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ON THE BOUNDARY VALUE PROBLEM FOR THE ELLIPTIC SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

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The purpose of this talk is to give the formulation of the boundary value problem for the elliptic system of linear differential equations. The formulation is closely related to the theory of microfunctions.

We use the same notations as in Kashiwara-Kawai [1], [2] and Sato-Kawai-Kashiwara [1] and do not give their definitions here.

The problem which we want to consider is the following :

Let $\mathfrak{M}$ be an admissible and elliptic system of linear differential equations on a real analytic manifold M. Consider a submanifold N of M with codimension d. What is the "boundary value problem" for $\mathfrak{M}$ with respect to N ?

The answer is given by using the comonoidal transform $\widetilde{N_M^*}$ of M with center N as follows. We denote by $\pi_{N|M}$ the projection from $\widetilde{N_M^*}$ to M.

<u>Theorem 1</u> : Assume that N is non-characteristic with respect to $\mathfrak{M}$. Let X and Y be a complex neighborhood of M and N respectively. Denote by p the canonical projection from $S_N^*X - S_Y^*X$ to $S_N^*Y = \sqrt{-1}\,S^*N$ and by q the canonical projection from $S_N^*X - \sqrt{-1}\,S^*M$ to S_N^*M . Then we have the following canonical isomorphism.

$$(1)\quad \mathbb{R}\Gamma_{S_N^*M}(\pi_{N/M}{}^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_M}(\mathfrak{M},\mathcal{B}_M))\ {}^{a}\!\otimes\omega_{N/M}$$

$$\cong \mathbb{R}q_*\,\mathbb{R}\,\mathcal{H}om_{\mathcal{P}_X}(\mathfrak{M},\mathcal{P}_{X\leftarrow Y}|_{S_N^*X}\overset{\mathbf{L}}{\underset{p^{-1}\mathcal{P}_Y}{\otimes}} p^{-1}\mathcal{C}_N)\,.$$

Here $\mathcal{B}_M$ denotes the sheaf of hyperfunctions on M, $\mathcal{C}_N$ denotes the sheaf of microfunctions on $\sqrt{-1}\,S^*N$ and $\omega_{N/M}$ denotes the locally constant sheaf $\mathcal{H}_N^d(\mathbb{C}_M)$.

In this talk we sketch the proof of the theorem.
The proof is given by introducing the auxiliary sheaf

$\mathcal{C}_{N/X} = \mathcal{H}^n_{S^*_N X}(\pi^{-1}_{N/X}\mathcal{O}_X)^a \otimes \omega_N$ on $S^*_N X$.

The following $\mathcal{D}_M$-linear homomorphisms follow obviously from the definition of the sheaf $\mathcal{C}_{N/X}$:

$$(2) \qquad \begin{array}{ccc} \mathbb{R}\Gamma_{S^*_N M}(\pi_{N/M}^{-1}\mathcal{A}_M)^a \otimes \omega_{N/M} & \leftarrow & \mathbb{R}q_!(\mathcal{C}_{N/X}|_{S^*_N X - \sqrt{-1}\, S^* M}) \\ \downarrow & & \downarrow \\ \mathbb{R}\Gamma_{S^*_N M}(\pi_{N/M}^{-1}\mathcal{B}_M)^a \otimes \omega_{N/M} & \rightarrow & \mathbb{R}q_*(\mathcal{C}_{N/X}|_{S^*_N X - \sqrt{-1}\, S^* M}) \end{array}$$

These homomorphisms immediately imply the following homomorphisms :

$$(3) \qquad \begin{array}{ccc} \mathbb{R}\Gamma_{S^*_N M}(\pi_{N/M}^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_M}(\mathfrak{M},\mathcal{A}_M))^a \otimes \omega_{N/M} & \leftarrow & \mathbb{R}q_!(\mathbb{R}\mathcal{H}om_{\mathcal{D}_M}(\mathfrak{M},\mathcal{C}_{N/X})|_{S^*_N X - \sqrt{-1} S^* M}) \\ \downarrow i & & \downarrow j \\ \mathbb{R}\Gamma_{S^*_N M}(\pi_{N/M}^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_M}(\mathfrak{M},\mathcal{B}_M))^a \otimes \omega_{N/M} & \rightarrow & \mathbb{R}q_*(\mathbb{R}\mathcal{H}om_{\mathcal{D}_M}(\mathfrak{M},\mathcal{C}_{N/X})|_{S^*_N X - \sqrt{-1} S^* M}) \end{array}$$

