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GENERAL CAUCHY FORMULAS IN $\mathbb{C}^n$

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General Cauchy Formulas in $\mathbb{C}^n$

by

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Introduction.

Let $K(x,t)$, $x = (x_1, \ldots, x_n)$, $t = (t_1, \ldots, t_n)$ be a holomorphic function (possibly multi-valued) in $\mathbb{C}^n \times \mathbb{C}^n$. We will say that K is a Cauchy kernel if there exists a non-empty domain $D \subset \mathbb{C}^n$, an n-cycle $\Gamma \subset \mathbb{C}^n$, and a holomorphic function $C(x)$, regular and non-vanishing in D such that for any function f holomorphic and regular in a domain $D_f \supset D \cup \Gamma$ we have

$$f(x) = \frac{1}{C(x)} \int_{\Gamma} f(y) K(x, y-x) dy_1 \wedge \ldots \wedge dy_n \tag{1}$$

for every $x \in D$. (1), as it stands, is not yet quite right, in general. In §1 we will indicate the exact changes and conditions under which the right hand expression is meaningful. The classical Cauchy formula corresponds to $K(x,t) \equiv t_1^{-1} t_2^{-1} \ldots t_n^{-n}$. The author proved (1) a few years ago for the kernel $K(x,t) \equiv (t_1^2 + \ldots + t_n^2)^{-n/2}$ in connection with his study of polyharmonic functions and Almansi expansions. This gave rise to the general theory which is presented here.

There are some other Cauchy formulas. For instance, those introduced by S. Bergman [5] and by A. Weil [10] in the early thirties. More recently, in the late fifties, J. Leray [8] introduced the Cauchy-Fantappiè formulas, based on results of L. Fantappiè (see [6] and [7]). These formulas are of different type than ours, and the difference, we believe, can be explained best in the following way. The above mentioned authors aimed at defining domains D such that functions, holomorphic on $\overline{D}$, can be represented by (1) with some kernel K and some cycle Γ. In the formulas of Bergman and Weil, the kernel K is not holomorphic on the whole of Γ, is only piecewise holomorphic (i.e., is represented by different holomorphic kernels on different portions of Γ). In the Cauchy-Fantappiè formulas the kernel K is actually holomorphic on the whole of Γ, but is given by an integral, not directly. In our concept of (1) we consider an explicitly given kernel K, find for it a suitable cycle Γ which determines the function C(x), and then find the corresponding domain D.[1)]

As illustration of the advantage of our approach, we prove, in §5, the existence of Almansi expansions for arbitrary functions, holomorphic around the origin.

Our main problem will be to characterize the Cauchy kernels, and if K is a Cauchy kernel, to find explicitly, if possible, a corresponding Γ, D and C(x).

After some preliminaries in §1, we give, in Theorem I, §2, a general necessary condition for K to be a Cauchy kernel. Theorem II shows that this condition is also sufficient for the important class of kernels which are homogeneous in the variables t. However, for general kernels K, we can get only sufficient conditions, which possibly are not far from necessary and sufficient (see §3).

The criterion that a kernel homogeneous in t be Cauchy is that the exterior differential form, $K(x,t)dt_1 \wedge \dots \wedge dt_n$, not be a differential on the Weierstrassian manifold $\mathfrak{M}_x$ in variables t of K(x,t) for some fixed x. It is proved that this criterion implies that K is homogeneous of degree -n. It is, in general, far from trivial to decide if the form $Kdt_1 \wedge \dots \wedge dt_n$ is a differential or not, even if K is homogeneous of degree -n in t. According to de Rham's theorem, this is equivalent to showing that for some n-cycle, the integral of the form is $\neq 0$. Actually, this cycle determines

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1) We usually look for the largest domain for which the formula is valid.

a cycle Γ in the corresponding Cauchy formula.

In §4 we show how to define explicitly such a cycle for certain classes of kernels. We construct also a cycle for certain kernels which are not in the classes just mentioned; this construction stresses the difficulty of finding a corresponding cycle for homogeneous kernels in general.

In §5, for certain restricted kinds of kernels, we obtain the corresponding Almansi development for an arbitrary holomorphic function.

§1. Preliminary definitions, notations and properties.

For an arbitrary holomorphic function $F(z)$ (in general, multi-valued), $z \in \mathbb{C}^n$, we define the corresponding Weierstrassian manifold $\mathfrak{W}$ of Taylor developments which can be obtained by analytic continuation from one another, and which form a holomorphic manifold over $\mathbb{C}^n$. The complete holomorphic multi-valued function $F(z)$ becomes then a single-valued function $F(\hat{z})$, $\hat{z} \in \mathfrak{W}$. The projection of $\hat{z}$ on $\mathbb{C}^n$ is given by the center of the Taylor development $\hat{z}$. For every $\hat{z} \in \mathfrak{W}$ there exists a small neighborhood, $\hat{U} \subset \mathfrak{W}$, on which the projection is an injection. Any domain $\hat{U}$ on which the projection is an injection determines a single-valued branch of the multi-valued function F, defined on the projection U of $\hat{U}$.

The kernel $K(x,t)$ considered here is a function of 2n-variables (x,t). Hence, it defines a Weierstrassian manifold $\mathfrak{M}$ over $\mathbb{C}^n \times \mathbb{C}^n$. Also, the kernel $K(x, y-x)$ as a function of variables x and y defines a Weierstrassian manifold $\mathfrak{M}'$ over $\mathbb{C}^n \times \mathbb{C}^n$. The two manifolds are isomorphic, the isomorphism being completely described by the following properties: if $\hat{z} \in \mathfrak{M}$ and $\hat{z}' \in \mathfrak{M}'$, and if the projections of $\hat{z}$ and $\hat{z}'$ are (x,t) and (x',y), then $\hat{z}'$ is the isomorphic image of $\hat{z}$ if $x' = x$, $y = x + t$ and the value of the function $K(x,t)$ at $\hat{z}$ is the same as the value of the function $K(x, y-x)$ at $\hat{z}'$.

Consider now a fixed point, $x \in \mathbb{C}^n$. $K(x,t)$ represents then a holomorphic function of t (possibly several of them). On the other hand, the set of points in $\mathfrak{M}$ which project on points of the form (x,t) can be written in the form $(x, \mathfrak{M}_x)$ where $\mathfrak{M}_x$ is a holomorphic manifold over $\mathbb{C}^n$, in general disconnected, whose components are isomorphic to the Weierstrassian manifolds corresponding to the functions of t given by $K(x,t)$, each of these manifolds being isomorphic to one or possibly several components of $\mathfrak{M}_x$. Similarly, we define, for $\mathfrak{M}'$, the manifold $\mathfrak{M}'_x$ with the same remark, relating it to the Weierstrassian manifold corresponding to the functions of y, given by $K(x, y-x)$.

We will assume that the cycle Γ, which is independent of x, is the projection of a well-determined connected n-cycle $\hat{\Gamma}(x) \subset \mathfrak{M}'_x$ (and, therefore, depending on x). Since $\hat{\Gamma}(x)$ is connected, it lies in a well-determined component of $\mathfrak{M}'_x$ which corresponds to a well-determined function of y, given by $K(x, y-x)$. It is clear now that if we write (1) in the form:

$$(1.1) \qquad f(x) = \frac{1}{C(x)} \int_{\hat{\Gamma}(x)} f(y)K(x, y-x)dy_1 \wedge \dots \wedge dy_n$$

the condition $D_f \supset D \cup \Gamma$ makes the formula meaningful (even though it is not always true).

