it yields the same results for the Neumann problem. Still more refined results on $a(\theta, \omega, k)$ -which permit a uniform expansion near $\theta = \omega$ -have been obtained by Melrose and M. E. Taylor. The construction given here seems sufficiently intuitive -at least to the author - that it may serve as a prologue to the results of Melrose and Melrose-Taylor.

Localization

Using the standard construction of geometric optics one can decompose $e^{-ikx \cdot \omega}$ into a sum of terms u_e , where

$$u_e = e^{-ikx \cdot \omega} \left(a_0 + \frac{a_1}{k} + \dots + \frac{a_M}{k^M} \right)$$

such that

$$i) \quad (\Delta + k^2)u_e = O(k^{-N})$$

ii) the projections of the supports of u_e onto $x \cdot \omega = 0$ can be made subordinate to any given cover of $x \cdot \omega = 0$.

The strategy here will be, given u_e to construct a u_s satisfying

- i) $(\Delta + k^2)u_s = O(k^{-N})$
- ii) $u_s = -u_e$ on ∂K
- iii) u_s satisfies the Sommerfeld condition.

Actually one has only to construct u_s on a neighborhood of ∂K in $\mathbf{R}^n \setminus K$ satisfying i) and ii) with wave fronts -or more precisely "frequency set" (see [1]) - over points near ∂K but strictly inside $\mathbf{R}^n \setminus K$ directed toward ∂K . Then u_s can be extended to satisfy the Sommerfeld condition by the outgoing Green's function for the laplacian on \mathbf{R}^n (see [4], pp.521-3).

If the projection of the support of u_e on $x \cdot \omega$ does not intersect Γ , the construction of u_s is a standard application of geometric optics. Hence from here on we consider only u_e whose support projects onto a neighborhood -which we may take as small as we wish - of a point on Γ .

The Ludwig-Melrose construction

The idea here is to find a representation of u_e in the form

$$(3) \quad u_e = \int_{\mathbf{R}^{n-1}} e^{ik\theta} (a \text{Ai}(-k^{2/3}\rho) + b \text{Ai}'(-k^{2/3}\rho)) d\xi + O(k^{-N})$$

where the integrand is an asymptotic solution to $(\Delta + k^2)w = 0$ uniformly in ξ , and one has additionally

$$(4) \quad \text{i) } \rho = \xi_1 \text{ and } \frac{\partial \rho}{\partial \nu} > 0 \text{ on } \partial K ,$$

$$(5) \quad \text{ii) } b = 0 \text{ on } \partial K.$$

The function Ai is the standard Airy function

$$\text{Ai}(s) = \int_{-\infty}^{\infty} e^{i(\beta s + \frac{\beta^3}{3})} d\beta ,$$

and a and b have the form

$$a = \sum_{i=0}^R a_i(x, \xi) k^{-i + \frac{n}{2} - \frac{1}{3}}, \quad b = \sum_{i=0}^R b_i(x, \xi) k^{-i + \frac{n}{2} - \frac{2}{3}}.$$

Once we have (3) - (5) the function u_s will be given by

$$(6) \quad u_s = - \int_{\mathbf{R}^{n-1}} e^{ik\theta} (aA(-k^{2/3}\rho) + bA'(-k^{2/3}\rho)) \frac{A_i(-k^{2/3}\xi_1)}{A(-k^{2/3}\xi_1)} d\xi,$$

where $A(s) = Ai(e^{-\frac{2\pi i}{3}s})$. Note that, since A satisfies Airy's differential equation, the integrand is automatically an asymptotic solution to $(\Delta + k^n)w = 0$ in $\mathbf{R}^n \setminus K$ - since $A(s)$ is exponentially increasing as $s \rightarrow +\infty$, we use the fact $\rho > \xi$ in $\mathbf{R}^n \setminus K$ here. The choice of A is made so that the frequency set of u_s is directed toward ∂K from $\mathbf{R}^n \setminus K$.

