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MICROLOCAL STRUCTURE OF A SINGLE LINEAR PSEUDODIFFERENTIAL EQUATION

by M. SATO

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§ 1. Let $P(x,D)u = 0$ be a single pseudodifferential equation of finite order m defined in a neighborhood of $(x_o, i\eta_o\infty)$, a point in the cosphere bundle $\sqrt{-1}\, S^*M$ of a real analytic manifold M of dimension n, and denote with V and $\overline{V}$ its characteristic variety and the complex conjugate thereof, namely the complex hypersurfaces in a complex neighborhood U of $(x_o, i\eta_o\infty)$ defined by $P_m(z,\zeta) = 0$ and $\overline{P}_m(z,\zeta)(= \overline{P_m(\overline{z},\overline{\zeta})}) = 0$, respectively, P_m denoting the principal symbol of P. If $f(z,\zeta) = 0$ be a reduced local equation for V, one can write $P_m(z,\zeta) = a(z,\zeta)(f(z,\zeta))^l$ with some integer $l > 0$ and non vanishing factor $a(z,\zeta)$.

Assumption 1 : $(x_o, i\eta_o\infty)$ is a non singular point of V as well as of $V\cap\overline{V}$.

Assumption 2 : The restriction onto $V\cap\overline{V}$ of the canonical 1-form $\omega = \zeta_1\, dz_1 + \dots + \zeta_n\, dz_n$ does not vanish at $(x_o, i\eta_o)$.

The codimension of $V\cap\overline{V}$ in U is either 1 or 2 according as $V=\overline{V}$ (the "real characteristics" case) or not. In the latter case, the degree of osculation of V and $\overline{V}$ is a constant integer, say $k(\geq 1)$, along $V\cap\overline{V}$ in a neighborhood of $(x_o, i\eta_o\infty)$. This case we classify further into two, according as $V\cap\overline{V}$ is involutory or not. Here $V\cap\overline{V}$ is said to be involutory if, together with the (reduced) local defining equations $f_1 = f_2 = 0$ of $V\cap\overline{V}$, their Poisson bracket $\{f_1,f_2\}$ vanishes on $V\cap\overline{V}$. (Of course, similar definition applies to a subvariety of an arbitrary codimension). In the opposite case of non-involutory $V\cap\overline{V}$, $(x_o,i\eta_o\infty)$ is a non degenerate point if $\{f_1,f_2\}(x_o,i\eta_o) \neq 0$.

Assumption 3 : In the case of non real V and non involutory $V\cap\overline{V}$, our $(x_o, i\eta_o\infty)$ be a non degenerate point of $V\cap\overline{V}$.

Note that in this case assuption 3 plus the first part of Assumption 1 implies Assumption 2 and the second part of Assumption 1.

Theorem 1 : Under the Assumptions 1, 2 (and 3, in the case (iii) below), the equation $P(x,D)u = 0$ is microlocally equivalent to one of the following equations, considered at $x = 0$, $\eta = (1, 0, 0, \dots, 0)$. (Note that our assumptions implies $n \geq 2$ in the cases (i), (iii) and $n \geq 3$ in the case (ii).)

(i) (The real characteristics case)

$$D_2^1 u = 0 \qquad (\text{or } x_2^1 u = 0, \text{ if one prefers}),$$

(ii) (The non real characteristics case, with involutory $V \cap \overline{V}$)

$$(D_1^{k-1} D_2 + iD_3^k)^1 u = 0$$

$$\left(\begin{array}{ll} \text{or } (D_2 + ix_3^k D_1)^1 u & = 0 \\ \text{or } (x_2 + ix_3^k)^1 u & = 0 \end{array} \right),$$

(iii) (The non real characteristics case, with non involutory $V \cap \overline{V}$)

$$(D_2 \pm ix_2^k D_1)^1 u = 0.$$

By virtue of the principles of microlocal analysis developed in [1], this theorem is readily reduced to the corresponding geometrical statement, namely to the following.

Theorem 2 : By a real contact transformation any hypersurface V satisfying assumptions 1, 2, 3 reduces microlocally to one of the following

(i) $\zeta_2 = 0$ (or $z_2 = 0$),

(ii) $\zeta_1^{k-1}\zeta_2 + i\zeta_3^k = 0$ (or $\zeta_2 + iz_3^k \zeta_1 = 0$ or $z_2 + iz_3^k = 0$),

(iii) $\zeta_2 \pm iz_2^k \zeta_1 = 0$.