If $\hat{\Gamma}(x)$, for every $x \in D$, lies in a domain $\hat{U} \subset \mathfrak{M}'$ which determines a branch of $K(x, y-x)$, then (1) has a meaning in its original form.

The isomorphism between $\mathfrak{M}'$ and $\mathfrak{M}$ transforms $\mathfrak{M}'_x$ onto $\mathfrak{M}_x$ and hence $\hat{\Gamma}(x)$ is transformed on a well-determined cycle in $\mathfrak{M}_x$ which can be denoted for obvious reasons as $\hat{\Gamma}(x)-x$. Its projection is $\Gamma - x$. By a change of variables we can write (1.1) in the equivalent form:

$$(1.1') \qquad f(x) = \frac{1}{C(x)} \int_{\hat{\Gamma}(x)-x} f(x+t)K(x, t)dt_1 \wedge \dots \wedge dt_n .$$

Denoting $f(x+t)$ by $\varphi_x(t)$ we get, finally, (1.1) in the form in which we will prove it under certain conditions on $K(x, t)$:

$$(1.1'') \qquad \varphi_x(0) = \frac{1}{C(x)} \int_{\hat{\Gamma}(x)-x} \varphi_x(t)K(x, t)dt_1 \wedge \dots \wedge dt_n .$$

We pass now to holomorphic exterior differential forms on a holomorphic manifold over $\mathbb{C}^n$. Specifically, we will consider the manifolds $\mathfrak{M}_x$. We can take on $\mathfrak{M}_x$, as local coordinates, $t_1, \ldots, t_n, \overline{t_1}, \ldots, \overline{t_n}$. The exterior differential forms, which are expressed only in terms of wedge-products of dt_k with coeficients holomorphic on some domain $V \subset \mathfrak{M}_x$, are called holomorphic in V.

A holomorphic differential form of order n has necessarily a vanishing differential (i.e., it is closed). For such a form $\hat{\Phi}$, for two homologous n-cycles $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$, by Stokes Theorem, we obtain $\int_{\hat{\Gamma}_1} \hat{\Phi} = \int_{\hat{\Gamma}_2} \hat{\Phi}$.

By the general deRham's theorem, a holomorphic form $\hat{\Phi}$ of order k is a differential if and only if, for every k-cycle $\hat{\Gamma}$ in its domain, $\int_{\hat{\Gamma}} \hat{\Phi} = 0$.

In our main Theorem II (§2) we will assume that the kernel $K(x, t)$ is homogeneous in t of certain order m (independent of x). Under this assumption, the manifold $\mathfrak{M}_x$ has the property that the homothetie operator $t \to rt$, for any positive r, can be lifted in a unique way from $\mathbb{C}^n$ to $\mathfrak{M}_x$ whenever t is a projection of an element in $\mathfrak{M}_x$. This remark will be used in our proofs.

§2. Main Theorems.

Theorem I. A necessary condition for $K(x,t)$ to be a Cauchy kernel for x belonging to a domain $D \subset \mathbb{C}^n$ is that for no $x \in D$, $K(x,t)dt_1 \wedge \ldots \wedge dt_n$ be a differential on $\mathfrak{M}_x$.

The proof is immediate since, otherwise, for $f(x) \equiv 1$, i.e., $\varphi_x(t) \equiv 1$, the right-hand of formula (1.1'') would give 0 instead of 1.

The next theorem shows that the above necessary condition is actually sufficient for a large and important class of kernels.

Theorem II. Let $K(x,t)$ be homogeneous in variables t of degree m independent of x. If, for some x_0, $\mathfrak{M}_{x_0}$ is not empty, and the form $K(x_0,t)dt_1 \wedge \ldots \wedge dt_n$ is not a differential on $\mathfrak{M}_{x_0}$, then $K(x,t)$ is a Cauchy kernel, $m = -n$, and, for any convex domain $G \ni x_0$, we can choose the n-cycle $\hat{\Gamma}(x)$ so that its projection Γ be on the boundary ∂G.

Proof. Since $K(x_0,t)$ is not a differential on $\mathfrak{M}_{x_0}$, by de Rham's theorem there exists a cycle $\hat{\Gamma}_1$ on $\mathfrak{M}_{x_0}$ such that

(2.1) $$\int_{\hat{\Gamma}_1} K(x_0,t)dt_1 \wedge \ldots \wedge dt_n \neq 0 .$$

We remark first that for $r > 0$, $r\hat{\Gamma}_1$ is homologous to $\hat{\Gamma}_1$ on $\mathfrak{M}_{x_0}$. Hence, by change of variables

$$\int_{\hat{\Gamma}_1} K(x_0,t)t_1 \wedge \ldots \wedge t_n = \int_{r\hat{\Gamma}_1} K(x_0,t)dt_1 \wedge \ldots \wedge dt_n = r^{n+m} \int_{\hat{\Gamma}_1} K(x_0,t)dt_1 \wedge \ldots \wedge dt_n .$$

It follows that $m + n = 0$, and

(2.2) $$m = -n .$$

As a consequence, the projection of $\mathfrak{M}_x$, for every x, does not contain 0.

The convex domain $G-x_0$ contains 0. For every $\hat{t} \in \hat{\Gamma}_1$ there exists a unique $r(\hat{t}) > 0$, such that $r(\hat{t})t \in \partial(G-x_0)$ where t is the projection of $\hat{t}$. The points $r(\hat{t})\hat{t}$ are obviously describing an n-cycle $\hat{\Gamma}_2 \subset \mathfrak{M}_{x_0}$ which is homologous to $\hat{\Gamma}_1$ on $\mathfrak{M}_{x_0}$. Therefore,

$$\int_{\hat{\Gamma}_2} K(x_0, t)dt_1 \wedge \ldots \wedge dt_n = \int_{\hat{\Gamma}_1} \ldots \neq 0 .$$

We will consider that $\hat{\Gamma}_1$ is already chosen as the $\hat{\Gamma}_2$, and we will have for the projection Γ_1 of $\hat{\Gamma}_1$

(2.3) $$\Gamma_1 \subset \partial(G-x_0) .$$

By the isomorphism of $\mathfrak{M}$ and $\mathfrak{M}'$, the cycle $\hat{\Gamma}_1$ is transformed onto the cycle $\hat{\Gamma}(x_0) \subset \mathfrak{M}'_{x_0}$ such that

(2.3') $$\Gamma = \Gamma_1 + x_0 \subset \partial G .$$

Since $\hat{\Gamma}(x_0)$ is compact, there exists a small neighborhood U_0 of 0 in $\mathbb{C}^n$ such that for every point $\hat{y} \in \hat{\Gamma}(x_0)$ with projection y, there is a unique neighborhood $\hat{U}(x_0, \hat{y})$ of $(x_0, \hat{y})$ on $\mathfrak{M}'$ which projects injectively on $(U_0 + x_0) \times (U_0 + y)$. We will assume that $\overline{U_0} \subset G-x_0$.

