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ON THE SINGULARITIES OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Let $P(D)$ be a differential operator with constant coefficients in $\mathbb{R}^n$, $D = -i\partial/\partial x$. We shall study the properties of the singular support of a solution of an equation $P(D)u = f \in C^\infty(X)$ where X is an open set in $\mathbb{R}^n$. For applications to existence theorems for the adjoint see [1].

When P is of principal type it is known that a closed set $F \subset X$ is the singular support of a distribution u in X with $P(D)u = f$ if and only if for every $x \in F$ there is a bicharacteristic B through x such that the component of $B \cap X$ containing x is in F. The bicharacteristics are of dimension 1 or 2. If p is the principal part of P then by definition

$$B = \{x + \operatorname{Re} zp'(\xi),\ z \in \mathbb{C}\}$$

for some $\xi \in \mathbb{R}^n \setminus 0$ with $p(\xi) = 0$. Thus the space of normals of B is a tangent of $P^{-1}(0)$ at infinity in the direction ξ.

We shall here give general results which are similar but less precise. To state them we must first give a suitable definition of tangent planes at infinity to the surface $P^{-1}(0)$. If V is a linear subspace of $\mathbb{R}^n$ we introduce

$$\widetilde{P}_V(\xi,t) = \sup\{|P(\xi+\theta)|;\ \theta \in V, |\theta| < t\}$$

with an arbitrary norm. When $V = \mathbb{R}^n$ we write $\widetilde{P}(\xi,t)$ instead of $\widetilde{P}_V(\xi,t)$ and note that with constants depending only on n and the degree of P we have

$$C_1\widetilde{P}(\xi,t) \leq \Sigma|P^{(\alpha)}(\xi)|\ t^{|\alpha|} \leq C_2\widetilde{P}(\xi,t),$$

so the notation agrees with the usual one. Now set

$$\sigma_P(V) = \inf_{t>1} \varliminf_{\xi\to\infty} \widetilde{P}_V(\xi,t)/\widetilde{P}(\xi,t).$$

This is a continuous function of V so it vanishes for a closed set of subspaces V which is clearly independent of the choice of norm in $\mathbf{R}^n$. In view of lemmas 8 and 9 below it is reasonable to consider V as a tangent of $P^{-1}(0)$ at ∞ in $\mathbf{R}^n$ precisely when $\sigma_P(V) = 0$.

<u>Theorem 1</u> : Let Γ be a closed convex set in $\mathbf{R}^n$ and V a linear subspace of $\mathbf{R}^n$ with $\Gamma + V = \Gamma$, that is, V belongs to the edge. If $\sigma_P(V') = 0$, where V' denotes the orthogonal space, one can for every non-negative integer k find $u \in C^k(\mathbf{R}^n)$ with $P(D)u = 0$, sing supp $u = \Gamma$ and $u \notin C^{k+1}(N)$ if N is any open set intersecting Γ.

<u>Theorem 2</u> : Let Γ be a closed convex set in $\mathbf{R}^n$ and let V be the largest vector space with $\Gamma + V = \Gamma$, that is, V is the edge of Γ. If $\sigma_P(V') \neq 0$ it follows that every $u \in \mathcal{D}'(\mathbf{R}^n)$ with $P(D)u \in C^\infty(\mathbf{R}^n)$ and sing supp $u \subset \Gamma$ is in $C^\infty(\mathbf{R}^n)$.

There is also a local uniqueness theorem :

<u>Theorem 3</u> : Let $\varphi_1,\dots,\varphi_k \in C^1(X)$ where X is an open set in $\mathbf{R}^n$, and let x^0 be a point in X where $d\varphi_1(x^0),\dots,d\varphi_k(x^0)$ are linearly independent. Assume that $\sigma_P(W) \neq 0$ for the space W spanned by $d\varphi_1(x^0),\dots,d\varphi_k(x^0)$. If $u \in \mathcal{D}'(X)$, $P(D)u \in C^\infty(X)$ and $u \in C^\infty(X_-)$,

$$X_- = \{x \in X; \varphi_j(x) < \varphi_j(x^0) \text{ for some } j = 1,\dots,k\},$$

then $u \in C^\infty$ in a neighborhood of x^0.