The case (i) is a classical result since Lagrange-Hamilton-Jacobi (see [1]). The case (iii) is proved in [2]. Here we shall supply a proof for the case (ii), by slightly modifying the proof of theorem 2.2.1 of [1] (which says that an involutory manifold V of an arbitrary codimension r which intersects transversally with its complex conjugate $\overline{V}$ at an involutory submanifold (of codimension 2r) and satisfies the Assumptions 1 and 2 above at $(x_0, i\eta_0\, \infty)$, can always be contact-transformed microlocally to $\zeta_2 + i\zeta_3 = 0, \ldots, \zeta_{2r} + i\zeta_{2r+1} = 0$ considered at $x = 0$, $\eta = (1, 0, \ldots, 0)$. We always have $2r+1 \leq n$).

Namely, we first prove Lemma 3 below, and thence our statement above (as well as theorem 2.2.1 of [1] cited above) will follow .

§ 2. Let V denote an involutory submanifold of codimension r in U, and V_o a submanifold of codimension 1 in V, both of them passing through $(x_o, i\eta_o)$. Their local defining equations will be given by $f_1 = \dots = f_r = 0$ and $f_1 = \dots = f_r = q = 0$, respectively. (Hence $q = 0$ defines a non singular hypersurface U_o in U passing through $(x_o, i\eta_o)$ which intersects transversally with V at V_o.) Here and in what follows, all functions to be considered on U are holomorphic functions in $(z,\zeta) = (z_1,\dots,z_n; \zeta_1,\dots,\zeta_n)$ which are <u>homogeneous</u> in variables ζ_j.

Let Λ denote an open set in $\mathbb{C}^r$ containing the origin whose point we denote by $\lambda = (\lambda_1,\dots,\lambda_r)$. Let $\Phi(\lambda) = \Phi(z,\zeta; \lambda)$ and $\psi(\lambda) = \psi(z,\zeta;\lambda)$ be holomorphic functions in $U\times\Lambda$ which vanish on $V\times\Lambda$. Hence we can write

$$\Phi(\lambda) = \Phi_1(\lambda)f_1 + \dots + \Phi_r(\lambda)f_r, \quad \psi(\lambda) = \psi_1(\lambda)f_1 + \dots + \psi_r(\lambda)f_r$$

with $\Phi_j(\lambda)$ and $\psi_j(\lambda)$ holomorphic in a neighborhood of $(x_o, i\eta_o;0)$ in $U\times\Lambda$. Finally, we denote with $\Delta(\lambda)$ the determinant of the following $r\times r$-matrix

$$\{q,\psi(\lambda)\}\left(\frac{\partial\Phi_j(\lambda)}{\partial\lambda_k}\right)_{j,k=1,\dots,r} - \{q,\Phi(\lambda)\}\left(\frac{\partial\psi_j(\lambda)}{\partial\lambda_k}\right)_{j,k=1,\dots,r} .$$

We note that the equation $\Delta(\lambda) = 0$ as well as the condition that $\Delta(\lambda)$ should be non vanishing for a generic vector λ, depends only on V, V_o, $\Phi(\lambda)$ and $\psi(\lambda)$ and is not affected by the ambiguity of the choice of f_j, q, $\Phi_j(\lambda)$ and $\psi_j(\lambda)$. We now state.

<u>Lemma 3</u> : <u>Let holomorphic functions</u> $h_{o1},\dots,h_{or}$ <u>which vanish at</u> $(x_o,i\eta_o)$ <u>be given on</u> U_o <u>so that</u> $\Delta(h_{o1},\dots,h_{or}) \not\equiv 0$ <u>on</u> V_o. <u>Then they can be prolonged to holomorphic functions</u> $h_1,\dots,h_r$ <u>in a neighborhood of</u> U_o <u>in</u> U <u>so that</u> $\{\psi(h_1,\dots,h_r),\Phi(h_1,\dots,h_r)\} = 0$ <u>holds identically.</u>

And indeed, one can construct such $h_1,\dots,h_r$ by solving a Kowalewskian system of (non-linear) first order partial differential equations, as will be seen in the below.