As we mentioned earlier the constructions here are strictly local. We assume that we are given $x_0 \in \partial K$ with $\omega \cdot \nu(x_0) = 0$ and, writing $\xi = (\xi_1, \xi')$, a ξ'_0 such that $\nabla\theta(x_0, 0, \xi'_0) = -\omega$. All the assertions (of existence etc...) in the constructions that follow are to be qualified by "for (x, ξ) in a neighborhood of $(x_0, 0, \xi'_0)$ " - even though this will always be omitted. Just how small the support of u_e must be is only determined at the end of the construction.

The representation (3) with conditions (4), (5) is the delicate part of the construction. One first determines θ and ρ and then a and b . In order that the integrand in (3) be an asymptotic solution to $(\Delta + k^2)w = 0$, θ and ρ must satisfy the "eichonal" equations :

$$(7) \quad a) \quad |\nabla_x \theta|^2 + \rho |\nabla_x \rho|^2 = 1$$

$$b) \quad \nabla_x \rho \cdot \nabla_x \theta = 0$$

on $\rho \geq 0$. These equations are solved by choosing a smooth family of strictly convex surfaces S_ξ with $S_\xi = \partial K$ when $\xi_1 = 0$, and defining $\rho(x, \xi) = 0$ on S_ξ . Note that, since we want $\nabla_x \rho \neq 0$, this implies $|\nabla_x \theta(x, \xi)|^2 = 1$ on S_ξ and $\nabla_x \theta(x, \xi)$ is tangent to S_ξ . Thus we must choose θ on S_ξ to be a solution of the surface eichonal on S_ξ . With these choices 7a) and 7b) determine θ and ρ uniquely for x outside S_ξ , i.e. in the region where we will have $\rho \geq 0$. The condition (4) implies and, modulo a change of variables in ξ , is equivalent to the following geometric

condition on S_ξ and $\theta \uparrow S_\xi$: if the straight line through $x_0 \in S_{\xi_0}$ with direction $\nabla_x \theta(x_0, \xi_0)$ hits ∂K at x' , then the reflection of this line in ∂K is, for some $x_1 \in S_{\xi_0}$, the line through x_1 with direction $\nabla_x \xi(x_1, \xi_0)$.

In [3] the surfaces S_ξ were only chosen so that (4) held up to an error which was $O(\xi_1^N)$ for all N . However, it is a direct consequence of [7] (the derivation is given in [8] that S_ξ and $\theta \uparrow S_\xi$ can be chosen so that (4) holds exactly. Then one completes the construction by extending θ and ρ as C^∞ functions in the complement of $\rho \geq 0$, maintaining (4).

If we replace A_i and A_i' by their integral representations, (3) becomes

$$u_e = \int e^{ik(\theta - \beta\rho + \frac{\beta^3}{3})} (a + ik^{1/3}\beta b) d\xi d\beta .$$

Note that, writing $\xi = (\xi_1, \xi')$, if $\det \frac{\partial^2 \theta}{\partial \xi' \partial \xi'} \neq 0$, then this integral can be expanded by the method of stationary phase. If the result of this expansion agrees with u_e , then we must have

$$(8) \quad -x \cdot \omega = \phi(x) \equiv (\theta - \beta\rho + \frac{\beta^3}{3}) \uparrow \xi = \xi(x), \beta = \beta(x),$$

where $\xi(x)$ and $\beta(x)$ are defined by

$$\theta_\xi - \beta\rho_\xi = 0 \quad \text{and} \quad -\rho + \beta^2 = 0$$

However, since ϕ is automatically a solution of the standard eichonal ($|\nabla \phi|^2 = 1$) it suffices to have (8) hold for x on a surface transverse to ω . The eichonal equations (7) and condition (4) remain valid if we replace θ by $\theta + \psi(\xi)$, and we must exploit this freedom to obtain $\det \frac{\partial^2 \theta}{\partial \xi' \partial \xi'} \neq 0$ and (8) .

