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Absolutely-p-summing operators in $\mathcal{L}_r$-spaces II

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ABSOLUTELY-p-SUMMING OPERATORS IN $\mathcal{L}_r$-SPACES II

by A. PIETSCH

Exposé N° 31 — 9 Juin 1971

§ 6. THE ν_p-NORM (cf. [10], [23], [24]).

In the following let us assume that at least one of the Banach spaces E and F has finite dimension. Then every operator $T\in\mathcal{L}(E,F)$ can be represented in the form

$$Tx = \sum_i \langle x, a_i\rangle y_i \quad \text{for all} \quad x\in E$$

with $a_1,\dots,a_n\in E'$ and $y_1,\dots,y_n\in F$. Now the ν_p-norm is defined by

$$\nu_p(T) := \inf\left[\{\sum_i \|a_i\|^p\}^{1/p} \sup_{\|b\|\le 1}\{\sum_i |\langle y_i, b\rangle|^{p'}\}^{1/p'}\right],$$

$1<p<\infty$, where the infimum is taken over all possible representations. In the case $p=1$ and $p=\infty$ we put

$$\nu_1(T) := \inf\left[\sum_i \|a_i\|\,\|y_i\|\right]$$

and

$$\nu_\infty(T) := \inf\left[\sup_i \|a_i\| \sup_{\|b\|\le 1}\sum |\langle y_i, b\rangle|\right].$$

It follows from the well-known relations

$$\pi_p(T) = \sup\{|\mathrm{trace}(ST)| : S\in\mathcal{L}(F,E),\ \nu_{p'}(S)\le 1\} \quad \text{for all } T\in\mathcal{L}(E,F)$$

and

$$\nu_{p'}(S) = \sup\{|\mathrm{trace}(ST)| : T\in\mathcal{L}(E,F),\ \pi_p(T)\le 1\} \quad \text{for all } S\in\mathcal{L}(F,E)$$

that the inequalities

$$\pi_p(T)\le c\,\pi_q(T) \quad \text{for all } T\in\mathcal{L}(E,F)$$

and

$$\nu_{q'}(S)\le c\,\nu_{p'}(S) \quad \text{for all } S\in\mathcal{L}(F,E)$$

are equivalent.

We have

$$\pi_p(T)\le \nu_p(T) \quad \text{for all } T\in\mathcal{L}(E,F),$$

and in the case $p = 2$,

$$\pi_2(T) = \nu_2(T) \quad \text{for all } T \in \mathcal{L}(E,F) .$$

If at least one of the Banach spaces E and F has the extension property then also the equation

$$\pi_p(T) = \nu_p(T) \quad \text{for all } T \in \mathcal{L}(E,F)$$

is valid. On the other side A. Pełczyński [21] has shown that there exists no constant $c > 0$ such that for every bounded linear operator T between arbitrary finite dimensional Banach spaces the inequality

$$\nu_p(T) \leq c \ \pi_p(T)$$

holds.

<u>Problem</u> : If $1 \leq r,\ s \leq \infty$ and $1 < p < \infty$, does there exists a constant $c_{rsp} > 0$ such that

$$\nu_p(T) \leq c_{rsp}\, \pi_p(T) \quad \text{for all } T \in \mathcal{L}(l_r^n, l_s^n) \ ?$$

Now we prove further results by duality.

<u>Theorem 1*</u> : Let $T \in \mathcal{L}(E, l_s^n)$. If $2 < s < p \leq \infty$ then

$$\nu_p(T) \leq c_{s'p'}\, c_{s'1}^{-1}\, \nu_\infty(T) \quad .$$

<u>Proof</u> : If $2 < s < \infty$ and $1 \leq p' < s'$ then by theorem 1 we have

$$\pi_1(S) \leq c_{s'p'}\, c_{s'1}^{-1}\, \pi_{p'}(S) \quad \text{for all } S \in \mathcal{L}(l_s^n, E) .$$

Consequently, there holds the dual inequality

$$\nu_p(T) \leq c_{s'p'}\, c_{s'1}^{-1}\, \nu_\infty(T) \quad \text{for all } T \in \mathcal{L}(E, l_s^n) .$$

Theorem 2* : Let $T \in \mathcal{L}(E, l_s^n)$. If $s = 1$, resp $1 < s \leq 2$, then

$$\nu_2(T) \leq c_G \nu_\infty(T), \quad \text{resp. } \nu_2(T) \leq c_{2s'} c_{21}^{-1} \nu_\infty(T) \quad .$$