We remark that, if $h_j^*(z,\zeta)$ denote <u>any</u> holomorphic extension of h_{oj} into a neighborhood of U_o in U, the restriction onto V: $\{q,\Phi(h^*)\}|_V$ coincides with $\{q,\Phi(\lambda)\}|_{\lambda\mapsto h_o}$ because one has

$$\{q,\Phi(h^*)\} = \{q,\Phi(\lambda)\}|_{\lambda\to h^*} + \sum_j \{q,h_j^*\} \frac{\partial\Phi(\lambda)}{\partial\lambda_j}\Big|_{\lambda\to h^*}$$

and $\dfrac{\partial\Phi(\lambda)}{\partial\lambda_j} \equiv 0 \bmod. f_1,\dots,f_r$.

<u>Proof of lemma 3</u> : Along with the ordinary Poisson bracket $\{\psi,\Phi\} = \sum_k \left(\dfrac{\partial\psi}{\partial\zeta_k}\dfrac{\partial\Phi}{\partial z_k} - \dfrac{\partial\psi}{\partial z_k}\dfrac{\partial\Phi}{\partial\zeta_k}\right)$, we have the following "prolonged" expression for the bracket of functions $\Phi(w) = \Phi(z,\zeta;w)$ and $\psi(w) = \psi(z,\zeta;w)$ involving functions $w_j = w_j(z,\zeta)$:

$$\{\psi(w),\Phi(w)\} = \sum_k \Big(\Big(\frac{\partial\psi}{\partial\zeta_k} + \sum_l (w_\zeta)_{l,k}\frac{\partial\psi}{\partial w_l}\Big)\Big(\frac{\partial\Phi}{\partial z_k} + \sum_l (w_z)_{l,k}\frac{\partial\Phi}{\partial w_l}\Big)$$

$$- \Big(\frac{\partial\psi}{\partial z_k} + \sum_l (w_z)_{l,k}\frac{\partial\psi}{\partial w_l}\Big)\Big(\frac{\partial\Phi}{\partial\zeta_k} + \sum_l (w_\zeta)_{l,k}\frac{\partial\Phi}{\partial w_l}\Big)\Big),$$

with $(w_z)_{l,k}$ and $(w_\zeta)_{l,k}$ denoting $\dfrac{\partial w_l}{\partial z_k}$ and $\dfrac{\partial w_l}{\partial\zeta_k}$, respectively. The right hand side expression will be denoted by $\Theta(w,w_z,w_\zeta) = \Theta(z,\zeta;w,w_z,w_\zeta)$. Since V is involutory, there exist holomorphic functions $\Theta_{oj}(\lambda)$ in a neighborhood of $(x_o,i\eta_o;0)$ in $U\times\Lambda$ so that we have $\{\psi(\lambda),\Phi(\lambda)\}$ $\left(= \sum_k \left(\dfrac{\partial\psi(\lambda)}{\partial\zeta_k}\dfrac{\partial\Phi(\lambda)}{\partial z_k} - \dfrac{\partial\psi(\lambda)}{\partial z_k}\dfrac{\partial\Phi(\lambda)}{\partial\zeta_k}\right)\right) = \Theta_{01}(\lambda)f_1 + \dots + \Theta_{0r}(\lambda)f_r$,

whence we obtain

$$\Theta(w,w_z,w_\zeta) = \Theta_1(w,w_z,w_\zeta)f_1 + \dots + \Theta_r(w,w_z,w_\zeta)f_r$$

by setting

$$\Xi_j(w,w_z,w_\zeta) = \Theta_{0j}(w) +$$

$$+\sum_{k,l}\Big((w_z)_{l,k}\Big(\frac{\partial\psi(w)}{\partial\zeta_k} + \frac{1}{2}\sum_p (w_\zeta)_{p,k}\frac{\partial\psi(w)}{\partial w_p}\Big) - (w_\zeta)_{l,k}\Big(\frac{\partial\psi(w)}{\partial z_k} + \frac{1}{2}\sum_p (w_z)_{p,k}\frac{\partial\psi(w)}{\partial w_p}\Big)\Big)\frac{\partial\Phi_j(w)}{\partial w_l}$$

$$-\sum_{k,l} \text{(the similar expression with } \Phi(w) \text{ and } \psi(w) \text{ interchanged).}$$

Let us further consider the case where $w_j = w_j(t) = w_j(z,\zeta;t)$ involve a parameter t and are holomorphic in $(z,\zeta;t) \in U\times \mathbb{C}$ in a neighborhood of $(x_o,i\eta_o;0)$. Of course we have $\{\psi(w(t)),\Phi(w(t))\} = \Theta(w(t),w_z(t),w_\zeta(t))$ as long as t is an independent parameter, while we obtain , when t is substituted by $q(z,\zeta)$, the following identity :

$$\{\psi(w(q)),\Phi(w(q))\} = \Big(\Theta(w(t),w_z(t),w_\zeta(q)) + \frac{\partial\psi(w(t))}{\partial t}\{q,\Phi(w(t))\} - \frac{\partial\Phi(w(t))}{\partial t}\{q,\psi(w(t))\}\Big)_{t\mapsto q} .$$