Introducing local coordinates (z, y) where $z = 0$ on ∂K , $\partial f / \partial z = \partial f / \partial \nu$ on ∂K and $y_1 = x \cdot \omega$ on ∂K , we can assume S_ξ is given by $z = \alpha(y, \xi)$. Writing $y = (y_1, y')$, it is a consequence of the constructions in [7] that S_ξ and $\theta \uparrow S_\xi$ can be chosen so that $\det \frac{\partial^2 \theta}{\partial y \partial \xi} \neq 0$ (this is used in [8]) and $\det \frac{\partial^2 \theta}{\partial y' \partial y'} \neq 0$. We let β denote $x \cdot \omega$ written as a function of (z, y) .

To achieve 8) we begin by solving $(\theta_z, \theta_y) = (\beta_z, \beta_y)$ on $z = \alpha(y, \xi)$ for $y = y(\xi)$. This is over determined, but since $|\nabla_x \theta| = |\nabla_x(x \cdot \omega)| = 1$ when $z = \alpha(y, \xi)$ it suffices to solve $(\theta_z, \theta_{y'}) = (\beta_z, \beta_{y'})$ on $z = \alpha(y, \xi)$. To check the hypothesis of the implicit function theorem, we set $\xi_1 = 0$ (so that $\alpha = \theta_z = 0$) and compute

$$\begin{pmatrix} \frac{\partial^2(\theta - \beta)}{\partial z \partial y_1} & \frac{\partial^2(\theta - \beta)}{\partial z \partial y'} \\ \frac{\partial^2(\theta - \beta)}{\partial y' \partial y_1} & \frac{\partial^2(\theta - \beta)}{\partial y' \partial y'} \end{pmatrix} = \begin{pmatrix} -\frac{\partial^2 \beta}{\partial z \partial y_1} & -\frac{\partial^2 \beta}{\partial z \partial y'} \\ \frac{\partial^2 \theta}{\partial y' \partial y_1} & \frac{\partial^2 \theta}{\partial y' \partial y'} \end{pmatrix}$$

Since $\partial^2 \beta / \partial z \partial y_1$ is nonzero by the strict convexity of ∂K , and we may assume $\partial^2 \theta / \partial y' \partial y_1$ vanishes at the base point, we conclude that

$(\theta_z, \theta_y) = (\beta_z, \beta_y)$ can be solved for $y(\xi)$ on $z = \alpha(y, \xi)$.

Now we defined ψ by the requirement

$$(9) \quad \theta_\xi(\alpha(y(\xi), \xi), y(\xi), \xi) + \psi_\xi = 0 \quad .$$

To check that

$$\theta_{\xi\xi} + \theta_{\xi z} \alpha_y y_\xi + \theta_{\xi y} y_\xi$$

is symmetric, we note that

$$\theta_{z\xi} + \theta_{zz} \alpha_y y_\xi + \theta_{zy} y_\xi = \beta_{zz} \alpha_y y_\xi + \beta_{zy} y_\xi$$

$$\theta_{y\xi} + \theta_{yz} \alpha_y y_\xi + \theta_{yy} y_\xi = \beta_{yz} \alpha_y y_\xi + \beta_{yy} y_\xi \quad .$$

Since (9) determines ψ up to an additive constant, we complete the construction of ψ by choosing this constant so that $\theta + \psi = -x \cdot \omega$ at the base point. Further work along exactly the same lines shows that $\det \partial^2(\theta + \psi) / \partial \xi' \partial \xi' \neq 0$ (This uses $\det \frac{\partial^2 \theta}{\partial y \partial \xi} \neq 0$) and that

$z = \alpha(y(\xi), \xi)$, $y = y(\xi)$ defines a surface transverse to ω . Then it follows that (8) holds when θ is replaced by $\theta + \psi$.

