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HYPSELLIPTIC OPERATORS WITH DOUBLE CHARACTERISTICS

par A. MENIKOFF

I want to discuss pseudo-differential operators of the form $P = P_m(x, D) + P_{m-1}(x, D) + \dots$ such that in a conic neighborhood Γ of $(x_0, \xi_0) \in \mathbb{R}^{2n}$, $P_m(x, \xi) \geq 0$ and vanishes to exactly second order on a submanifold Σ of codimension 1 transverse to the fiber axis. P is a classical pseudo-differential operator of order m . These assumptions imply that $P_m = QU^2$ where $Q \neq 0$ and $d_\xi U \neq 0$ near (x_0, ξ_0) , Q and U are real and homogeneous of order $m-2$ and 1 respectively. Recall that the definition of subprincipal symbol of P is

$$P_{m-1}^S(x, \xi) = P_{m-1}(x, \xi) - \frac{1}{2i} \sum_{j=1}^n \frac{\partial^2 P_m}{\partial x_j \partial \xi_j}.$$

The first two results I want to talk about concern necessary and sufficient conditions for P to be locally solvable and hypoelliptic.

Theorem 1 : Suppose that $\operatorname{Re} P_{m-1}^S \neq 0$ in Γ and that whenever $\operatorname{Re} P_{m-1}^S < 0$ at a point of Σ then along the null bicharacteristic of U through that point $\operatorname{Im} P_{m-1}^S$ has only zeros of even order $\leq k$.

(A) Then given any $\varphi \in S^0(\Gamma)$, $\varphi(x_0, \xi_0) \neq 0$ there exist operators E_i , and R_i , $i = 1, 2$ such that

- (1) $E_i : H_S(\Gamma) \rightarrow H_{s+m-2+(k+2)/2(k+1)}(\Gamma)$ and
- (2) $R_i : H_S(\Gamma) \rightarrow H_{s+m-2+1/k+1}(\Gamma)$ are bounded and
- (3) $PE_1 = \varphi(x_1 D)I + R_1$
- (4) $E_2 P = \varphi(x_1 D)I + R_2$

(B) P is hypoelliptic and locally solvable in Γ

In the converse direction there is

Theorem 2 : If at $(x_0, \xi) \in \Sigma$, $\operatorname{Re} P_{m-1}^S(x_0, \xi) < 0$ and $\operatorname{Im} P_{m-1}^S$ changes sign and has a zero of finite order on the null bicharacteristic of U through (x_0, ξ_0) then $P(x, D)$ is not locally solvable at x_0 .

Note that since the hypotheses of theorem 2 also hold for P^* , P is also not hypoelliptic.

Theorem 2 is due to P. Wenston [5] for partial differential operators. Theorems 1 and 2 are due to P. Popivanov [2] and myself [1].

X.2

By means of Fourier integral operators the problem of studying P may be reduced to considering

$$(5) \quad Q = D_t^2 + a(t, x, D_x) + b(t, x, D_x, D_t)D_t$$

where $a \in S^1$, $b \in S^0$. This is a consequence of the invariance of the assumptions and conclusions of theorem 2 under multiplication by real elliptic factors or conjugation by Fourier integral operators. Noting that $a(t, x, \xi)$ is the sub-principal symbol of Q the assumptions of theorem 1 become that if $\operatorname{Re} a < 0$ then $\operatorname{Im} a(t, x, \xi)$ has only zeros of even order $\leq k$ as a function of t . This, of course, implies that a has constant sign.