The case k = 1 is an analogue of Holmgren's uniqueness theorem with supports replaced by singular supports and the principal part p replaced by $\sigma_P(N)$ where N denotes the one dimensional space containing $N \in \mathbf{R}^n\setminus 0$. It is therefore possible to use theorem 3 and theorem 1 to give the following analogue of theorem 5.3.3 in [2] :

Theorem 4 : Let $X_1 \subset X_2$ be open convex sets in $\mathbb{R}^n$. Then an open set $X \subset X_2$ has the property

$$u \in \mathcal{D}'(X_2),\ Pu \in C^\infty(X_2),\ u \in C^\infty(X_1) \Rightarrow u \in C^0(X)$$

if and only if for every hyperplane H with $\sigma_P(H') = 0$ the set X_1 intersects every affine hyperplane parallel to H which meets X.

Theorem 3 also implies the following result :

Theorem 5 : Let V be a linear subspace of $\mathbb{R}^n$ such that $\sigma_P(V') = 0$ but $\sigma_P(W') \neq 0$ for every linear subspace W strictly contained in V. If $P(D)u \in C^\infty$ and sing supp $u \subset V$ it follows that either sing supp $u = V$ or $u \in C^\infty$.

On the other hand we know from theorem 1 that one can find u with $P(D)u = 0$ and sing supp $u = V$. Minimal linear subspaces V with $\sigma_P(V') = 0$ therefore play to a large extent the same role as the bicharacteristics for operators of principal type. However, examples show that the singular support of a distribution with $P(D)u = 0$ is not always a union of such spaces as in the case of operators of principal type.

Theorem 6 : If P_1 and P_2 are equally strong then $\sigma_{P_1}(V) = 0$ is equivalent to $\sigma_{P_2}(V) = 0$.

This follows easily from the definition.

We shall now give a brief sketch of the proofs of theorems 1 and 3. First of all one must reformulate the condition $\sigma_P(V) = 0$ or $\sigma_P(V) \neq 0$ using the Tarski-Seidenberg theorem.

<u>Lemma 7</u> : If $\sigma_P(V) = 0$ it follows that there are positive constants b, β, r_1, ρ such that for any $t > 1$ and $r > r_1 t^\rho$ one can find $\xi \in \mathbb{R}^n$ with $|\xi| = r$ and

$$\widetilde{P}_V(\xi,t)/\widetilde{P}(\xi,t) < b\,t^{-\beta}.$$

If $\sigma_P(V) \neq 0$ on the other hand one can find b, r_1, ρ such that

$$\widetilde{P}_V(\xi,t)/\widetilde{P}(\xi,t) > b > 0 \text{ if } t > 1 \text{ and } |\xi| > r_1 t^\rho.$$

To prove theorem 1 the next step is to express the smallness of $\widetilde{P}_V(\xi,t)/\widetilde{P}(\xi,t)$ in terms of the zeros of P. In doing so we assume that V is defined by $x' = 0$ where $x = (x',x'')$, $x' = (x_1,\dots,x_\nu)$ and $x'' = (x_{\nu+1},\dots,x_n)$ is a splitting of the coordinates in two groups.

<u>Lemma 8</u> : For suitable positive constants ε_0, C, γ (depending only on n and the degree m of P) the inequality $\widetilde{P}_{V'}(\xi,t)/\widetilde{P}(\xi,t) \leq \varepsilon < \varepsilon_0$ implies that there exists an analytic map $\theta \to \zeta(\theta)$ from the ball $\Omega = \{\theta \in \mathbb{C}^\nu,\ |\theta| < \gamma t\}$ to $\mathbb{C}^n$ such that

(i) $\zeta'(\theta) = \xi_0' + \theta$ where $\xi_0' \in \mathbb{R}^\nu$ and $|\xi_0' - \xi'| \leq t$

(ii) $|\zeta''(\theta) - \xi''| < C\,t\,\varepsilon^{1/m}$, $\theta \in \Omega$,

(iii) $P(\zeta(\theta)) = 0$.

