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## **Propagation of singularities along gliding rays**

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PROPAGATION OF SINGULARITIES  
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ALONG GLIDING RAYS  
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Let  $\Omega = \{x \in \mathbb{R}^n; f(x) > 0\}$  be a domain with  $C^\infty$  boundary and let  $P = P(x, D)$  be a second order differential operator of real principal type with coefficients in  $C^\infty(\bar{\Omega})$ . The subject of this lecture is to discuss the singularities of distributions  $u$  satisfying

$$Pu \in C^\infty(\bar{\Omega}), \quad u|_{\partial\Omega} \in C^\infty(\partial\Omega).$$

If  $\mathcal{P}$  denotes the set of zeros of the principal symbol  $p(x, \xi)$ ,  $\mathcal{F} = \partial\Omega \times \dot{\mathbb{R}}^n$  and

$$(1) \quad \{p, f\} \neq 0 \text{ in } \mathcal{P} \cap \mathcal{F} \quad (\text{microlocally}),$$

then this problem has been treated by several authors (see [2], [8]). We shall consider a case where (1) is violated. If the boundary  $\partial\Omega$  is non-characteristic, then

$$(2) \quad \{p, f\}(z) = 0 \Rightarrow \{f, \{f, p\}\}(z) \neq 0, \quad z \in \mathcal{P} \cap \mathcal{F}.$$

We shall also assume that  $\Omega$  is pseudo-convex with respect to  $P$ , i.e.

$$(3) \quad \{p, f\}(z) = 0 \Rightarrow \{p, \{p, f\}\} < 0, \quad z \in \mathcal{P} \cap \mathcal{F}.$$

The opposite case, when  $\Omega$  is pseudo-concave, has been treated in [4], [6] and [9].

When (2) and (3) are satisfied, it is possible to define so called boundary bicharacteristics for  $P$  as follows. Consider the restriction  $\tilde{p}$  of  $p$  to the symplectic manifold  $\Delta_{\mathcal{F}} = \{z \in \mathcal{F}; \{p, f\}(z) = 0\}$ . The projection, onto  $T^*(\partial\Omega)$  of the null-bicharacteristics for  $\tilde{p}$  are called boundary bicharacteristics for  $P$ . It is easy to check that the null-bicharacteristics for the restriction of  $f$  to  $\Delta_{\mathcal{P}} = \{z \in \mathcal{P}; \{p, f\}(z) = 0\}$  give the same curves as the null-bicharacteristics for  $\tilde{p}$ . In fact, the tangent of such a curve lies in the plane spanned by  $H_p$  and  $H_f$  and is orthogonal to the gradient of  $\{p, f\}$ .

In order to localize the concept of "regularity up to the boundary", we assume that  $\Omega$  is given by  $x_n > 0$  and denote points on the boundary  $\partial\Omega$  by  $x' = (x', 0)$ . If  $(\bar{x}', \bar{\xi}') \in T^*(\partial\Omega)$ , we say that  $u \in H_s(\bar{\Omega})$  at  $(\bar{x}', \bar{\xi}')$  if there is a homogeneous symbol  $\psi(x', \xi')$  such that  $\psi(x', \xi') = 1$  in a neighborhood of  $(\bar{x}', \bar{\xi}')$  and  $\psi(x', D')u(x', x_n)$  belongs to  $H_s(\bar{\Omega})$  close to  $\bar{x}'$ . Here it is implicitly assumed that  $x_n \mapsto u(\cdot, x_n)$  is well-defined. In the same manner we say that  $(\bar{x}', \bar{\xi}') \notin WF(u; \bar{\Omega})$  if, for some  $\psi$  as above,  $\psi(x', D')u \in C^\infty(\bar{\Omega})$  close to  $\bar{x}'$ . This definition of microlocal regularity up to the boundary has also been suggested by Chazarain [3].

**Theorem** : Suppose that  $\Omega$  is pseudo-convex with respect to  $P$  and let  $\gamma$  be a boundary bicharacteristic for  $P$ . If  $u \in \mathcal{D}'(\bar{\Omega})$ ,  $Pu \in C^\infty(\bar{\Omega})$  and  $\gamma \cap WF(u|_{\partial\Omega}) = \emptyset$ , then either  $\gamma \subset WF(u; \bar{\Omega})$  or  $\gamma \cap WF(u; \bar{\Omega}) = \emptyset$ .

**Sketch of proof** : In order to avoid technical complications, we only consider the case when  $P$  is hyperbolic. We also assume that  $\Omega$  is given by  $x_n > 0$ . Let  $(\bar{x}', \bar{\xi})$  be a point in  $\mathcal{P} \cap \mathcal{F}$  where  $\{p, f\} = 0$  and denote by  $\gamma = \gamma(\bar{x}', \bar{\xi}')$  the boundary bicharacteristic through the projection  $(\bar{x}', \bar{\xi}')$  of  $(\bar{x}', \bar{\xi})$  onto  $T^*(\partial\Omega)$ . The main step in the proof is to construct a suitable symbol  $a(x, \xi)$  of order zero such that

$$(4) \quad \{p, a\}(x, \xi) = 0$$

$$(5) \quad a(x', \xi) = r(x', \xi)p(x', \xi) + q(x', \xi')$$

and  $a(x, \xi) = 1$  on a conic neighborhood of  $(\bar{x}', \bar{\xi})$ . Here the condition (5) is required to be satisfied for some choice of  $r$  and  $q$ .

