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CLASSES OF SPATIALLY INHOMOGENEOUS
PSEUDODIFFERENTIAL OPERATORS

by R. BEALS and C. FEFFERMAN

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Abstract. One can obtain sharp information on a pseudodifferential operator $p(x, D)$ by embedding the symbol p in a symbolic calculus specially designed to reflect the behavior of p . We sketch the development of symbolic calculi arising in this connection, and use our results to give simple proofs of the sharp Gårding inequality and of sufficiency of Nirenberg-Treves' condition (P) for local solvability of equations of principal type.

I. A Calculus of Symbols.

Let ρ and φ be positive smooth functions on $\mathbb{R}^n \times \mathbb{R}^r$ satisfying the inequalities

$$(1) \quad c \leq \rho(x, \xi) \leq C(1 + |\xi|^2)^{1/2}, \quad c(1 + |\xi|^2)^{-1/2} \leq \varphi(x, \xi) \leq C.$$

$$(2) \quad \rho\varphi \geq c.$$

$$(3) \quad |D_x \varphi| \leq C, \quad |D_\xi \rho| \leq C, \quad |D_x \rho(x, \xi)| \leq C \frac{\rho(x, \xi)}{\varphi(x, \xi)},$$

$$|D_\xi \varphi(x, \xi)| \leq C \frac{\varphi(x, \xi)}{\rho(x, \xi)}.$$

$$(4) \quad c \frac{\rho(x, \xi)}{\varphi(x, \xi)} \leq \frac{\rho(y, \eta)}{\varphi(y, \eta)} \leq C \frac{\rho(x, \xi)}{\varphi(x, \xi)} \quad \text{if} \quad |\xi - \eta| < \frac{1}{2} |\xi|.$$

For real numbers M and m we define the class $S_{\phi, \varphi}^{M, m}$ of symbols to consist of all smooth functions $a(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ which satisfy the conditions $|D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} \phi^{M-|\beta|}(x, \xi) \varphi^{m-|\alpha|}(x, \xi)$ for all multi-indices α and β .

Example: Take $\phi = (1 + |\xi|^2)^{\rho/2}$ and $\varphi = (1 + |\xi|^2)^{-\delta/2}$ with $0 \leq \delta \leq \rho \leq 1$ ($\delta \neq 1$). Then $S_{\phi, \varphi}^{M, m}$ is essentially Hörmander's class $S_{\rho, \delta}^{M\rho - m\delta}$ (see [1]).

Corresponding to $a(x, \xi) \in S_{\phi, \varphi}^{M, m}$ we define an operator $a(x, D)$ from the Schwartz class \mathcal{L} to itself by the standard formula

$$a(x, D) u(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) d\xi,$$

where \wedge denotes the Fourier transform.

Theorem 1. Let $a \in S_{\phi, \varphi}^{M, n}$ and $b \in S_{\phi, \varphi}^{M', m'}$ be symbols. Then

(A) The operator $a(x, D) \circ b(x, D)$ defined on \mathcal{L} is the pseudodifferential operator arising from a symbol $a \circ b \in S_{\phi, \varphi}^{M+M', m+m'}$. Moreover, $a \circ b$ has the asymptotic expansion $a \circ b \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} a(x, \xi) \cdot D_x^{\alpha} b(x, \xi)$ in the sense that for any $N > 0$,

$$a \circ b - \sum_{|\alpha| < N} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} a \cdot D_x^{\alpha} b \in S_{\phi, \varphi}^{M+M' - N, m+m' - N}$$

(B) Similarly, the adjoint operator $a^*(x, D)$ restricted to \mathcal{L} is the pseudo-differential operator arising from a symbol $a^{\#}(x, \xi) \in S_{\phi, \varphi}^{M, m}$. Moreover

$$a^{\#} \text{ has the asymptotic expansion } a^{\#} \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} D_x^{\alpha} \bar{a}(x, \xi), \text{ i.e.,}$$

$$a^{\#} - \sum_{|\alpha| < N} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} D_x^{\alpha} \bar{a} \in S_{\phi, \varphi}^{M-N, m-N}$$

Theorem 2. If $a(x, \xi)$ is a symbol in $S_{\phi, \varphi}^{0,0}$, then $a(x, D)$ is bounded on L^2 .

Remarks: 1. Theorem 2 leads to an extension of itself in which one defines analogues of the Sobolev spaces H^s relative to ϕ and φ .

2. If, in addition to (1)-(4) one assumes the estimate

$$(5) \quad \phi(x, \xi) \geq c |\xi| \varphi(x, \xi),$$

then one can prove a theorem on the behavior of operators in $S_{\phi, \varphi}^{M, m}$ under smooth changes of co-ordinates.

