J. Wermer

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ECOLE POLYTECHNIQUE
CENTRE DE MATHEMATIQUES
17, rue Descartes
75230 Paris Cedex 05

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SOME RATIONALLY CONVEX SETS

by J. WERMER

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We consider a compact Hausdorff space X and on X a uniform algebra $\mathfrak{A}$. That means that $\mathfrak{A}$ is an algebra of continuous complex-valued functions on X, closed under uniform convergence on X, separating the points of X, and containing the constants.

With norm

$$\|f\| = \max_X |f| \quad ,$$

$\mathfrak{A}$ is than a commutative Banach algebra with unit. According to Gelfand, $\mathfrak{A}$ possesses a spectrum $\mathfrak{M}(\mathfrak{A})$, i.e. the space of all non-trivial homomorphisms of $\mathfrak{A} \to \mathbb{C}$. $\mathfrak{M}(\mathfrak{A})$ is a compact Hausdorff space, in Gelfand's topology.

There is a natural injection of X into $\mathfrak{M}(\mathfrak{A})$, namely the map sending each point x into the functional of evaluation at x. This injection may or may not be onto, i.e. we may have $\mathfrak{M}(\mathfrak{A}) = X$ or $\mathfrak{M}(\mathfrak{A})$ larger than X.

When $\mathfrak{A} = C(X)$, one has $\mathfrak{M}(C(X)) = X$. We have

Problem : Let $\mathfrak{A}$ be a uniform algebra on X such that $\mathfrak{M}(\mathfrak{A}) = X$. What additional condition assures that $\mathfrak{A} = C(X)$?

Of course, one has the classical condition of Stone :

$$f \in \mathfrak{A} \Rightarrow \overline{f} \in \mathfrak{A} \quad .$$

But in problems of uniform approximation in the complex domain this condition is usually difficult to verify.

In 1959, E. Bishop in [1] introduced the notion of a peak point. Let X now be metrizable, $\mathfrak{A}$ a uniform algebra on X. Fix $x_o \in X$.

x_o is a peak point for $\mathfrak{A}$ if $\exists\, f \in \mathfrak{A}$ with $f(x_o) = 1$ and $|f| < 1$ on $X \setminus \{x_o\}$.

Evidently, when $\mathfrak{A} = C(X)$ every point of X is peak point. When $\mathfrak{A}$ is the disk algebra of functions analytic in the open unit disk and continuous in $|z| \leq 1$, $\mathfrak{M}(\mathfrak{A})$ is the full disk while the peak points are exactly the points on the boundary. In general, the set of peak points does not coincide with the Šilov boundary of $\mathfrak{A}$, but in fact coincides with the Choquet boundary.

Let now X be a compact subset of $\mathbb{C}$. We denote by

$$R(X)$$

the uniform algebra on X which is the closure on X of the set of rational functions of z which are holomorphic on X.

It was pointed out by Mergelyan that there exist sets X without interior points such that $R(X) \neq C(X)$. In [1] Bishop proved the following

<u>Theorem</u> : $R(X) = C(X)$ <u>if and only if each point of</u> X <u>is a peak point for</u> $R(X)$.

The question now arose to what extent this result was a general property of uniform algebras. It is not easy to find, among examples arising in a natural way, uniform algebras distinct from C(X), yet such that the spectrum of the algebra consists entirely of peak points.

In 1968, in his Yale thesis Brian Cole gave a very general construction of uniform algebras $\mathfrak{A}$ with the property that every element of $\mathfrak{A}$ has a square root in $\mathfrak{A}$, and used this construction to produce an example of an $\mathfrak{A}$ with $\mathfrak{M}(\mathfrak{A}) = X$, every point of X is a peak point, yet $\mathfrak{A} \neq C(X)$. Later on, he modified his construction to obtain an example which is doubly generated.

It remained of interest, however, to exhibit concrete and simple examples of such algebras. I want to discuss such a construction, due to Richard Basener and contained in his thesis, Brown University (1971).

Let X now be a compact set in $\mathbb{C}^n$. We define R(X), in analogy with the case $n = 1$, as the closure in C(X) of the set of quotients $\frac{P}{Q}$ where P, Q are polynomials in $z_1,\dots,z_n$ and $Q \neq 0$ on X.