We will not discuss the construction of the amplitudes $a(x, \xi, k)$ and $b(x, \xi, k)$ here. In [3] a and b are constructed so that, given the preceding construction of θ and ρ , (3) holds and in place of (5) one has $b = O(\xi_1^N)$ for any N on ∂K . The modifications needed to improve this to (5), i.e. $b = 0$ on ∂K are substantially simpler than those that were used in obtaining (4) - no use of [7] is involved. Actually, imposing (5) for all $x \in \partial K$ (or even the weaker condition $b = O(\xi_1^N)$) would make it impossible to keep the intersection of the support of a and b with ∂K strictly inside the set where θ and ρ are defined. This is a turn would prevent us from making the integrand in (3) an asymptotic solution to $(\Delta + k^2)w = 0$ on a neighborhood of ∂K in $\mathbf{R}^n \setminus K$. However, we only impose (5) for (x, ξ) in a small neighborhood of the base point. Provided the projection of the support of u_e is made sufficiently small, one still has $u_s(x, k) + u_e(x, k) = O(k^{-N})$ for $x \in \partial K$ in this case.

The representation of $\partial\Phi/\partial\nu$

Away from the intersection of ∂K with the pre-image of Γ the expansion of $\partial\Phi/\partial\nu$ is easy to compute from geometric optics; the leading term is

$$\frac{\partial\Phi}{\partial\nu} = \begin{cases} -2ik\omega \cdot \nu e^{-ikx \cdot \omega} & \text{if } \omega \cdot \nu < 0 \\ 0 & \text{if } \omega \cdot \nu > 0 \end{cases}$$

(the "Kirchhoff approximation"). The next term is $O(1)$ and it does not contribute to the second term in (1) and (2).

In a neighborhood of a point $x_0 \in \partial K$ where $\omega \cdot \nu(x_0) = 0$, i.e. a point that projects to Γ , one can combine (3)-(6) to get

$$\frac{\partial\Phi}{\partial\nu} = \int_{\mathbf{R}^{n-1}} e^{ik\theta} \left(-k^{2/3} \frac{\partial\rho}{\partial\nu} a + \frac{\partial b}{\partial\nu} \right) F(-k^{2/3} \xi_1) d\xi + O(k^{-N})$$

where $F(x) = Ai'(x) - \frac{A'(x)}{A(x)} Ai(x)$. Expanding by stationary phase in the variable ξ' , this can be further simplified to a representation in the form

$$(10) \quad \frac{\partial\Phi}{\partial\nu}(x, k) = \int_{\mathbf{R}} e^{ik\check{\theta}(x, \xi_1)} G(x, \xi_1, k) F(-k^{2/3} \xi_1) d\xi_1 + O(k^{-N})$$

where G has the form

$$G = \sum_{i=0}^M G_i(x, \xi_1) k^{\frac{1}{6} + i}$$

Substituting (10) and the analogous expression (derived from (3)),

$$e^{-ikx \cdot \omega} = \int_{\mathbf{R}} e^{ik \tilde{\theta}(x, \xi_1)} H(x, \xi_1, k) \text{Ai}(-k^{2/3} \xi_1) d\xi_1 + O(k^N)$$

into the integral formulas for ds/dk and $a(\omega, \omega, k)$ one derives (1) and (2). The crucial advantage here of (10) over the formulas that could be obtained from [3] is that one can eliminate the oscillatory $e^{ik\tilde{\theta}}$ factors by an integration in an x -variable without disturbing the Airy functions. At the final stage in the derivation of (1) and (2) one must expand integrals of the form

$$\int_{\mathbf{R}} H(s) G(-k^{2/3} s) ds$$

where G is a polynomial in A'/A , \bar{A}'/\bar{A} , Ai and their derivatives, it is here that the $k^r \log k$ terms seem to appear in the asymptotic expansions. It is known that there are no logarithms (and only integral powers of k) in the expansion of ds/dk (see [5]), but unknown whether logarithms actually appear in the expansion of $a(\theta, \theta, k)$.

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