If  $(x', \xi')$  is close to  $(\bar{x}', \bar{\xi}')$  then there is either no root or else two roots  $\xi_n^\pm$  of the equation  $p(x', \xi', \xi_n) = 0$ . These roots coincide when  $\{p, f\} = 0$ . Since  $q$  is independent of  $\xi_n$  it follows from (5) that one must have

$$(5') \quad a(x', \xi', \xi_n^+) = a(x', \xi', \xi_n^-).$$

Now (4) means that  $a$  is constant along the bicharacteristics for  $P$  so (5') implies that  $a$  has to be constant along the successively reflected

bicharacteristics. In particular it follows that  $a$  must be constant along the lifting  $\tilde{\gamma}$  to  $\mathcal{P} \cap \mathcal{F}$  of the boundary bicharacteristic  $\gamma = \gamma(\bar{x}', \bar{\varepsilon}')$ . If we denote by  $\Gamma(\bar{x}', \bar{\varepsilon}')$  the flow-out along  $H_P$  from  $\tilde{\gamma}$ , the results of [7] imply that  $a$  can be chosen to satisfy (4) and (5) and to have support in an arbitrary small conic neighborhood of  $\Gamma(\bar{x}', \bar{\varepsilon}')$ . Since  $x_n = 0$  is non-characteristic, we can in particular assume that  $a$  has the transmission property (see [1]).

Let  $r$  be a homogeneous symbol of order  $-2$  which has the transmission property and satisfies (5) and denote by  $R$  the corresponding operator. Let furthermore  $Q$  and  $A$  be operators with symbols  $q$  and  $a + a_{-1}$ , where

$$(6) \quad a_{-1} = p_1 r + i \left[ \frac{\partial}{\partial \varepsilon_n} \frac{\partial}{\partial x_n} (pr - a) - \sum_{k=1}^n \frac{\partial p}{\partial \varepsilon_k} \frac{\partial r}{\partial x_k} \right]$$

Here  $p_1$  denotes the symbol of order one of  $P$ .

Assume now that  $u \in H_S(\bar{\Omega})$  along  $\gamma$  and put

$$v = Au^0 + R((Pu)^0 - Pu^0),$$

where  $u^0$  denotes the extension of  $u$  which vanishes outside  $\Omega$ . Since  $\gamma \cap \text{WF}(u|_{\partial\Omega}) = \emptyset$ , it follows from (4) - (6) that

$$(7) \quad Pv = A((Pu)^0) \mod H_S(\bar{\Omega}).$$

Moreover (5) gives that

$$(8) \quad v|_{\partial\Omega} = (R((Pu)^0) + Qu)|_{\partial\Omega} \mod H_{s+1/2}(\partial\Omega).$$

Because  $A$  and  $R$  have the transmission property,  $Pu \in C^\infty(\bar{\Omega})$  and  $\gamma \cap \text{WF}(u|_{\partial\Omega}) = \emptyset$ , we get from (7) and (8) that

$$Pv \in H_S(\bar{\Omega}), v|_{\partial\Omega} \in H_{s+1/2}(\partial\Omega).$$

If now some point on  $\gamma(\bar{x}', \bar{\varepsilon}')$  is outside  $\text{WF}(u; \bar{\Omega})$  and  $a(x, \varepsilon)$  has support sufficiently close to  $\Gamma(\bar{x}', \bar{\varepsilon}')$ , it follows that we can assume that  $v$  has initial data in  $C^\infty$ . Note that the pseudo-convexity

means that the bicharacteristics in  $\Gamma(\bar{x}', \bar{\varepsilon}')$  leave  $\Omega$ . Well-known regularity theorems for the mixed problem now implies that  $v \in H_{s+1/2}(\bar{\Omega})$ .

Since  $a(x, \varepsilon) = q(x', \varepsilon') = 1$  in a neighborhood of the zeros of  $p(x, \varepsilon)$ , when  $(x', \varepsilon')$  is close to  $\gamma(\bar{x}', \bar{\varepsilon}')$  and  $x_n$  is small, we can assume that  $r(x, \varepsilon)$  satisfies

$$(9) \quad a(x, \varepsilon) = r(x, \varepsilon)p(x, \varepsilon) + q(x', \varepsilon')$$

there. Remember that (5) is only required to be satisfied when  $x_n = 0$ . From (9) it follows that

$$v = R((Pu)^0) + Qu \bmod H_{s+1}(\bar{\Omega}),$$

close to  $\gamma$ . Thus

$$Qu = H_{s+1/2}(\bar{\Omega}),$$

close to  $\gamma$ , and the proof is finished.

Remark : Just before this lecture we received a manuscript from G. Eskin [5] where a parametrix is constructed for certain mixed hyperbolic problems with gliding rays. For second order Dirichlet problems such a parametrix has also been constructed by one of the authors (Melrose). This construction as well as a fuller account of the argument described above will be published elsewhere.

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