To understand the hypotheses of theorems 1 and 2 consider the ordinary differential operator

$$(6) \quad L = D_t^2 + a(t, x, \xi)$$

depending on $(x, \xi) \in \mathbb{R}^{2n}$ as parameters. Considering functions which are oscillatory in x , the local solvability of Q^* is related to whether or not an estimate of the form

$$(7) \quad \| \| u \| \| \leq C \| Lu \|$$

holds for some pair of norms on $C_0^\infty(\mathbb{R})$. Using the Green-Liouville approximation, $Lu \sim 0$ has solutions

$$(8) \quad u_{\pm}(t, x, \xi) = \frac{1}{a^{1/4}(t, x, \xi)} e^{\pm \int_{T(x, \xi)}^t \sqrt{a(t', x, \xi)} dt'}$$

If $\operatorname{Re} a < 0$ and $\operatorname{Im} a(t, x, \xi)$ changes signs at $T(x, \xi)$ then $\operatorname{Re} \sqrt{a}$ will also have a change of sign. This means that $Lu \sim 0$ has a solution in \mathcal{S} and consequently an estimate of form (7) cannot hold.

To construct parametrices under the assumptions of theorem 1, I will use the notion of vector-valued pseudo-differential operators. The idea of using pseudo-differential operators whose symbols are operators between Hilbert space originated in Trèves [4]. I shall use an extension of this idea due to Sjöstrand [3] which allows the norms on the Hilbert spaces to vary.

Let H_1 and H_2 be a pair of Hilbert spaces whose norms depend on $\xi \in \mathbb{R}^n$ such that

$$(9) \quad c \|u\|_{H_i} \leq \|u\|_{H_i(\xi)} \leq C(1 + |\xi|)^{m_i} \|u\|_{H_i}$$

for $i = 1, 2$. Let $\mathcal{L}(H_1(\xi), H_2(\xi))$ be the space of bounded operators from $H_1 \rightarrow H_2$ with the uniform operator norm, also varying with ξ . I will define the symbol in class $S_{\rho, \delta}^m(\Omega \times \mathbb{R}^n; H_1(\xi), H_2(\xi))$ to be the class of functions $A(x, \xi) : \Omega \times \mathbb{R}^n \rightarrow \mathcal{L}(H_1(\xi), H_2(\xi))$ such that

$$(10) \quad \left\| \partial_x^\alpha \partial_\xi^B A(x, \xi) \right\|_{\mathcal{L}(H_1(\xi), H_2(\xi))} \leq C(1 + |\xi|)^{m+\delta} |\alpha| - \rho |\beta|$$

The corresponding class of operators given by

$$A(x, D)u(x) = (2\pi)^{-n} \int e^{ix\xi} A(x, \xi) \hat{u}(\xi) d\xi$$

will be denoted by $L_{\rho, \delta}^m(\Omega, H_1(D), H_2(D))$, where $\Omega \subset \mathbb{R}^n$. These operators are at least maps

$$A(x, D) : C_0^\infty(\Omega, H_1) \rightarrow C^\infty(\Omega, H_2)$$

Furthermore, the standard calculus of pseudo-differential operators still holds for these operators.

The special case of such operators I will use is obtained by taking $H = L^2(\mathbb{R}, dt)$ and $B(\xi)$ to be the completion of $C_0^\infty(\mathbb{R})$ in the norm

$$\|u\|_{B(\xi)} = \left\| (1 + |D_t|^2 + |D_x|)^{(k+2)/2(k+1)} u \right\|_{L^2}$$

An example of the boundlessness of results for this class of operators is that if $A \in L_{\rho, \delta}^0(\mathbb{R}^n; B(D), H)$ and $0 \leq \delta < 1/2 < \rho \leq 1$ then for any compact subset K of Ω

$$\|Au\|_{L^2(\mathbb{R}^{n+1})} \leq C \left\| (1 + |D_t|^2 + |D_x|)^{(k+2)/2(k+1)} u \right\|$$

for all $u \in C_0^\infty(K \times \mathbb{R})$.