The ellipticity of $\mathfrak{M}$ implies that the homomorphism i is an isomorphism, since $\mathbb{R}\mathcal{H}om_{\mathcal{D}_M}(\mathfrak{M},\mathcal{A}_M) \cong \mathbb{R}\mathcal{H}om_{\mathcal{D}_M}(\mathfrak{M},\mathcal{B}_M)$. Moreover it also implies that the homomorphism j is an isomorphism because $\mathrm{Supp}\,(\mathcal{P}_X \underset{\mathcal{D}_X}{\otimes} \mathfrak{M}) \to S^*_N M$ turns out to be a proper map. Note that

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_M}(\mathfrak{M},\mathcal{C}_{N/X}) = \mathbb{R}\mathcal{H}om_{\mathcal{P}_X}(\mathcal{P}_X \underset{\mathcal{D}_X}{\otimes} \mathfrak{M},\mathcal{C}_{N/X}) .$$

On the other hand the horizontal arrows in (3) are isomorphisms because of ellipticity of $\mathfrak{M}$. In fact it is sufficient to prove this under the assumption that $\mathfrak{M}$ is a single equation, while the proof in this ca can be performed by using the results of Kotaké-Narasimhan [1] and Komatsu [1].

Therefore it is sufficient to show the following lemma to prove the theorem.

<u>Lemma 2</u> : Let $\mathfrak{M}$ be a system of pseudo-differential equations with respect to which N is non-characteristic. Then we have the following isomorphism :

$$(4) \quad \mathbb{R}\mathcal{H}om_{\mathcal{P}_X}(\mathfrak{M},\mathcal{C}_{N/X}) \cong \mathbb{R}\mathcal{H}om_{\mathcal{P}_X}(\mathfrak{M}, \mathcal{P}_{X \leftarrow Y}) \overset{L}{\underset{p^{-1}\mathcal{P}_Y}{\otimes}} p^{-1}\mathcal{C}_N \, .$$

In order to prove this lemma we use the fact that the sheaf $\mathcal{C}_{N/X}$ is isomorphic to the sheaf of microfunctions with holomorphic parameters outside $S^*_Y X \cap S^*_N X$, i.e. solution sheaf of the partial Cauchy-Riemann equations. More precisely, we can transform the statement of Lemma 2 in the following form (Lemma 2') by the aid of the "quantized" contact transformation. Then the situation results as follows.

Consider a smooth holomorphic map $f : X \to Y$ with fiber dimension d. Assume that Y is a complexification of a real analytic manifold N and set $M = f^{-1}(N)$. Then, using the comonoidal transformation

$$\pi_{M/X} : \widetilde{M^*_X} \longrightarrow X,$$

we define the sheaf $\tilde{\mathcal{C}}_M$ by $\mathcal{H}^d_{S^*_M X}(\pi^{-1}_{M/X}\,\mathcal{O}_X)^a \otimes \omega_N$. Note that $\tilde{\mathcal{C}}_M$ is the subsheaf of $\mathcal{C}_M$ which is the solution sheaf of the Cauchy-Riemann equation along the fibers of f.

Now Lemma 2 may read as follows :

<u>Lemma 2'</u> : Let $\mathfrak{M}$ be a system of pseudo-differential equations on X with respect to which f is non-characteristic, that is, $\operatorname{Supp}\mathfrak{M} \cap M \times_Y P^*Y \to P^*Y$ is a finite morphism. Then we have the following isomorphism :

$$(5) \quad \mathbb{R}\mathcal{H}om_{\mathcal{P}_X}(\mathfrak{M},\tilde{\mathcal{C}}_M) \cong \mathbb{R}\mathcal{H}om_{\mathcal{P}_X}(\mathfrak{M}, \mathcal{P}_{X\to Y}) \overset{L}{\underset{f^{-1}\mathcal{P}_Y}{\otimes}} f^{-1}\mathcal{C}_N .$$

In order to see that Lemma 2 is equivalent to Lemma 2', we use the following proposition, which implies that the sheaf $\mathcal{C}_{N/X}$ is isomorphic to the sheaf of microfunctions with holomorphic parameters.