It follows that for each $x \in U_0 + x_0$ and each $\hat{y} \in \hat{\Gamma}(x_0)$, there exist a unique point $\hat{y}(x) \in \mathfrak{M}'_x$ such that proj. $\hat{y}(x)$ = proj. $\hat{y}$ and $(x, \hat{y}(x)) \in \hat{U}(x_0, \hat{y})$. For $\hat{y}$ describing the cycle $\hat{\Gamma}(x_0)$, $\hat{y}(x)$ describes a cycle $\hat{\Gamma}(x)$ in $\mathfrak{M}'_x$.

By continuity and in view of (2.1), we can choose U_0 so small that, for $x-x_0 \in U_0$,

$$\int_{\hat{\Gamma}(x)} K(x, y-x)dy_1 \wedge \ldots \wedge dy_n = C(x) \neq 0 .$$

Using again the isomorphism between $\mathfrak{M}'$ and $\mathfrak{M}$, we have to prove the Cauchy formula in the form (1.1'') where

$$(2.4) \qquad C(x) = \int_{\hat{\Gamma}(x)-x} K(x,t)dt_1 \wedge \ldots \wedge dt_n \neq 0 .$$

We will assume that the function $f(x)$ is regular in a domain D_f containing $U_0 + x_0$ and all the closed segments joining x with points of Γ for every $x \in U_0 + x_0$. Consequently, for every $x \in U_0 + x_0$, $\varphi_x(t)$ is regular in a domain of t containing all the closed segments joining 0 with points of $\Gamma - x$.

We use now a classical argument to prove the formula (1.1''). We notice again that $r(\hat{\Gamma}(x)-x)$ is homologous to $\hat{\Gamma}(x)-x$ for every $r > 0$ and we write

$$\frac{1}{C(x)} \int_{\hat{\Gamma}(x)-x} \varphi_x(t)K(x,t)dt_1 \wedge \ldots \wedge dt_n = \frac{1}{C(x)} \int_{r(\hat{\Gamma}(x)-x)} \varphi_x(t)K(x,t)dt_1 \wedge \ldots \wedge dt_n =$$

$$= \frac{1}{C(x)} \int_{\hat{\Gamma}(x)-x} \varphi_x(0)K(x,t)dt_1 \wedge \ldots \wedge dt_n + \int_{\hat{\Gamma}(x)-x} \varphi_x(rt) - \varphi_x(0)K(x,t)dt_1 \wedge \ldots \wedge dt_n .$$

For $r \searrow 0$, $\varphi_x(rt)$ converges uniformly to $\varphi_x(0)$ and we obtain in the limit (1.1'') for $x \in U_0 + x_0$.

Let us now come back to (1.1). Consider all the paths $x(s)$ in $\mathbb{C}^n$, $x(0) = x_0$, $0 \leqq s \leqq s_1$, such that, for every $\hat{y} \in \hat{\Gamma}(x_0)$, there exists a path $(x(s), \hat{y}(s))$ on $\mathfrak{M}'$, starting with $(x(0), \hat{y})$, the projection of each $\hat{y}(s)$ being y. It is then clear that every function $f(x)$ represented by the Cauchy formula in a neighborhood of x_0 will have an analytic continuation along the path $x(s)$, and along each such path still will be represented by the Cauchy formula. All the paths $x(s)$ will describe a multiply covered domain $\hat{D}$ (a manifold over $\mathbb{C}^n$) on which the, in general, multiple-valued function f will be single-valued. The manifold $\hat{D}$ is the natural domain of validity of the Cauchy formula. If we want a domain D for the Cauchy formula satisfying the requirements stated in the introduction, we can choose a maximal domain D in $\mathbb{C}^n$, containing x_0, such that:

1° every open segment joining a point in D with a point of Γ lies in D,

2° for any path $x(s)$, $0 \leqq s \leqq s_1$, joining in D $x_0 = x(0)$ with $x(s_1)$, and for any $\hat{y} \in \hat{\Gamma}(x_0)$ there exists a corresponding path $(x(s), \hat{y}(s))$ in $\mathfrak{M}'$ with $\hat{y}(0) = \hat{y}$ and proj. $\hat{y}(s)$ = proj. $\hat{y}$, such that $\hat{y}(s_1)$ depends only on $x(s_1)$ and not on the choice of the path $x(s)$.

The beginning part of the proof of Theorem II gives the following useful lemma:

Lemma. If $K(x,t)$ is homogeneous in t of degree $\neq -n$, then the form $K(x,t)dt_1 \wedge \ldots \wedge dt_n$ is a differential on $\mathfrak{M}_x$.

§3. Sufficient conditions for general kernels $K(x,t)$.

For kernels $K(x,t)$ which are not homogeneous in t, the condition of Theorem I is not, in general, sufficient. As a simple example, we can take the kernel $K(x,t) \equiv K(t) = \frac{1}{t_1 t_2} + \frac{1}{t_1^2 t_2}$ for $t \in \mathbb{C}^2$.

In the next two theorems we will give sufficient conditions in order that $K(x,t)$ be a Cauchy kernel. In the first, the conditions will be weaker, but, in general, rather difficult to verify. In the second the condition will be essentially stronger, but easier to verify.

Theorem III. The kernel $K(x,t)$ is a Cauchy kernel with cycle $\hat{\Gamma}(x) \subset \mathfrak{M}'_x$ and domain $D \subset \mathbb{C}^n$ if

1° for every $x \in D$

$$\int_{\hat{\Gamma}(x)-x} K(x,t)dt_1 \wedge \ldots \wedge dt_n = C(x) \neq 0.$$

2° For every $x \in D$ there exist a sequence of n-cycles $\hat{\Gamma}_k(x) \subset \mathfrak{M}_x$ and (n+1)-chains $\hat{A}_k \subset \mathfrak{M}_x$, $k = 1, 2, \ldots$ such that $\partial(\hat{A}_k) = (\hat{\Gamma}(x)-x)-\hat{\Gamma}_k(x)$, proj. $\hat{A}_k \setminus (\Gamma - x) \subset D-x$, and maximal distance from proj. $\hat{\Gamma}_k(x)$ to 0 converges to 0 for $k \nearrow \infty$.

3° $\int_{\hat{\Gamma}_k(x)} |K(x,t)dt_1 \wedge \ldots \wedge dt_n| \leqq b$, b independent of k.

Proof. The argument is modeled on the basic elements of the proof of Theorem II. For $\varphi_x(t)$ regular in a domain containing $(D-x) \cup (\Gamma - x)$, the exterior differential form $\varphi_x(t)K(x,t)dt_1 \wedge \ldots \wedge dt_n$ is defined on $\hat{A}_k$. Hence,

$$\frac{1}{C(x)} \int_{\hat{\Gamma}(x)-x} \varphi_x(t)K(x,t)dt_1 \wedge \ldots \wedge dt_n =$$

$$\frac{1}{C(x)} \int_{\hat{\Gamma}(x)-x} \varphi_x(0)K(x,t)dt_1 \wedge \ldots \wedge dt_n + \frac{1}{C(x)} \int_{\hat{\Gamma}_k(x)} (\varphi_x(t)-\varphi_x(0))K(x,t)dt_1 \wedge \ldots \wedge dt_n .$$

The first integral in the last term gives $\varphi_x(0)$, whereas the last integral, in view of 3° and the last condition in 2°, converges to 0. Hence, (1.1'').