We forgo writing down the explicit results.

Our proofs of theorems 1 and 2 rely heavily on ideas from a recent paper of A. P. Calderón and R. Vaillancourt [2].

II. Applications. We begin with a simple proof of the sharp Gårding inequality [3], which asserts that $\operatorname{Re}(a(x, D)u, u) \geq -C \|u\|^2$ for all $u \in L^2$, whenever $a(x, \xi)$ is a non-negative classical first-order symbol. Clearly, the estimate is unaffected (only the constant C changes) if we replace a by $a+1$, so we may and shall assume that $a(x, \xi) \geq 1$. Now set $\phi(x, \xi) = (1 + |\xi|^2)^{1/4} a^{1/2}(x, \xi)$ and $\varphi(x, \xi) = (1 + |\xi|^2)^{-1/4} a^{1/2}(x, \xi)$ and observe that inequalities (1)-(4) hold, and that $a^{1/2}(x, \xi)$ belongs to $S_{\phi, \varphi}^{1/2, 1/2}$. Thus for $T = a^{1/2}(x, D)$ we have $T^*T = a(x, D) +$ a pseudo-differential operator with symbol in $S_{\phi, \varphi}^{0,0}$ by Theorem 1; and we know that the error

term is bounded, by Theorem 2. That is, $a(x, D) = T^* T +$ a bounded error, from which the sharp Gårding inequality is obvious. Note that we could have replaced T by $T' = (a^{1/4}(x, D)) \circ (a^{1/4}(x, D))$ to obtain a positive self-adjoint pseudo-differential operator whose square differs from $a(x, D)$ only by a bounded error. A slight generalization of this procedure produces approximate square roots of operators with non-negative symbols in $S_{\phi, \varphi}^{1,1}$ classes; the square roots have symbols in $S_{\Psi, \psi}^{1/2, 1/2}$ for suitable Ψ, ψ .

Next we sketch a new proof of sufficiency of Nirenberg-Treves' condition (P) for local solvability of partial differential equations of principal type. In their paper [4], Nirenberg and Treves reduced the whole problem to proving the following result: Lemma NT: Let $p_t(x, \xi)$ be a first-order classical symbol depending smoothly on the real parameter t ; and suppose that for each fixed (x, ξ) , the function $t \rightarrow p_t(x, \xi)$ does not change sign. Then the corresponding operator $p_t(x, D)$ may be written in the form $p_t(x, D) = A_t B + C_t$, where B is a fixed (unbounded) self-adjoint operator, A_t is self-adjoint, bounded and non-negative, and C_t is a bounded error.

Ideally, one should prove the lemma simply by writing the symbol $p_t(x, \xi)$ in the form

(*) $p_t = a_t \cdot b + c_t$, where $a_t \geq 0$ and a_t, b, c_t are classical symbols of order $0, 1, 0$ respectively.

If p_t could be so expressed, the conclusion of Lemma NT would follow instantly from the classical symbolic calculus and sharp Gårding inequality. Nirenberg and Treves carried this out in [6] for the case of real-analytic symbols p_t , but unfortunately p_t cannot in general be written in the form (*) using classical symbols. (The unpublished counterexample is due to Mather.) In our original solution [5] to the sufficiency problem, the authors circumvented direct use of Lemma NT by a complicated argument. Now, however, we can use $S_{\phi, \varphi}^{M, m}$ classes to give a rather simple direct proof of Lemma NT: we simply pick the proper ϕ and φ to enable us to write $p_t(x, \xi)$ in the form (*) with a_t, b, c_t in $S_{\phi, \varphi}^{0,0}$, $S_{\phi, \varphi}^{1,1}$, $S_{\phi, \varphi}^{0,0}$ respectively. Essentially, we can take

$$\varphi(x, \xi) = (1 + |\xi|^2)^{-1/4} + \sup_t \left\{ |D_{\xi} p_t(x, \xi)| + \frac{1}{|\xi|} |D_x p_t(x, \xi)| \right\} + \sup\{\delta > 0 \mid p_t(y, \eta)$$

does not change sign as a function of (t, y, η) in the region

$$|y-x| < \delta, \quad |\xi-\eta| < \delta |\xi|\}, \text{ and } \phi(x, \xi) = (1 + |\xi|^2)^{1/2} \varphi(x, \xi).$$

The construction of a_t, b, c_t is not hard, and can be made without mentioning the elaborate Whitney decompositions of [5], though the latter still lurk in the background.

Detailed proofs of the results mentioned in this note will appear in [6].

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