This gives a precise sense to the statement that $\sigma_P(V') = 0$ means that V' is a tangent to $P^{-1}(0)$ at ∞

For any positive integer N one can find a function $\Phi^N(\theta)$ with support in the real part of Ω and integral 1 such that the derivatives of order $|\alpha| \leq N$ can be estimated by $(CN/t)^{|\alpha|}$. With such functions we form a solution of the equation $P(D)u = 0$ by taking the average

$$u(x) = \int e^{i\langle x,\zeta(\theta)\rangle}\Phi(\theta)\,d\theta.$$

For a suitable choice of the parameters ξ, t, N one can make u very small outside V although u(0) = 1, and the proof of theorem 1 follows easily.

We shall only sketch the proof of theorem 3 in the case k = 1 in order to simplify the notations. The first step is again to express a lower bound for $\tilde{P}_W(\xi,t)/\tilde{P}(\xi,t)$ as a property of the zeros of P when W is a line in $\mathbb{R}^n$ generated by the unit vector η^0.

Lemma 9 : Let δ, c be fixed positive constants, $\delta < 1$. Then there exists positive constants c_1, γ depending only on δ, c, n and the degree of P such that $\tilde{P}_W(\xi,t)/\tilde{P}(\xi,t) > c$ implies that for some r with $0 < r < \delta t$ we have

$$|P(\xi+(i\,t+z)\eta^0+\zeta)| \geq c_1\tilde{P}(\xi,t) \text{ if } z \in \mathbb{C},\ |z| = r,\ |\zeta| < \gamma t .$$

The converse is also true and the proof is elementary.

To construct a fundamental solution of P one usually interprets the integral

$$(2\pi)^{-n} \int e^{i<x,\zeta>} P(\zeta)^{-1}\, d\zeta$$

by taking it over some cycle which avoids the zeros of P and is close to $\mathbb{R}^n$. Sometimes the cycle is taken close to the cycle defined by

$$\xi \to \xi + i\lambda(\log|\xi|)\eta^0$$

instead, where η^0 is a unit vector in $\mathbb{R}^n$ and λ is large. The modulus of the exponential is then $|\xi|^{-\lambda<x,\eta^0>}$ so the fundamental solution becomes roughly $\lambda<x,\eta^0>$ times differentiable at x (thus a distribution of order $-\lambda<x,\eta^0>$ when $<x,\eta^0> < 0$). The conclusion is that if $P(D)u \in C^\infty$ and if the singular support of u has a compact intersection with a half space $\{x;<x,\eta^0> > a\}$, then the intersection is in fact empty.

If $\sigma_p(\eta^0) \neq 0$ it follows from lemma 7 that outside a compact set we have on this cycle a lower bound for $\tilde{P}_W(\xi,t)/\tilde{P}(\xi,t)$ when $t = \lambda \log|\xi|$ We can therefore replace the Dirac measure at $\xi + i t \eta^0$ by a mean value over the zero free region given by lemma 3. More precisely we use the measure

$$\int u(\zeta)\, d\mu^N_{\xi,t}(\zeta) = (2\pi)^{-1} \int_0^{2\pi} d\phi \int u(\xi + (i\, t + re^{i\phi})\eta^0 + \tau)\Phi^N(\tau)\, d\tau$$

where $|\tau| < \gamma t/2$ in supp Φ^N and the derivatives of Φ^N of order $k \leq N$ can be estimated by $(CN/t)^k$. We choose N to be the integral part of εt. This gives a fundamental solution which for any ν is in C^ν for large λ in the set defined by

$$(1-\delta) < x, \eta^0 > \; > - \gamma|x|/20, \quad 3\varepsilon e/\gamma < |x| < 6\varepsilon e/\gamma .$$

The proof of theorem 3 is then a routine matter.

For the details of proof and additional statements we refer to a paper with the same title to be published in connection with the symposium on linear and partial differential equations in Jerusalem June 1972.

BIBLIOGRAPHIE

[1] L. Hörmander : On the existence and the regularity of solutions of linear pseudo-differential equations. L'enseignement Math.17(1971), 99-163.

[2] L. Hörmander : Linear partial differential operators. Springer Verlag, Berlin-Göttingen-Heidelberg 1963.