<u>Proposition 3</u> : Let L be a real analytic manifold and Z be its complexification. Consider a complex submanifold X of Z with codimension d such that X (regarded as a real manifold) and L intersect transversally. Denote $L \cap X$ by N. We can identify $S_N^* X$ with $S_L^* Z \times_L N$ by the assumption of transversality of X and L. Then we have the following isomorphism :

$$(6) \qquad \mathbb{R}\mathcal{H}om_{\mathcal{P}_Z}(\mathcal{P}_{Z \hookleftarrow X}, \mathcal{C}_L) \cong \mathcal{C}_{N/X} .$$

As a special case of Proposition 3 we consider the following situation :

$$Z = \mathbb{C}^d \times \bar{\mathbb{C}}^d \times \mathbb{C}^{n-d}, \quad L = \mathbb{C}^d \times_{\mathbb{C}^d} \bar{\mathbb{C}}^d \times \mathbb{R}^{n-d} \qquad \text{and}$$

$$X = \mathbb{C}^d \times \{0\} \times \mathbb{C}^{n-d}.$$

Here $\bar{\mathbb{C}}^d$ denotes the complex conjugate of $\mathbb{C}^d$. We denote a point in L by $(z,s) \in \mathbb{C}^d \times \mathbb{R}^{n-d}$ and a point in Z by $(z,\bar{z},\sigma) \in \mathbb{C}^d \times \bar{\mathbb{C}}^d \times \mathbb{C}^{n-d}$.

In this case $\mathcal{P}_{Z \hookleftarrow X}$ is nothing but the equation $\mathcal{N}$ defined by the relation $\bar{z}_j u = 0$ $(j = 1,\ldots,d)$. The system $\mathcal{N}$ is an equation of Cauchy-Riemann type (in the terminology of Sato-Kawai-Kashiwara [1] Chapter III) on $S_N^* X - S_Y^* X$, where Y is a complexification of N.

Therefore the application of the "quantized" Legendre transformation reduces the situation discussed in Lemma 2 to that in Lemma 2'.

In passing the proof of lemma 2' is performed by the reduction to the case where $d = 1$ and $\mathfrak{M}$ is a single equation. In this cas Lemma 2' is proved by applying the division theorem of Späth's type for pseudo-differential operators (Sato-Kawai-Kashiwara Chapter II Theorem 2.2.6). This completes the proof of Theorem 1.

At the end of this talk we give the formulation of the generalized boundary value problems. This formulation is useful when one treats the so-called "non-elliptic boundary value problems." As for the concrete arguments we refer to Kashiwara-Kawai [2].

In order to treat the generalized boundary value problems we assume the following condition :

(7) The restriction of the map q to $Z = \operatorname{Supp} \mathfrak{M} \cap S_N^* X$ decomposes into the composite of the mapping p and the mapping t from p(Z) to $S_N^* M$.

This assumption allows us to rewrite the right hand side of (4) in the following form :

$$\mathbb{R} t_* \, \mathbb{R}\mathcal{H}om_{\mathcal{P}_N}(p_*(\mathcal{P}_{Y \hookrightarrow X} \underset{\mathcal{P}_X}{\otimes} \mathfrak{M}|_{S_N^* X}), \mathcal{C}_N)[-d]. \tag{8}$$

Now consider an admissible $\mathcal{P}_N$ - submodule $\mathfrak{N}$ of $\mathfrak{M}_o = p_*(\mathcal{P}_{Y \hookrightarrow X} \underset{\mathcal{P}_X}{\otimes} \mathfrak{M}|_{S_N^* X})$ such that the quotient sheaf $\mathcal{L} = \mathfrak{M}_o / \mathfrak{N}$ is also admissible.

Then we clearly have the following exact sequence :

$$\begin{array}{ccc} & \mathbb{R}t_* \, \mathbb{R}\mathcal{H}om_{\mathcal{P}_N}(\mathcal{L}, \mathcal{C}_N) & \\ \swarrow & & \nwarrow {\scriptstyle +1} \\ \mathbb{R}t_* \, \mathbb{R}\mathcal{H}om_{\mathcal{P}_N}(\mathfrak{M}_o, \mathcal{C}_N) & \xrightarrow{B} & \mathbb{R}t_* \, \mathbb{R}\mathcal{H}om_{\mathcal{P}_N}(\mathfrak{N}, \mathcal{C}_N) \end{array} \tag{9}$$

The generalized boundary value problem is by definition the investigation of the range of B. Thus the study of $\mathbb{R}t_* \, \mathbb{R}\mathcal{H}om_{\mathcal{P}_N}(\mathcal{L}, \mathcal{C}_N)$ gives the answer to the generalized boundary value problem.

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