X.4

To construct a parametrix of $Q = D_t^2 + a(t, x, D_x)$ define

$$(11) \quad e(x, \xi, t, s) = \begin{cases} \frac{1}{2(a(t, x, \xi)a(s, x, \xi))^{1/4}} e^{-\int_t^s \sqrt{a(t', x, \xi)} dt'} & \text{if } t \leq s \\ e(x, \xi, s, t) & \text{if } s \leq t \end{cases}$$

Suppose $\text{Re } a < 0$, and $\text{Im } a \geq 0$, then $a^{1/2}$ is a smooth function of all its variables. Choose the square root in (11) so that $\text{Re } a^{1/2} \geq 0$. Define the corresponding integral operator with kernel g as

$$(12) \quad E_g(x, \xi)f(t) = \int_{-\infty}^{\infty} g e(x, \xi, t, s)f(s)ds$$

where $g \in S^0(\mathbb{R}^{2n+2})$. My candidate for a parametrix of Q is $E_1(x, D)$.

A calculation will show that

$$(13) \quad L(x, t, D_t) E(x, \xi) = I + E_g$$

where

$$(14) \quad g = \left(\frac{a''}{4a} - \frac{5}{16} \frac{a'^2}{a} \right) \in S^0$$

Equation (13) may be thought of as a statement about the multiplication of the symbols of Q and $E(x, D)$. The crucial step in proving theorem 1 is

Lemma 3 : With the above assumptions if $g \in S^0(\mathbb{R}^{2n+2})$ then $E_g(x, \xi) \in S_{\rho, \delta}^0(\Omega; H, B(\xi))_s$ where

$$\frac{1}{2} - \frac{1}{2(k+1)} = \delta < 1/2 < \rho = \frac{1}{2} + \frac{1}{2(k+1)}$$

To prove lemma 3, I must show that

$$(15) \quad \|(1 + (D_t)^2 + |\xi|^2)^{(k+2)/2} E_g\|_{L^2} \leq C \|g\|_{L^2}$$

and similar estimates for $\partial_x^\alpha \partial_\xi^\beta E(x, \xi)$. I will need the following lemmas.

Lemma 4 : Suppose $k(t, s)$ is a measurable function on \mathbb{R}^2 such that

$$\int |k(t,s)| ds \leq B \quad \text{and} \quad \int |k(t,s)| dt \leq B .$$

Then $\mathcal{K} f(t) = \int k(t,s)f(s)ds$ is a bounded operator on $L^2(\mathbb{R})$ and $\|\mathcal{K}\| \leq B$.

Lemma 5 : If $0 \leq a(t,x,\xi) \in S^0(\mathbb{R}^{n+1} \times \mathbb{R}^n)$ and has zeros in t of order $\leq k$, then there is a constant $c > 0$ such that

$$(16) \quad c|t-s|^{k+1}|\xi| \leq \int_t^s a(t',x,\xi) dt'$$

For a proof see the appendix of [4].

Lemma 6 : Given $C > 0$ there is a constant $C' > 0$ such that for any complex number z $|\operatorname{Im} z| \leq C|\operatorname{Re} z|$ implies $|\operatorname{Re} z|^{1/2} \geq C'|\operatorname{Im} z|/|\operatorname{Re} z|^{1/2}$.

To prove (15), I will estimate

$$I = \int_t^\infty |e(x,\xi,t,s)| ds \quad \text{and} \quad \int_{-\infty}^t |e(x,\xi,t,s)| ds .$$

Using the definition of E ,

$$(17) \quad I \leq \int_t^\infty \frac{1}{|a(t)a(s)|^{1/4}} e^{-\int_s^t \sqrt{a} dt} ds .$$

Since $\operatorname{Re} a \neq 0$ $|a(s)a(t)|^{1/4} \geq C|\xi|^{1/2}$. Combining, lemmas 5 and 6 give

$$\operatorname{Re} \int_t^s \sqrt{a(t')} dt' \geq C|\xi|^{1/2}|t-s|^{k+1}$$

Using the last two inequalities in (17), I get

$$I \leq C|\xi|^{-1/2} \int_t^\infty e^{-c|\xi|^{1/2}|t-s|^{k+1}} ds = C|\xi|^{-\frac{1}{2} - \frac{1}{2(k+1)}} .$$

Lemma 4, then yields the bound

$$\| |\xi|^{(k+1)/2(k+2)} E(x,\xi)u \| \leq C \|u\| .$$

Since $D_t^2 E = -a(t,x,\xi)E +$ bounded operators and $a \in S^1$ we also may

estimate twice as many t-derivatives as x-derivatives. This gives (15).