Theorem 3* : Let $T \in \mathcal{L}(l_r^n, F)$ If $1 \leq r \leq 2$ and $1 < p \leq 2$ then

$$\nu_p(T) \leq c_{2p'} c_{21}^{-1} \nu_2(T) \quad .$$

Theorem 4* (CONJECTURE) : Let $T \in \mathcal{L}(l_r^n, F)$. If $2 < r < \infty$ and $1 < p < q < r'$ then, with a constant $c_{r.pq} > 0$,

$$\nu_p(T) \leq c_{r,pq} \nu_q(T) \quad .$$

Finally, we formulate some special cases of theorem 1* and 2*.

Proposition 4 (S Kwapień [7]) : Let $T \in \mathcal{L}(l_\infty^n, l_s^n)$. If $2 < s < p < \infty$ then

$$\nu_p(T) \leq c_{s'p'} c_{s'1}^{-1} \|T\| \, .$$

Proposition 5 (J. Lindenstrauss and A. Pełczyński [8]) : Let $T \in \mathcal{L}(l_\infty^n, l_s^n)$. If $s = 1$, resp. $1 < s \leq 2$, then

$$\nu_2(T) \leq c_G \|T\| \, , \quad \text{resp} \quad \nu_2(T) \leq c_{2s'} c_{21}^{-1} \|T\| \quad .$$

Proof : The results follow from the fact that

$$\nu_\infty(T) = \|T\| \quad \text{for all} \quad T \in \mathcal{L}(l_\infty^n, F) \quad .$$

Remark : It is easy to prove the following stronger form of lemma 4. Let $T \in \mathcal{L}(E, l_s^n)$. Then

$$\nu_s(T) \leq \pi_s(T) \quad .$$

One can obtain further results by using this inequality.

§ 7. IDENTITY OPERATORS IN l_r^n-SPACES.

Let I_n be the identity operator from l_r^n into l_s^n. We define the limit order $\lambda_I(r,s,\pi_p)$ to be the infimum of all real numbers λ for which there exists a constant $c_{rs,p} > 0$ such that the inequality

$$\pi_p(I_n : l_r^n \to l_s^n) \le c_{rs,p}\, n^\lambda$$

for all $n = 1,2,\ldots$ holds. The limit order $\lambda_I(r,s,\nu_p)$ is defined in the same way.

Historical remark : The π_p- and ν_p-norm of the identity operator from l_r^n into itself was determined or estimated by D.J.H. Garling and Y. Gordon (cf. [16], [17], [18]). In the cases ν_∞ and π_1 the first result was proved by B. Grünbaum [19] and D. Rutovitz [22]. A. Tong [26] has given necessary and sufficient conditions for a diagonal operator from l_r into l_s to be nuclear (cf. also L. Schwartz [25]).

Lemma 5 : If $\alpha + \beta \le 1$,

$$\lambda_I(r,s,\pi_p) \le \alpha \quad \text{and} \quad \lambda_I(s,r,\nu_{p'}) \le \beta$$

then

$$\lambda_I(r,s,\pi_p) = \alpha \quad \text{and} \quad \lambda_I(s,r,\nu_{p'}) = \beta \quad .$$

Proof : Since

$$n = \operatorname{trace}(I_n) \le \pi_p(I_n : l_r^n \to l_s^n)\, \nu_{p'}(I_n : l_s^n \to l_r^n)$$

we have

$$1 \le \lambda_I(r,s,\pi_p) + \lambda_I(s,r,\nu_{p'}) \le \alpha+\beta = 1 \quad .$$

Consequently, identity holds.

Lemma 6 :

$$\lambda_I(r,s,\|.\|) \le \begin{cases} 1/s - 1/r & \text{if} \quad r \ge s \\ 0 & \text{if} \quad r \le s \end{cases} .$$

Proof : The result follows from the well-known inequality

$$\|I_n : l_r^n \to l_s^n\| \le \begin{cases} n^{1/s - 1/r} & \text{if } r \ge s \\ 1 & \text{if } r \le s \end{cases} .$$

Lemma 7 :

$$\lambda_I(1, \infty, \nu_1) \le 0 \quad .$$

Proof : If $e = (\varepsilon_i)$ ranges over the set of all n-dimensional vectors with $\varepsilon_i = \pm 1$ then the identity operator I_n has the representation

$$I_n x = 2^{-n} \sum_e \langle x, e \rangle e \quad \text{for all } x \in l_1^n .$$