Theorem IV. $K(x,t)$ is a Cauchy kernel with domain $D \subset \mathbb{C}^n$ and cycle $\hat{\Gamma}(x)$ for $x \in D$ if

1° $\hat{\Gamma}(x) \subset \mathfrak{M}'_x$, proj. $\hat{\Gamma}(x) = \Gamma \subset \partial D$, every open segment joining $x \in D$ with $y \in \Gamma$ is contained in D.

2° For every $x \in D$, and $\hat{t} \in \hat{\Gamma}(x)-x$, $K(x,t) = \sum_{\ell=0}^{\infty} K_\ell(x,t)$ where $K_\ell(x,t)$ is homogeneous in t of degree m_ℓ, $m_0 = -n$, $m_\ell < m_{\ell+1}$ the series converging uniformly in $x \in D$ and $\hat{t} \in \hat{\Gamma}(x)-x$, the kernel $K_0(x,t)$ being a Cauchy kernel relative to D and $\hat{\Gamma}(x)$.

Proof. We have

$$\int_{\hat{\Gamma}(x)-x} K(x,t)dt_1 \wedge \ldots \wedge dt_n = \sum_{\ell=0}^{\infty} \int_{\hat{\Gamma}(x)-x} K_\ell(x,t)dt_1 \wedge \ldots \wedge dt_n .$$

Since $m_\ell > -n$ for $\ell > 0$, by the lemma of §2 this series reduces to $\int_{\hat{\Gamma}(x)-x} K_0(x,t)dt_1 \wedge \ldots \wedge dt_n$. We can put

$$\int_{\hat{\Gamma}(x)-x} K(x,t)dt_1 \wedge \ldots \wedge dt_n = \int_{\hat{\Gamma}(x)-x} K_0(x,t)dt_1 \wedge \ldots \wedge dt_n = C(x) \neq 0 .$$

Since, for $x \in D$, $\varphi_x(t)$ is regular on $(D-x) \cup (\Gamma-x)$, we can write

$$\int_{\hat{\Gamma}(x)-x} \varphi_x(t)K(x,t)dt_1 \wedge \ldots \wedge dt_n =$$

$$= \sum_{\ell=0}^{\infty} \int_{\hat{\Gamma}(x)-x} \varphi_x(t)K_\ell(x,t)dt_1 \wedge \ldots \wedge dt_n = \sum_{\ell=0}^{\infty} \int_{r(\hat{\Gamma}(x)-x)} \varphi_x(t)K_\ell(x,t)dt_1 \wedge \ldots \wedge dt_n .$$

for every r with $0 < r < 1$. We take r small enough so that $r(\Gamma-x)$ is contained in a ball $\mathcal{B}$ with center 0 such that $\overline{\mathcal{B}} \subset D-x$. For $t \in \overline{\mathcal{B}}$ and x in any compact contained in D, $\varphi_x(t) = \sum_{k=0}^{\infty} P_k(t)$ where $P_k(t)$ are homogeneous polynomials of degree k (with coefficients depending on x), the series converging absolutely and uniformly. Therefore,

$$\int_{r(\hat{\Gamma}(x)-x)} \varphi_x(t) K_\ell(x,t) dt_1 \wedge \dots \wedge dt_n = \sum_{k=0}^{\infty} \int_{r(\hat{\Gamma}(x)-x)} P_k(t) K_\ell(x,t) dt_1 \wedge \dots \wedge dt_n .$$

$P_k(t)K_\ell(x,t)$ is homogeneous in t of degree $m_\ell + k$, hence the degree is greater than $-n$, except when $\ell = k = 0$. Again by the lemma of §2, all the terms of the last series are 0 except when $\ell = k = 0$. We get, therefore, that

$$\frac{1}{C(x)} \int_{\hat{\Gamma}(x)-x} \varphi_x(t) K(x,t) dt_1 \wedge \dots \wedge dt_n = \frac{1}{C(x)} \int_{r(\hat{\Gamma}(x)-x)} P_0(t) K_0(x,t) dt_1 \wedge \dots \wedge dt_n = \varphi_x(0)$$

and (1.1'') is proved.

As illustration of the last theorem, consider the case when $K(x,t)$ is a meromorphic function in the whole space $\mathbb{C}^n \times \mathbb{C}^n$ of the form $K(x,t) = \frac{F(x,t)}{G(x,t)}$, F and G being entire functions of the form $F(x,t) = 1 + \sum_{k=1}^{\infty} P_k(x,t)$ and $G(x,t) = \sum_{k=n}^{\infty} Q_k(x,t)$, where $P_k(x,t)$ and $Q_k(x,t)$ are entire in x and homogeneous polynomials in t of degree k.

Let now $\frac{1}{Q_n(x,t)}$ be a Cauchy kernel relative to D and $\hat{\Gamma}(x)$. Then taking any point $x_0 \in D$ and replacing D and $\hat{\Gamma}(x)$ by $x_0 + r(D-x_0)$ and $x + r(\Gamma(x)-x)$, we will obtain, for r sufficiently small, a suitable development $K(x,t) = \sum_{\ell=0}^{\infty} K_\ell(x,t)$ by using the development of

$$\frac{F(x,t)}{G(x,t)} = \frac{1 + \sum P_k(x,t)}{Q_n(x,t)\left(1 + \frac{\sum_{n+1}^{\infty} Q_k(x,t)}{Q_n(x,t)}\right)} .$$

§4. Construction of cycles for certain classes of kernels.

If we restrict ourselves to kernels $K(x,t)$ homogeous in t, then by Theorem II and its proof, we notice that it will be enough to find such a cycle for a kernel $K(x_0, t)$ with fixed x_0, i.e., for a kernel $K(t)$ independent of x. In this case all the $\mathfrak{M}_x$ coincide. We will denote them by $\mathfrak{M}^0$. $\mathfrak{M}$ is then simply $\mathbb{C}^n \times \mathfrak{M}^0$.[1)]

The construction of a cycle $\hat{\Gamma}_1 \subset \mathfrak{M}^0$ for which $\int_{\hat{\Gamma}_1} K(t)dt_1 \ldots dt_n \neq 0$, i.e., a cycle which shows that $K(t)$ is not a differential, and which will figure in the corresponding Cauchy formula, is, in general, not a trivial task. We are going to mention first two simple cases where such a construction is available:

Case 1). Suppose that the variables $t_1 \ldots t_n$ are divided in k consecutive groups, $t_1, \ldots, t_{\ell_1}$, $t_{\ell_1+1}, \ldots, t_{\ell_1+\ell_2}, \ldots, t_{\ell_1+\ldots+\ell_{k-1}+1}, \ldots, t_{\ell_1+\ldots+\ell_k}$, with $\ell_1 + \ldots + \ell_k = n$. Denote by $t^{(j)}$ the point in $\mathbb{C}^{\ell_j}$ whose coordinates are the variables of the j^{th} group. If then we have, for each $j = 1, \ldots, k$, a homogeneous kernel $K_j(t^{(j)})$ of degree $-\ell_j$ for which we have a corresponding cycle $\hat{\Gamma}^{(j)}$ in the corresponding manifold $\mathfrak{M}_j^0$, then the product $K(t) = K_1(t^{(1)}) \ldots K_k(t^{(k)})$ will have for corresponding cycle the product cycle $\hat{\Gamma}^{(1)} \times \ldots \times \hat{\Gamma}^{(k)}$.
That is how the product of circumferences is obtained as a cycle corresponding to $K(t) = \frac{1}{t_1} \ldots \frac{1}{t_n}$.