I am now in a position to complete the proof of theorem 1 by applying the calculus of pseudo-differential operators. Since D_t^2 is independent of t the symbol of $D_t^2 \circ E(x, D_x)$ is D_t^2 applied to the symbol of E. To compute the composition of $a(t, x, D)$ and E, consider a as being in the class $S^1(\Omega \times \mathbf{R}^n; B(\xi), B(\xi))$. I then have

$$a(t, x, D_x) \circ E(x, D_x) = (aE)(x, D) \text{ mod } L^{-\min(\rho, 1-\delta)}(H, B(\xi)).$$

Combining the above observations with (13) gives

$$Q(t, x, D_t, D_x) \circ E(x, D_x) = I + R \quad ,$$

where $R \in L^{k/2(k+1)}(\Omega; H, B(D))$.

For R, there will be the estimate

$$\| (1 + |D_t|^2 + |D_x|^2)^{1/k+1} R u \| \leq C \| u \|, \quad u \in C_0^\infty(\mathbf{R}^{n+1}) .$$

This completes the proof of theorem 1.

An extension of the argument used for theorem 1 will give

Theorem 7 : Let $L = D_t^2 + a(t, x, \xi) + b(t, x, D_t, D_x) D_t + \dots$ where $a \in S^1$, $b \in S^0$ and suppose that $\text{Re } a \neq 0$ and if $\text{Re } a < 0$ then $\text{Im } a$ has a constant sign as a function of t and $\text{Im } a$ never vanishes on an open t-interval for fixed x and ξ .

Then, L is locally solvable and for any $\varepsilon > 0$ there is a neighborhood ω of sufficiently small diameter such that

$$\| u \|_{s+m-3/2} \leq \varepsilon \| PL u \|_s + C \| u \|_{s+m-2} \quad , \quad u \in C_0^\infty(\omega) .$$

Operators whose sub-principal symbol vanishes on the characteristic variety may also be treated. Now let

$$Q = D_t^2 + b(t, x, D_x, D_t)$$

where $b(0, x, \xi) = t^k a(t, x, \xi)$, $a \neq 0$.

A parametrix for L may be attempted to be constructed in the same way as the one in theorem 1 was. Consider the ordinary differential operator $L = D_t^2 + t^k a(t, x, \xi)$. A solution of $Lu \sim 0$ may be sought of the form

$$u = g(t, x, \xi) V(\phi(t, x, \xi))$$

where V is a solution of $-V'' + s^k V(s) = 0$. This will force that

$$\phi = (k+2/2 \int_0^t \sqrt{t^k a} dt)^{2/k+2},$$

and $g = \phi^{-1/2}$. Then $Lu = (-\frac{d^2 g}{dt^2}) V(\phi)$.

If $Lu \sim 0$ has two independent solutions which increase and decrease exponentially on opposite sides of the t -axis, a parametrix may again be constructed. For instance, I have shown

Theorem 8 : If either $a(0, x_0, \xi_0)$ is not real, ϕ or k is even and $a(0, x_0, \xi_0) > 0$ then L is hypoelliptic and locally solvable and has left and right parametrices.

Theorem 9 : If $a(0, x_0, \xi_0)$ is real and $\text{Im} a$ has a zero of first order then Q is hypoelliptic.

Theorem 10 : If $a(0, x, \xi)$ is real and $\text{Im} b(0, x, \xi) \neq 0$, then

$$Q = D_t^2 + t^k (a(t, x, D_x) + t^\ell b(t, x, D_x))$$

is locally solvable when $\ell < k+2$.

I don't know what the situation is for the operators of theorem 10 if $\ell \geq k+2$, nor do I have non-solvability results if $k > 1$ in theorem 8 etc...

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