The expression inside the bracket on the right hand side is again a linear form of $f_1,\dots, f_r$, and, by equating to 0 each of the coefficients we form a system of equations.

$$\Theta_j(z,\zeta;w,w_z,w_\zeta) + \frac{\partial\psi_j(w)}{\partial t}\{q,\Phi(w)\} - \frac{\partial\Phi_j(w)}{\partial t}\{q,\psi(w)\} = 0 ,$$

or equivalently

$$\Theta_j(w,w_z,w_\zeta) + \sum_k \Big(\{q,\Phi(w)\}\frac{\partial\psi_j(w)}{\partial w_k} - \{q,\psi(w)\}\frac{\partial\Phi_j(w)}{\partial w_k}\Big)\frac{\partial w_k}{\partial t} = 0 \quad , \ (j = 1,\dots,r)$$

This is a determined system of first order differential equations for unknown functions $w_1,\dots,w_r$ in $(z,\zeta;t)$, and, under the assumptions of the lemma, one has a well-posed Cauchy problem if one assigns to $w_j(t)$ initial data at $t = 0$ such that $\Delta(w(0)) \neq 0$. Therefore, existence of prolongations h_j of h_{oj} with the properties claimed in the lemma is implied if one first choose an arbitrary holomorphic extension h_j^* of h_{oj} to a neighborhood of U_o in U, then solves the above system of equations by assigning h_j^* as initial data (see the remark following the lemma) to obtain the local solutions $w_j(z,\zeta;t)$ and finally, defines h_j by $h_j(z,\zeta) = w_j(z,\zeta;q(z,\zeta))$. Note that h_j and h_j^* coincide on U_o because we have $w_j(q) \equiv w_j(0) \bmod q$. (q.e.d.)

<u>Remark 1</u> : If Φ_j, ψ_j, h_{oj} are all of real coefficients (i.e. $\Phi_j(\bar{z},\bar{\zeta};\bar{\lambda}) = \Phi_j(z,\zeta;\lambda)$, etc.) h_j can also be chosen real-coefficiented.

<u>Remark 2</u> : If W is another involutory submanifold of codimension $s(\le r)$ in U containing V as submanifold (i.e. $V \subset W \subset U$), if our defining equation $f_1 = 0,\ldots,f_r = 0$ of V is so chosen that the first s equations define W, and if $\Phi(\lambda)$ vanishes on $W \times \Lambda$ so that it has the form $\Phi(\lambda) = \Phi_1(\lambda)f_1 + \ldots + \Phi_s(\lambda)f_s$, then we have

$$\psi(w(t))|_W = \psi(w(0))|_W \text{ and hence, } \psi(h)|_W = \psi(h^*)|_W ,$$

provided that $\{q,\Phi(w(0))\} \not\equiv 0$ at $(x_o, i\eta_o)$. In particular, if initial data h^* are so chosen that $\{f_j,\psi(h^*)\}|_W = 0$ holds for $j = 1,\ldots,s$, then one has $\{f_j,\psi(h)\}|_W = 0$ for $j = 1,\ldots,s$, because for a holomorphic function g on U, $\{f_j,g\}|_W$, $j = 1,\ldots,s$ is completely determined by $g|_W$ (and hence one can naturally talk about $\{f_j,g_o\}|_W$ for a holomorphic function g_o <u>on</u> U_o).

<u>Proof</u> : Combining the equations

$$\{\psi(w),\Phi(w)\} = \Theta(w,w_z,w_\zeta)$$
$$\Theta(w,w_z,w_\zeta) + \frac{\partial\psi(w)}{\partial t}\{q,\Phi(w)\} - \frac{\partial\Phi(w)}{\partial t}\{q,\psi(w)\} = 0$$

and taking into account the congruence $\Phi(w) \equiv 0 \pmod{f_1,\ldots,f_s}$ we have

$$\{q,\Phi(w)\}\frac{\partial\psi(w)}{\partial t} + \{\psi(w),\Phi(w)\} \equiv 0 \pmod{f_1,\ldots,f_s}$$

and this we regard as a differential equation <u>on</u> W, satisfied by an unknown function $\psi(w) = \psi(z,\zeta;w(z,\zeta;t))$ of $(z,\zeta;t)$ modulo $f_1,\ldots,f_s$. (Φ is regarded as known). Then the given $\psi(w(t))$ as well as t independent $\psi(w(0))$ both constitute holomorphic solutions to this equation corresponding to the same initial data $\psi(w(0))(\text{mod.} f_1,\ldots,f_s)$. Therefore by uniqueness of holomorphic solutions they coincide. (q.e.d.)