Case 2). Let $K(t)$ be a homogeneous kernel with corresponding cycle $\hat{\Gamma}_1$ and let L be a linear mapping of $\mathbb{C}^n$ onto $\mathbb{C}^n$. Then $K(Lt)$ is a homogeneous kernel with corresponding cycle $L^{-1}\hat{\Gamma}_1$.[2)]

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1) However, $\mathfrak{M}'_x$ will not coincide for different x's.

2) We use here the well-determined lifting of the mapping L^{-1} to the manifold $\mathfrak{M}^0$.

Remark 1. If we have a kernel $K(t)$ which is not homogeneous, and for which we have a cycle $\hat{\Gamma}_1$ with $\int_{\hat{\Gamma}_1} K(t)dt_1\wedge\ldots\wedge dt_n \neq 0$, then for any holomorphic homeomorphism of $\mathbb{C}^n$ onto $\mathbb{C}^n$, we will have, obviously, $\int_{L^{-1}\hat{\Gamma}_1} K(Lt)dt_1\wedge\ldots\wedge dt_n \neq 0$. What is more, it can be proved without great difficulty that if $K(t)$ is a Cauchy kernel, then so is also $K(Lt)$.

The next theorem defines a large class of homogeneous kernels for which we can explicitly construct a corresponding cycle.

Theorem V. Let $K(t)$ be a homogeneous kernel of degree $-n$ for which there exists a branch with domain containing $\mathbb{R}^n\setminus(0)$ such that $K(x) > 0$ for $x\in\mathbb{R}^n\setminus(0)$. Then, for every $R > 0$, we get a cycle $\hat{\Gamma}_1$ corresponding to $K(t)$, if we consider the n-cycle $S_R^{n-1}\times S_1^1$,[3] and define points of $\hat{\Gamma}_1$ by $xe^{i\theta}$, $x\in S_R^{n-1}$ and $0\leqq\theta\leqq 2\pi$.

Proof. We have to show that

$$\int_{\hat{\Gamma}_1} K(t)dt_1\wedge\ldots\wedge dt_n \neq 0 . \tag{4.1}$$

We consider the x_n-axis in $\mathbb{R}^n$ as vertical and divide the sphere S_R^{n-1} into lower and upper half-spheres for which $x_n\leqq 0$ or $\geqq 0$ respectively. For the two half-spheres we will use local coordinates $\tau_1,\ldots,\tau_{n-1}$ such that $\tau_1^2+\ldots+\tau_{n-1}^2\leqq 1$. Then the points of the lower (or upper) half-sphere will be given by $x=(R\tau_1,\ldots,R\tau_{n-1},-R\sqrt{1-\tau_1^2-\ldots-\tau_{n-1}^2})$ (or $x=R\tau_1,\ldots,R\tau_{n-1},R\sqrt{1-\tau_1^2-\ldots-\tau_{n-1}^2}))$. The natural orientation of $\mathbb{R}^{n-1}$

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3) S_R^k is the sphere of radius r in $\mathbb{R}^{k+1}$ with center 0.

of variables $\tau_1, \ldots, \tau_{n-1}$ gives then the orientation of S_R^{n-1} on the lower half-sphere and gives the opposite orientation to the one of S_R^{n-1} on the upper half-sphere. It follows that when using the coordinates $\tau_1, \ldots, \tau_{n-1}$ in the integration, we will have to multiply by -1 the integral over the upper half-sphere. We can now write:

$$t_1 = R\tau_1 e^{i\theta}, \ldots, t_{n-1} = R\tau_{n-1} e^{i\theta},$$
$$t_n = \mp R\sqrt{1-\tau_1^2-\ldots-\tau_{n-1}^2}\, e^{i\theta} \quad ,$$

the sign - in t_n being for the lower half-sphere, and + for the upper one. We have then

$$dt_k = Re^{i\theta} d\tau_k + Rie^{i\theta}\tau_k d\theta \quad \text{for } k = 1, \ldots, n-1,$$
$$dt_n = \mp R\left[\sum_{k=1}^{n-1} e^{i\theta} \frac{-\tau_k}{\sqrt{1-\tau_1^2-\ldots-\tau_{n-1}^2}} d\tau_k + ie^{i\theta}\sqrt{1-\tau_1^2-\ldots-\tau_{n-1}^2}\, d\theta\right],$$
$$dt_1\wedge\ldots\wedge dt_n = \mp iR^n e^{in\theta} d\tau_1\wedge\ldots\wedge d\tau_{n-1}\wedge d\theta \frac{1}{\sqrt{1-\tau_1^2-\ldots-\tau_{n-1}^2}} .$$

For $\tau = (\tau_1, \ldots, \tau_{n-1})$ denote by $R\xi^-(\tau)$ the corresponding points of the lower half-sphere, and by $R\xi^+(\tau)$, the points of the upper half-sphere. Then, for $t = R\xi^{\mp}(\tau)e^{i\theta}$ we have $K(t) = R^{-n}e^{-in\theta}K(\xi^{\mp}(\tau))$. Putting in the left hand of (4.1), we get:

$$\int_{\hat{\Gamma}_1} K(t)dt_1\wedge\ldots\wedge dt_n = \int_{|\tau|<1}\int_0^{2\pi} -i\{K(\xi^-(\tau))+K(\xi^+(\tau))\}\frac{1}{\sqrt{1-\tau_1^2-\ldots-\tau_{n-1}^2}} d\tau_1\ldots d\tau_{n-1} d\theta$$
$$= -2\pi i\int_{|\tau|<1}\{K(\xi^-(\tau))+K(\xi^+(\tau))\}\frac{1}{\sqrt{1-\tau_1^2-\ldots-\tau_{n-1}^2}} d\tau_1\ldots d\tau_{n-1} .$$

By the condition of the theorem , $K(\xi^{\mp}(\tau)) > 0$. Hence (4.1) is proved.

An interesting example of Theorem V is the case of the kernel $K(t) = (t_1^2 + \ldots + t_n^2)^{n/2}$ which is important for the study of polyharmonic functions and their Almansi expansions. Another more general case is given by the following corollary:

Corollary V'. The kernel $K(t) = (Q(t))^{n/2}$ where $Q(t) = \sum_{(k,\ell)=1}^{n} a_{k\ell} t_k t_\ell$ is a non-degenerate[4)] quadratic polynomial with constant complex coefficients, is a Cauchy kernel.

Proof. We use here a classical algebraic theorem which says that if $Q_0(t) = t_1^2 + \ldots + t_n^2$, then there exists a linear homeomorphism L such that $Q(t) = Q_0(Lt)$, and we apply Case 2) from the beginning of the present section.

The two simple constructions given at the beginning of this section, together with the class of kernels defined in Theorem V, allow us to construct explicitly, cycles corresponding to a large class of kernels. However, for all the homogeneous kernels $K(t)$ outside of this class, we do not know of any explicit construction of a corresponding cycle, i.e., of any explicit way of checking if the form $K(t)dt_1 \wedge \ldots \wedge dt_n$ is a differential on $\mathfrak{M}^0$. To show the difficulty of this general problem, we will construct such cycles for certain kernels not belonging to the above mentioned class where we will have to use very special properties of these kernels.