Fix $m \in \mathfrak{M}(R(X))$. Put

$$a = (m(z_1),\dots,m(z_n)),\quad a \in \mathbb{C}^n .$$

We claim :

$$(*)\quad \begin{cases} \text{For every polynomial } Q : \\ Q(a) = 0 \Rightarrow Q \text{ vanishes somewhere on } X. \end{cases}$$

For if not, $\exists$ Q, $Q(a) = 0$, $\frac{1}{Q} \in R(X)$. Then

$$1 = m\left(\frac{1}{Q}\cdot Q\right) = m\left(\frac{1}{Q}\right) m(Q) = 0 ,$$

since $m(Q) = Q(a)$. So (*) holds.

<u>Definition</u> : $h_r(X) = \{a \in \mathbb{C}^n \mid (*) \text{ holds}\}$.

$h_r(X)$ is called the <u>rationally convex envelop</u> of X. To each $m \in \mathfrak{M}(R(X))$ there corresponds, as we have just seen, a point $a \in h_r(X)$. The map is easily seen to be bijective, and we may identify $\mathfrak{M}(R(X))$ with $h_r(X)$. We note that when $n = 1$, $h_r(X)$ evidently coincides with X. For $n > 1$, $h_r(X)$ may be larger than X.

Fix now a closed subset S of the open disk $|z| < 1$ in the z-plane. Denote by B the ball : $|z|^2 + |w|^2 \leq 1$ in $\mathbb{C}^2$ and by ∂B its boundary. Put

$$X_S = \{(z,w) \in \partial B \mid z \in S\} .$$

Thus X_S is the set of those points on ∂B which project into S.

Note that if $z \in S$, the entire circle

$$\Gamma_z = \{(z, \sqrt{1-|z|^2}\cdot e^{i\theta}) \mid 0 \leq \theta < 2\pi\}$$

lies in X_S. Thus X_S is, in a sense, a fibrespace with base S and fiber a circle.

Basener's result is the following :

<u>Theorem</u> : $\exists$ S such that the algebra $R(X_S)$ has the properties :

(a) $R(X_S) \neq C(X_S)$.

(b) $h_r(X_S) = X_S$.

(c) Each point of X_S is a peak point for $R(X_S)$.

The proof of (c) is trivial.
Let $(z_o, w_o) \in \partial B$. Put

$$P(z,w) = \frac{1}{2}\{z\bar{z}_o + w\bar{w}_o + 1\} \quad .$$

Then $P(z_o, w_o)$ and $|P| < 1$ on the rest of ∂B. So (c) holds.

To obtain (a) we only need S such that $R(S) \neq C(S)$. For then $\exists$ complex measure μ on S, $\mu \neq 0$, with $\mu \perp R(S)$. For each $F \in C(X_S)$, put

$$L(F) = \int_S d\mu(z)\left\{\int_{\Gamma_z} F\, dm_z\right\} \quad ,$$

where m_z is normalized Lebesgue measure on Γ_z. The L is a bounded linear functional on $C(X_S)$, and $\neq 0$.

If F is holomorphic in some neighborhood of X_S, it is easily verified that $\int_{\Gamma_z} F\, dm_z$ is holomorphic in Z in a neighborhood of S, and so $\in R(S)$. Hence $L(F) = 0$. It follows that L vanishes on $R(X_S)$, and so (a) holds.

To obtain (b) we must restrict S rather severely, and we do not give the details here. They are given in Basener's forthcoming paper [2], and also in [3], pp. 202-203. The crucial point in the proof of (b) is the notion of a <u>Jensen measure</u>.

Let $\mathfrak{A}$ be a uniform algebra on a space X and $m \in \mathfrak{M}(\mathfrak{A})$. A Jensen measure μ_m for m is a probability measure on X such that Jensen's inequality

$$\log |\hat{f}(m)| \leq \int_X \log |f| \, d\mu_m$$

holds for all $f \in \mathfrak{A}$. Concerning Jensen measures, see [3] or [4].

Cole's work, discussed above, also is treated in [3] and [4].

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