§ 3. Proof of theorem 2

We can assume without loss of generality that the reduced principal symbol $f(z,\zeta)$ be of the form $f = f_1 + if_2^k$ (cf. [2]). The involutory $V \cap \overline{V}$ is defined by $f_1 = f_2 = 0$. Letting a homogeneous polynomial A of u, v be given by

$$(u+v)^k = u^k + A(u,v).v \quad (\text{i.e. } A(u,v) \underset{\text{def}}{=} \Sigma_{\nu=1}^{k} \tbinom{k}{\nu} u^{k-\nu} v^{\nu-1})$$

we define Φ, Φ_j, ψ, ψ_j as follows :

$$\Phi(\lambda) = \Phi(z,\zeta;\lambda) = \lambda_1^k f_1 - A(\lambda_1 f_2, \lambda_2 f_1)\lambda_2 f_2^k$$

$$\Phi_1(\lambda) = \lambda_1^k, \qquad \Phi_2(\lambda) = - A(\lambda_1 f_2, \lambda_2 f_1)\lambda_2 f_2^{k-1}$$

$$\psi(\lambda) = \lambda_1 f_2 + \lambda_2 f_1, \qquad \psi_1(\lambda) = \lambda_2, \qquad \psi_2(\lambda) = \lambda_1,$$

so that we have

$$(\lambda_1^k + i\,A(\lambda_1 f_2, \lambda_2 f_1)\lambda_2)(f_1 + if_2^k) = \Phi(\lambda) + i(\psi(\lambda))^k,$$

$$\Phi(\lambda) = \Phi_1(\lambda) f_1 + \Phi_2(\lambda) f_2, \qquad \psi(\lambda) = \psi_1(\lambda) f_1 + \psi_2(\lambda) f_2,$$

and apply lemma 3 to it. The matrix $(\partial \psi_j/\partial\lambda_k)_{j,k}$ is equal to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ while $(\partial\Phi_j/\partial\lambda_k)_{j,k}$ is congruent to $\begin{pmatrix} k\lambda_1^{k-1} & 0 \\ 0 & 0 \end{pmatrix}$ (resp. to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$) modulo f_1 and f_2 if $k \geq 2$ (resp. $k = 1$). Also we have $\{q,\Phi(\lambda)\} \equiv \lambda_1^k\{q,f_1\}$ (mod. f_1, f_2).

Hence $\Delta(\lambda)|_V$, which is the determinant of

$$\{q,\psi(\lambda)\}\begin{pmatrix} k\lambda_1^{k-1} & 0 \\ 0 & 0 \end{pmatrix} - \{q,\Phi(\lambda)\}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

is given by $-(\lambda_1^k\{q,f_1\})^2$ for $k \geq 2$. (Similarly we have $\Delta(\lambda) = -(\lambda_1^2 + \lambda_2^2)(\{q,f_1\}^2 + \{q,f_2\}^2)$ for $k = 1$).