We will consider kernels of the form

$$K(t) = \frac{1}{t_1^n + \ldots + t_n^n}. \tag{4.2}$$

When n is even, these kernels belong to the class defined in Theorem V. However, when n is odd and $\geqq 3$, they do not belong to the above defined

4) i.e., $\operatorname{Det}\{a_{k\ell}\} \neq 0$.

class for which we gave an explicit construction of a corresponding cycle. For such an n we can proceed as follows: Consider the sphere S_1^{n-1}; we can consider S_1^{n-1} as the union of 2^n topological simplices $\wp_{m_1,\dots,m_n}$ with $m_k = 0$ or 1, $k = 1,\dots,n$, defined by the local coordinates $\tau = (\tau_1,\dots,\tau_{n-1})$, $\tau_k \geqq 0$, $|\tau| \leqq 1$, by the equation

(4.3) $$x \in \wp_{m_1,\dots,m_n} \ \underline{\text{if and only if}}\ x = (e^{\pi i m_1}\tau_1,\dots,e^{\pi i m_{n-1}}\tau_{n-1}, e^{\pi i m_n}\sqrt{1-|\tau|^2}\,).$$

The orientation of $\wp_{m_1,\dots,m_n}$ in S_1^{n-1} is the orientation given by the representation (4.3), multiplied by the factor $e^{\pi i(m_1+\dots+m_n)}$. Each $\wp_{m_1,\dots,m_n}$ is transformed homeomorphically on a topological simplex $Q_{m_1,\dots,m_n} \subset \mathbb{C}^n$ by the mapping

(4.4) $$x = \left(e^{\pi i m_1}\tau_1,\dots,e^{\pi i m_{n-1}}\tau_{n-1}, e^{\pi i m_n}\sqrt{1-|\tau|^2}\right) \to$$
$$\to y = \left(e^{\frac{2\pi i m_1}{n}}\tau_1,\dots,e^{\frac{2\pi i m_{n-1}}{n}}\tau_{n-1}, e^{\frac{2\pi i m_n}{n}}\sqrt{1-|\tau|^2}\right).$$

The orientation of $\wp_{m_1,\dots,m_n}$ in S_1^{n-1} determines an orientation of $Q_{m_1,\dots,m_n}$, and with this orientation

(4.5) $$Q = \bigcup_{(m_1,\dots,m_n)} Q_{m_1,\dots,m_n}$$

becomes a cycle, homeomorphic to S_1^{n-1}. We put now $\hat{\Gamma}_1 = Q \times S_1^1$, $t \in \hat{\Gamma}_1 \Leftrightarrow t = e^{i\theta}y$, $y \in Q$, $0 \leqq \theta \leqq 2\pi$.

We calculate now $\int_{Q_{m_1,\dots,m_n}\times S_1^1} K(t)dt_1\wedge\dots\wedge dt_n$. We notice first that

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5) We take for $\sqrt{1-|\tau|^2}$ the non-negative value.

$$K(t) = \frac{e^{-in\theta}}{\tau_1^n + \dots + \tau_{n-1}^n + \left(\sqrt{1-|\tau|^2}\right)^n} .$$

Further,

$$dt_1 \wedge \dots \wedge dt_n = \frac{-ie^{in\theta} e^{\frac{2\pi i}{n}(m_1 + \dots + m_n)}}{\sqrt{1-|\tau|^2}} d\tau_1 \wedge \dots \wedge d\tau_{m-1} \wedge d\theta .$$

Hence, multiplying by $e^{\pi i(m_1 + \dots + m_n)}$ to return to the orientation of Q, we obtain

$$\int_{Q_{m_1, \dots, m_n} \times S_1^1} K(t) dt_1 \wedge \dots \wedge dt_n =$$

$$-2\pi i e^{\pi i(1+\frac{2}{n})(m_1 + \dots + m_n)} \int_\tau \frac{1}{\sqrt{1-|\tau|^2}} \frac{d\tau_1 \dots d\tau_{n-1}}{\tau_1^n + \dots + \tau_{n-1}^n + \left(\sqrt{1-|\tau|^2}\right)^n} .$$

It follows

$$\int_{Q \times S_1^1} K(t) dt_1 \wedge \dots \wedge dt_n = (-2\pi i)(1 - e^{\frac{2\pi i}{n}})^n \int_\tau \frac{1}{\sqrt{1-|\tau|^2}} \frac{d\tau_1, \dots, d\tau_{n-1}}{\tau_1^n + \dots + \tau_{n-1}^n + \left(\sqrt{1-|\tau|^2}\right)^n} \neq 0$$

as required.

§5. Almansi Expansions.

An Almansi expansion of a function $f(x)$ around 0 is a development

$$f(x) = \sum_{k=0}^{\infty} (x_1^2+\ldots+x_n^2)^k h_k(x), \tag{5.1}$$

where $h_k(x)$ are harmonic functions in some neighborhood D of 0 and the development converges uniformly on compacts in D.

E. Almansi [1] constructed such developments for polyharmonic functions of finite degree (i.e. satisfying $\Delta^m f = 0$ for finite m) in a domain in $\mathbb{R}^3$. M. Nicolesco (see [9]) extended this result to $\mathbb{R}^n$, $n > 3$. The author introduced the notion of general polyharmonic functions of infinite degree (originally under the name of harmonic functions of infinite order, see [2]), and a few years later he was able to prove the existence of development (5.1) for general polyharmonic functions [3]. It was only much later, in the sixties, that the author proved the corresponding Cauchy formula, which allows us to construct development (5.1) for any function $f(x)$ holomorphic around the origin in $\mathbb{C}^n$.

This result was only mentioned, without proof, in the author's preliminary notes to his lecture on "Traces of analytic solutions of the heat equation", (see [4]). We will give here the proof of this result without insisting on the developments related to the study of polyharmonic functions, which make the study of Almansi expansions of special interest.[1)]

The relevant kernel for the Almansi expansion in $\mathbb{C}^n$ is $K(t) = (t_1^2+\ldots+t_n^2)^{-n/2}$

1) The general theory of polyharmonic functions, with all its ramifications, will be presented in a monograph of the Colloquium Publications of the American Mathematical Society, which we hope will appear soon.

Following the construction in Theorem V (§4), the corresponding cycle $\hat{\Gamma}(x) \subset \mathfrak{M}_x$ can be identified with its projection Γ_R, since it lies in a domain $\hat{U} \subset \mathfrak{M}'$ determining a branch of K. We have

(5.2) $y \in \Gamma_R \Leftrightarrow (y = e^{i\theta}\xi,\ 0 \leq \theta \leq 2\pi,\ \xi \in S_R^{n-1} \subset \mathbb{R}^n \setminus (0),\ R > 0),$

(5.3) $$K(y-x) = K(\xi - e^{-i\theta}x)e^{-in\theta} .$$

The relevant Cauchy formula is then

(5.4) $$f(x) = \frac{1}{C}\int_{\Gamma_R} f(y)K(y-x)dy_1 \wedge \ldots \wedge dy_n ,$$

where $C = \int_{\Gamma_R} K(y)dy_1 \wedge \ldots \wedge dy_n$.