Consequently,

$$\nu_1(I_n : l_1^n \to l_\infty^n) \le 1 \quad .$$

Lemma 8 : If $1 < p \le \infty$ then

$$\lambda_I(1, 2, \nu_p) \le 0 \quad .$$

Proof : We represent the identity operator I_n in the form

$$I_n x = 2^{-n} \sum_e \langle x, e \rangle e \quad \text{for all } x \in l_1^n .$$

Then

$$\{\sum_e \|e\|_\infty^p\}^{1/p} = 2^{n/p} .$$

On the other hand, it follows from Littlewood's inequality (cf. [20]) that

$$\sup_{\|b\|_2 \le 1} \{\sum_e |\langle e, b \rangle|^{p'}\}^{1/p'} \le 2^{n/p'} c_{p'} \quad .$$

Therefore,

$$\nu_p(I_n : l_1^n \to l_2^n) \le c_{p'}$$

Lemma 9 :

$$\lambda_I(1,2,\pi_1) \le 0 .$$

Proof : From Littlewood's inequality we have

$$\|x\|_2 \le c_L 2^{-n} \sum_e |\langle x, e\rangle| .$$

Consequently, if $x_1,\dots,x_m \in l_1^n$

$$\sum_i \|x_i\| \le c_L \sup_{\|a\|_\infty \le 1} \sum |\langle x_i, a\rangle| ,$$

and therefore,

$$\pi_1(I_n : l_1^n \to l_2^n) \le c_L .$$

Remark : Lemma 9 follows also from proposition 2[G]

Lemma 10 :

$$\lambda_I(\infty,p,\nu_p) \le 1/p .$$

Proof : If $e_1,\dots,e_n$ are the usual unit vectors we can represent the identity operator I_n in the form

$$I_n x = \sum_i \langle x, e_i\rangle \quad \text{for all} \quad x \in l_\infty^n .$$

Since

$$\{\sum_i \|e_i\|_1^p\}^{1/p} = n^{1/p} \quad \text{and} \quad \sup_{\|a\|_{p'} \le 1} \{\sum_i |\langle e_i, a\rangle|^{p'}\}^{1/p'} = 1$$

we obtain

$$\nu_p(I_n : l_\infty^n \to l_p^n) \le n^{1/p} .$$

Lemma 11 : If $1 \le s < 2$ then

$$\lambda_I(s',s,\pi_1) \le 1/s$$

Proof : In the case $s=1$ the result follows from lemma 10. Now we assume $1<s<2$ Then there exists ε with $0<\varepsilon<s-1$. By lemma 3 and 10 we obtain

$$\begin{aligned}\pi_1(I_n:l^n_{s'}\to l^n_s) &\le c_{s\,s-\varepsilon}\, c^{-1}_{s1}\, \pi_{s-\varepsilon}(I_n:l^n_{s'}\to l^n_s)\\ &\le c_{s\,s-\varepsilon}\, c^{-1}_{s1}\|I_n:l^n_{s'}\to l^n_\infty\|\pi_{s-\varepsilon}(I_n:l^n_\infty\to l^n_{s-\varepsilon})\|I_n:l^n_{s-\varepsilon}\to l^n_s\|\\ &\le c_{s\,s-\varepsilon}\, c^{-1}_{s1}\; n^{1/(s-\varepsilon)} .\end{aligned}$$

Consequently,

$$\lambda_I(s',s,\pi_1) \le 1/(s-\varepsilon) .$$

The result follows since ε can be made as small as we please.

Remark : It should be possible to determine the exact asymptotic behaviour of $\pi_p(I_n:l^n_{s'}\to l^n_s)$ as n tends to infinity by using the relation

$$\pi_p(I_n:l^n_{s'}\to l^n_s) = c^{-1}_{sp}\,\{\int_{\mathbb{R}^n}\|x\|^p_s\, d\,\mu^n_s(x)\}^{1/p} ,\quad 1\le p<s .$$

The limit orders $\lambda_I(r,s,\|.\|)$ and $\lambda_I(s,r,\nu_1)$

By lemma 6 we have

$$(1)\qquad \lambda_I(r,s,\|.\|) \le \begin{cases} 1/s - 1/r & \text{if } r\ge s\\ 0 & \text{if } r\le s .\end{cases}$$