So, in the case of $k \geq 2$, by choosing a real-coefficiented $q(z,\zeta)$ such that $q(x_o, i\eta_o) = 0$, $\{q, f_1\}(x_o, i\eta_o) \neq 0$ which of course exists, and initial data h_{oj}, $j = 1,2$, such that $h_{01}(x_o, i\eta_o) \neq 0$ (e.g. $h_{o1} = 1$, $h_{o2} = 0$), the condition $\Delta\ (h_{01}, h_{02}) \neq 0$ holds at $(x_o, i\eta_o)$ and h_{oj} are prolonged to such h_j that satisfy $\{\psi(h_1,h_2), \Phi(h_1,h_2)\} = 0$. The homogeneous degree of $\Phi(h_1,h_2)$, and $\psi(h_1,h_2)$ in ζ-variables can be adjusted (to 0, for example) by a corresponding adjustment to the initial data h_{oj}. The property that $h_{o1} \neq 0$ at $(x_o, i\eta_o)$ also implies that $\Phi(h_1,h_2) + i(\psi(h_1,h_2))^k = 0$ is equivalent to $f_1 + if_2^k = 0$ as a reduced defining equation of V, and $\Phi(h_1,h_2) = \psi(h_1,h_2) = 0$ to $f_1 = f_2 = 0$ as reduced defining equations of $V \cap \overline{V}$. Consequently $d\Phi$, $d\psi$ and ω are linearly independent at $(x_o, i\eta_o)$. The classical Jacobi theory now tells that $\Phi(h_1,h_2)$ and $\psi(h_1,h_2)$ go to z_2 and z_3 by a suitable contact transformation which is real coefficiented and sends $(x_o, i\eta_o)$ to $(0, i(1,0,\ldots,0))$. Then the defining equation of V assumes the form $z_2 + iz_3^k = 0$ and our theorem is proved. In place of (z_2, z_3) one may as well choose $(\zeta_2/\zeta_1, z_3)$ or $(\zeta_2/\zeta_1, \zeta_3/\zeta_1)$ to result $\zeta_2 + iz_3^k\,\zeta_1 = 0$ or $\zeta_1^{k-1}\zeta_2 + i\zeta_3^k = 0$ as the standard form of defining equation of V. (q. e. d.)

Finally we show how the key Lemma 2.2.2 to the theorem 2.2.1 of [1] is derived from lemma 3. Let again V be an involutory manifold of codimension s whose local defining equations $f_1 = \ldots = f_s = 0$ have the property that $df_1, \ldots, df_s$, df_1^c, $\ldots$, df_s^c, ω are linearly independent in the neighborhood of $(x_o, i\eta_o)$. (Whence V intersects with its complex conjugate <u>transversally</u>), and assume $V \cap \overline{V}$ <u>is also</u> involutory (of codimension 2s). Here f_j^c is defined by $f_j^c(z,\zeta) \underset{\text{def}}{=} \overline{f_j(\bar{z},\bar{\zeta})}$.

Choose first a $G(z,\zeta)$ such that $\{G, f_j\}|_V = 0$ (i.e. $\{G, f_j\} \equiv 0$ mod. $f_1, \ldots, f_s$) for $j = 1, \ldots, s$ and such that dG, $df_1, \ldots, df_s$, ω are linearly independent at $(x_o, i\eta_o)$. Choose then a real coefficiented function $q(z,\zeta)$ so that $q(x_o, i\eta_o) = 0$ and $\{G, q\}(x_o, \eta_o) \neq 0$ hold. Define $\Phi(\lambda)$ and $\Phi^c(\bar{\lambda})$ by $\Phi(\lambda) = \lambda_1 f_1 + \ldots + \lambda_s f_s$ and $\Phi^c(\bar{\lambda}) = \bar{\lambda}_1 f_1^c + \ldots + \bar{\lambda}_s f_s^c$, respectively. This means in particular that V, r, $\lambda = (\lambda_1, \ldots, \lambda_r)$, $f = (f_1, \ldots, f_r)$ and (Φ, ψ) in lemma 3 are now replaced by $V \cap \overline{V}$, 2s

$(\lambda,\bar{\lambda}) = (\lambda_1,\ldots,\lambda_s;\ \bar{\lambda}_1,\ldots,\lambda_s)$, $(f,f^c) = (f_1,\ldots,f_s;\ f_1^c,\ldots,f_s^c)$ and (Φ,Φ^c), respectively. Under these circumstances $\Delta(\lambda)$ in lemma 3, as the determined of the matrix

$$\{q,\Phi^c(\bar{\lambda})\}\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} - \{q,\Phi(\lambda)\}\begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix},$$

takes the form $\Delta(\lambda,\bar{\lambda}) = (-\{q,\Phi(\lambda)\}\{q,\Phi^c(\bar{\lambda})\})^s = (-1)^s|\{q,\Phi(\lambda)\}|^{2s}$.

Hence, by lemma 3 and remark 2 to lemma 3, we can conclude that by a suitable choice of $h_j(t)$ we have

$$\{\Phi^c(h^c(q)),\ \Phi(h(q))\} = 0, \text{ and } \{\Phi^c(h^c(q)),f_j\} \equiv 0 \pmod{f_1,\ldots,f_s}$$

while $d\Phi(h(q))$, $d\Phi^c(h^c(q))$ and ω are linearly independent at $(x_o,i\eta_o)$. This is lemma 2.2.2 of [1].

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