We would like now to determine the largest domain $\hat{D}$ such that (5.4) is valid for all functions f holomorphic in a domain $D_f \supset \hat{D} \cup \Gamma_R$. From the proof of Theorem II, we know that (5.4) is valid in a domain D' containing the origin. Therefore, by analytic continuation, it has to be valid (or give the analytic continuation of f) at any point which can be joined to zero by a path $x(s)$ on which $(x_1-y_1)^2 + \ldots + (x_n-y_n)^2 \neq 0$ for each x in the path and each $y \in \Gamma_R$. We will determine the set of all x's for which this is not true, and will find that the complement of this set in $\mathbb{C}^n$ is the desired $\hat{D}$.

We will need now a lemma which belongs to the theory of polyharmonic functions. The contents of the lemma are mentioned in [4] without proof.

Lemma. Let B_R be the ball of radius R and center 0 in $\mathbb{R}^n$. Let $\tilde{B}_R$ be the open set in $\mathbb{C}^n$ formed by all the points $z \in \mathbb{C}^n$ such that $z = \xi + i\eta$, ξ and η in $\mathbb{R}^n$ with

(5.5) $$|\xi|^2 + |\eta|^2 + 2\sqrt{|\xi|^2|\eta|^2 - (\xi,\eta)^2} < R^2 .$$

$\tilde{B}_R$ is a circled domain, i.e., with every point z, it contains all points $e^{i\theta}z$ with θ real. With the above notations, every function h, harmonic in B_R is analytically continuable to $\tilde{B}_R$, and $\tilde{B}_R$ is the largest domain in $\mathbb{C}^n$ with the last property.

Proof. To prove that $\tilde{B}_R$ is a circled domain, take any $z = \xi + i\eta \in \tilde{B}_R$, ξ and η in $\mathbb{R}^n$, then $e^{i\theta} z = \xi_1 + i\eta_1$ where $\xi_1 = \xi \cos\theta - \eta \sin\theta$, $\eta_1 = \xi \sin\theta + \eta \cos\theta$. An immediate checking shows that $|\xi_1|^2 + |\eta_1|^2 = |\xi|^2 + |\eta|^2$ and $|\xi_1|^2 |\eta_1|^2 - (\xi_1, \eta_1)^2 = |\xi|^2 |\eta|^2 - (\xi, \eta)^2$ which shows that $e^{i\theta} z \in \tilde{B}_R$.

Let h be a harmonic function in B_R. We use the Poisson formula for the ball B_{R_1} with $0 < R_1 < R$

$$h(x) = \frac{1}{R_1 C_0} \int_{S^{n-1}_{R_1}} \frac{R_1^2 - (x_1^2 + \ldots + x_n^2)}{\left((y_1 - x_1)^2 + \ldots + (y_n - x_n)^2\right)^{n/2}} h(y)\, d\sigma(y) . \tag{5.6}$$

If we take a path in $\mathbb{C}^n$, starting with the origin, it is clear from the formula that h will certainly have an analytic continuation along this path as long as $(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2$ does not vanish for any x on the path and any $y \in S^{n-1}_{R_1}$. Take the complementary set of such x's, i.e., that for some $y \in S^{n-1}_{R_1}$, $(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2 = 0$. Let $x = \xi + i\eta$, ξ and η in $\mathbb{R}^n$. Then the last equality is equivalent to

$$|\xi|^2 + |y|^2 - 2(\xi, y) = |\eta|^2, \quad (\xi - y, \eta) = 0 . \tag{5.7}$$

The existence of $y \in S^{n-1}_{R_1}$ for which the last equality in (5.7) is valid, is equivalent to the condition:

$$(\xi, \eta) \leqq |\eta| R_1 ,$$

and the y can be any point of $S^{n-1}_{R_1}$ lying in the hyperplane passing through ξ and orthogonal to η. The first equation in (5.7) can be written

$$2(\xi, y) = R_1^2 + |\xi|^2 - |\eta|^2 .$$

Using the projection $\frac{(\xi, \eta)\eta}{|\eta|^2}$, of ξ on the line of the vector η, we can write:

$$2(\xi, y) = 2\left(\left(\xi - \frac{(\xi,\eta)\eta}{|\eta|^2}\right) + \frac{(\xi,\eta)\eta}{|\eta|^2}, \left(y - \frac{(\xi,\eta)\eta}{|\eta|^2}\right) + \frac{(\xi,\eta)\eta}{|\eta|^2}\right)$$

$$= 2\left(\xi - \frac{(\xi,\eta)\eta}{|\eta|^2}, \; y - \frac{(\xi,\eta)\eta}{|\eta|^2}\right) + 2\,\frac{(\xi,\eta)^2}{|\eta|^2} \quad .$$

Therefore, in order to satisfy, for given $x = \xi + i\eta$, the two equalities in (5.7), y must satisfy, in addition to the preceding conditions, the condition

$$R_1^2 + |\xi|^2 - |\eta|^2 - 2\,\frac{(\xi,\eta)^2}{|\eta|^2} = 2\left(\xi - \frac{(\xi,\eta)\eta}{|\eta|^2}, \; y - \frac{(\xi,\eta)\eta}{|\eta|^2}\right) .$$

This last equality can be satisfied if and only if

$$\left(R_1^2 + |\xi|^2 - |\eta|^2 - 2\,\frac{(\xi,\eta)^2}{|\eta|^2}\right)^2 \leq 4\left|\xi - \frac{(\xi,\eta)\eta}{|\eta|^2}\right|^2 \left|y - \frac{(\xi,\eta)\eta}{|\eta|^2}\right|^2 =$$

$$= 4\left(|\xi|^2 - \frac{(\xi,\eta)^2}{|\eta|^2}\right)\left(R_1^2 - \frac{(\xi,\eta)^2}{|\eta|^2}\right) .$$

Developing the extreme terms of this inequality we obtain

$$(5.8) \qquad R_1^4 + |\xi|^4 + |\eta|^4 - 2R_1^2|\xi|^2 - 2R_1^2|\eta|^2 - 2|\xi|^2|\eta|^2 + 4(\xi,\eta)^2 \leq 0$$

This inequality implies, in particular, $|\xi,\eta| \leq R_1|\eta|$. Hence, (5.8) is the necessary and sufficient condition for the existence of $y \in S^{n-1}_{R_1}$ such that $(x_1-y_1)^2 + \ldots + (x_n-y_n)^2 = 0$. The opposite inequality, therefore, guarantees that there is no such y . The opposite inequality, however, is equivalent to

$$|R_1^2 - |\xi|^2 - |\eta|^2| > 2\sqrt{|\xi|^2|\eta|^2 - (\xi,\eta)^2} \; .$$

Since this condition must be satisfied for all x's on a path starting with zero, $R_1^2 - |\xi|^2 - |\eta|^2$ must be constantly >0 on this path. Hence, the harmonic function h has an analytic extension to every point $x = \xi + i\eta$ satisfying

$$|\xi|^2+|\eta|^2 + 2\sqrt{|\xi|^2|\eta|^2-(\xi,\eta)^2} < R_1^2 .$$

R_1 being arbitrary, but smaller than R, we see that h is analytically extendable to $\tilde{B}_R$.