On the other hand it follows from lemma 6, 7 and 10 that

$$\lambda_I(s,r,\nu_1) \le \lambda_I(s,1,\|.\|) + \lambda_I(1,\infty,\nu_1) + \lambda_I(\infty,r,\|.\|) \le 1/s' + 1/r$$

and

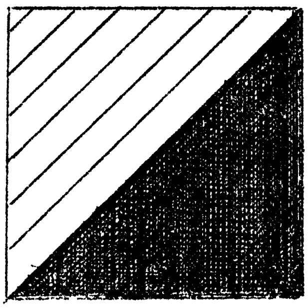
$$\lambda_I(s,r,\nu_1) \le \lambda_I(s,\infty,\|.\|) + \lambda_I(\infty,1,\nu_1) + \lambda_I(1,r,\|.\|) \le 1 .$$

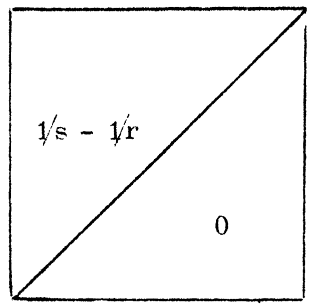
In each case, choosing the best result we obtain

$$(1^*)\qquad \lambda_I(s,r,\nu_1) \le \begin{cases} 1/s' + 1/r & \text{if } r\ge s\\ 1 & \text{if } r\le s\end{cases}$$

Finally, lemma 5 implies that identity holds in (1) and (1*). In what follows we illustrate our results with pairs of diagrams in the unit square with coordinates $1/r$ and $1/s$. In the left hand diagram we plot the level curves of $\lambda_I(r,s,\pi_p)$. In the right hand diagrams we indicate the algebraic expression for $\lambda_J(r,s,\pi_p)$.

$\underline{\lambda_I(r,s,\|\ \|)}$





<u>The limit orders</u> $\lambda_I(r,s,\pi_2)$ <u>and</u> $\lambda_I(s,r,\nu_2)$

By lemmas 6, 9 and 10 we have

$$\lambda_I(r,s,\pi_2) \le \lambda_I(r,\infty,\|\ \|) + \lambda_I(\infty,2,\pi_2) + \lambda_I(2,s,\|.\|)$$

$$\le 0 + 1/2 + \begin{cases} 1/s - 1/2 & \text{if} \quad 1 \le s \le 2 \\ 0 & \text{if} \quad 2 \le s \le \infty \end{cases}$$

and

$$\lambda_I(r,s,\pi_2) \le \lambda_I(r,1,\|.\|) + \lambda_I(1,2,\pi_2) + \lambda_I(2,s,\|.\|)$$

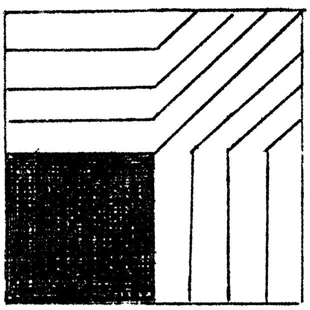
$$\le 1/r' + 0 + \begin{cases} 1/s - 1/2 & \text{if} \quad 1 \le s \le 2 \\ 0 & \text{if} \quad 2 \le s \le \infty . \end{cases}$$

Consequently,

$$\lambda_I(r,s,\pi_2) \le \begin{cases} 1/r' + 1/s - 1/2 & \text{if} \quad 1 \le r \le 2,\ 1 \le s \le 2, \\ 1/s & \text{if} \quad 2 \le r \le \infty,\ 1 \le s \le 2, \\ 1/r' & \text{if} \quad 1 \le r \le 2,\ 2 \le s \le \infty, \\ 1/2 & \text{if} \quad 2 \le r \le \infty,\ 2 \le s \le \infty . \end{cases} \tag{2}$$

Since $\lambda_I(s,r,\nu_2) = \lambda_I(s,r,\pi_2)$ it follows from lemma 5 that identity holds in (2).

$\underline{\lambda_I(r,s,\pi_2)}$



$\frac{1}{s}$	$\frac{1}{r'}+\frac{1}{s}-\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{r'}$

<u>The limit orders</u> $\lambda_I(r,s,\pi_p)$ <u>and</u> $\lambda_I(s,r,\nu_{p'})$ <u>with</u> $1 \le p < 2$

Since by theorem 2 and 2* for $1 \le r \le 2$ we have

$$\lambda_I(r,s,\pi_p) = \lambda_I(r,s,\pi_2) \quad \text{and} \quad \lambda_I(s,r,\nu_{p'}) = \lambda_I(s,r,\nu_2)$$

in the following we need only consider the case $2 < r \le \infty$.