To show that $\tilde{B}_R$ is the largest domain where all harmonic functions in B_R can be extended, it is enough to show that for each boundary point $z \in \partial\tilde{B}_R$ there exists a harmonic function h regular in B_R which has a singularity at z. But since $z \notin \tilde{B}_R$, there exists a $y \in S_R^{n-1}$ such that $(z_1-y_1)^2+\ldots+(z_n-y_n)^2 = 0$. Hence, the harmonic function

$$\frac{R^2-(x_1^2+\ldots+x_n^2)}{((y_1-x_1)^2+\ldots+(y_n-x_n)^2)^{n/2}}$$ is regular in B_R with singularity at z.

<u>Remark 1.</u> $\tilde{B}_R$ is called the <u>harmonicity hull</u> of B_R. It is shown in the theory of polyharmonic functions that for every domain $D \subset \mathbb{R}^n$ there exists a well-determined harmonicity hull $\tilde{D}$ such that every harmonic single-valued function in D has an analytic extension to $\tilde{D}$ and $\tilde{D}$ is the largest such domain. However, in many cases, this "domain" $\tilde{D}$ may be a manifold over $\mathbb{C}^n$.

We can now state the following theorem:

<u>Theorem VI.</u> <u>Each holomorphic function in a neighborhood of the origin in $\mathbb{C}^n$, can be developed into an Almansi expansion converging uniformly on compacts inside $\tilde{B}_R$ for the largest R for which f is holomorphic in $\tilde{B}_R$.</u>

<u>Proof.</u> Consider the largest $\tilde{B}_R$ in which f is holomorphic. Let A be a compact set contained in $\tilde{B}_R$. Hence, there is an $R_1 < R$ such that $A \subset \tilde{B}_{R_1}$. We can write now, by (5.2), (5.3), and (5.4), the Cauchy formula:

$$f(x) = \frac{1}{C}\int_{\Gamma_{R_1}} f(y)\frac{1}{((y_1-x_1)^2+\ldots+(y_n-x_n)^2)^{n/2}}\,dy_1\wedge\ldots\wedge dy_n =$$

$$=\frac{1}{C}\int_{\Gamma_{R_1}} f(e^{i\theta}\xi)\frac{R_1^2e^{2i\theta}-(x_1^2+\ldots+x_n^2)}{((\xi_1-e^{-i\theta}x_1)^2+\ldots+(\xi_n-e^{-i\theta}x_n)^2)^{n/2}}\;\frac{e^{-in\theta}}{R_1^2e^{2i\theta}-(x_1^2+\ldots+x_n^2)}\,dy_1\wedge\ldots\wedge dy_n .$$

In view of our Lemma, the right-hand side represents a holomorphic function in $\tilde{B}_{R_1}$. Since the formula is valid for x in the neighborhood of the origin, it is valid in the whole of $\tilde{B}_{R_1}$. Furthermore, for $x \in \tilde{B}_{R_1}$, $|x_1^2+\ldots+x_n^2| < R_1^2$. Hence, we can put the development

$$\frac{1}{R_1^2e^{i\theta}-(x_1^2+\ldots+x_n^2)} = \sum_{k=0}^{\infty}\frac{(x_1^2+\ldots+x_n^2)^k}{(R_1^2e^{2i\theta})^{k+1}}$$

in the last integral, and noting that for $y \in \Gamma_{R_1}$, $R_1^2e^{2i\theta} = y_1^2+\ldots+y_n^2$, we can write the formula for $f(x)$ in $\tilde{B}_{R_1}$ as

$$(5.9)\qquad f(x) = \sum_{k=0}^{\infty}(x_1^2+\ldots+x_n^2)^k\frac{1}{C}\int_{\Gamma_{R_1}} f(y)\frac{(y_1^2+\ldots+y_n^2)-(x_1^2+\ldots+x_n^2)}{((y_1-x_1)^2+\ldots+(y_n-x_n)^2)^{n/2}}\frac{1}{(y_1^2+\ldots+y_n^2)^{k+1}}\,dy_1\wedge\ldots\wedge dy_n .$$

The coefficient h_k of $(x_1^2+\ldots+x_n^2)^k$ in this series is a function of x independent of R_1 (since the cycles Γ_{R_1} are all mutually homologous), and we can write

$$(5.10)\; h_k(x) = \frac{1}{C}\int_{\Gamma_{R_1}} f(e^{i\theta}\xi)\frac{(R_1^2-(e^{-2i\theta}x_1^2+\ldots+e^{-2i\theta}x_n^2))e^{-i(n-2)\theta}}{((\xi_1-e^{-i\theta}x_1)^2+\ldots+(\xi_n-e^{-i\theta}x_n)^2)^{n/2}}\,(R_1^2e^{2i\theta})^{-k-1}dy_1\wedge\ldots\wedge dy_n .$$

For fixed y in Γ_{R_1}, we can put $x' = e^{-i\theta}x$, and the integrand up to a factor independent of x' becomes the Poisson kernel. Hence, it is a harmonic function of x', and, therefore, also of x. It follows that $h_k(x)$ is a harmonic function, and, thus, we have got the Almansi development (5.1) valid on any compact in $\tilde{B}_R$.

Remark 2. It is not difficult to prove that the harmonic coefficients h_k in the Almansi-development are uniquely determined by the function f.[2] In the theory of polyharmonic functions, it is proved that if we restrict the Almansi development to $\mathbb{R}^n$, the development for any polyharmonic function f is uniformly convergent on compacts in the largest star-domain centered at the origin in which the function f is regular.

Corollary VI'. Let $P(t)$ be a non-degenerate homogeneous polynomial of degree 2 with complex coefficients

(5.11) $$P(t) = \sum_{j,\ell=1\ldots n} a_{j,\ell} t_j t_\ell , \quad a_{j,\ell} = a_{\ell,j} , \quad \det\{a_{j,\ell}\} \neq 0 .$$

Let L be a linear homeomorphism of $\mathbb{C}^n$ onto $\mathbb{C}^n$, $t' = Lt$ such that $P(t) = \sum_{j=1}^{n} t'^2_j$. If $f(x)$ is holomorphic in $L^{-1}(\tilde{B}_R)$, it has a development into a series

(5.12) $$f(x) = \sum_{k=0}^{\infty} P(x)^k h_k(x)$$

where $h_k(x)$ are holomorphic solutions of the equation

(5.13) $$\sum_{j,\ell=1\ldots n} a_{j,\ell} \frac{\partial^2 h}{\partial x_j \partial x_\ell} = 0 .$$

the series in (5.12) converging uniformly on compacts in $L^{-1}(\tilde{B}_R)$.

The corollary is an immediate consequence of Theorem VI if we change variables x into $x' = Lx$. As a special case, we can consider $P(t) = t_1^2 + t_2^2 + t_3^2 - t_4^2$. The corresponding equation (5.13) is the wave-equation (with t_4 as time) and hence, we have a development of any analytic function into solutions of the wave-equation multiplied by successive powers of $x_1^2 + x_2^2 + x_3^2 - x_4^2$.

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2) We give briefly the proof. In (5.1) change x into y and restrict it to $\mathbb{R}^n$. Multiply both sides by the Poisson kernel corresponding to B_R and integrate over S_R^{n-1}. The left hand side gives a function of R and x completely determined by f. The right-hand side becomes $\sum_0^\infty R^{2k} h_k(x)$, a power series in R^2 with coefficients $h_k(x)$ uniquely determined.

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