By lemma 6 and 11 we obtain

$$\lambda_I(r,s,\pi_p) \le \lambda_I(r,s',\|.\|) + \lambda_I(s',s,\pi_p) \le 1/s \qquad \text{if} \quad r \le s' \quad \text{and} \quad 1 \le s \le 2,$$

and

$$\lambda_I(r,s,\pi_p) \le \lambda_I(r,r',\pi_p) + \lambda_I(r',s,\|.\|) \le 1/r' \qquad \text{if} \quad r' \le s \quad \text{and} \quad 1 \le r' \le 2$$

On the other hand it follows from lemma 6 and 10 that

$$\lambda_I(r,s,\pi_p) \le \lambda_I(r,\infty,\|.\|) + \lambda_I(\infty,p,\pi_p) + \lambda_I(p,s,\|.\|)$$

$$\le 0 + 1/p + \begin{cases} 1/s - 1/p & \text{if} \quad p \ge s, \\ 0 & \text{if} \quad p \le s \quad . \end{cases}$$

In each case, choosing the best result we obtain

$$(3) \qquad \lambda_I(r,s,\pi_p) \le \begin{cases} 1/s & \text{if} \quad p' \le r \le \infty,\ 1 \le s \le p, \\ 1/p & \text{if} \quad p' \le r \le \infty,\ p \le s \le \infty, \\ 1/s & \text{if} \quad 2 \le r \le p',\ 1 \le s \le r', \\ 1/r' & \text{if} \quad 2 \le r \le p',\ r' \le s \le \infty \quad . \end{cases}$$

By lemma 6 and 11

$$\lambda_I(s,r,\nu_{p'}) \le \lambda_I(s,\infty,\|.\|) + \lambda_I(\infty,p',\nu_{p'}) + \lambda_I(p',r,\|.\|)$$

$$\le 0 + 1/p' + \begin{cases} 1/r - 1/p' & \text{if} \quad p' \ge r, \\ 0 & \text{if} \quad p' \le r \quad . \end{cases}$$

Moreover,

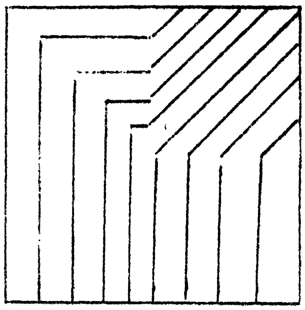
$$\lambda_I(s,r,\nu_{p'}) \le \lambda_I(s,r,\nu_2) = 1/s' \quad \text{if} \quad 1 \le s \le 2 \ .$$

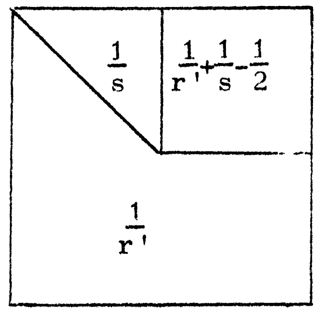
Consequently,

$$(3^*) \qquad \lambda_I(s,r,\nu_{p'}) \le \begin{cases} 1/s' & \text{if} \quad p' \le r \le \infty,\ 1 \le s \le p, \\ 1/p' & \text{if} \quad p' \le r \le \infty,\ p \le s \le \infty, \\ 1/s' & \text{if} \quad 2 \le r \le p',\ 1 \le s \le r', \\ 1/r & \text{if} \quad 2 \le r \le p',\ r' \le s \le \infty \ . \end{cases}$$

Finally, lemma 5 implies that identity holds in (3) and (3*).

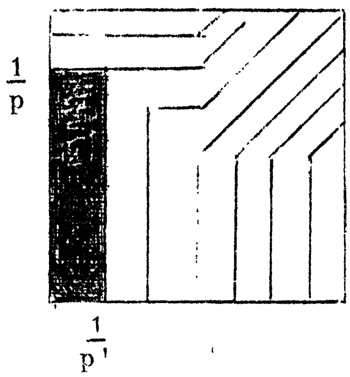
$\underline{\lambda_I(r,s,\pi_1)}$

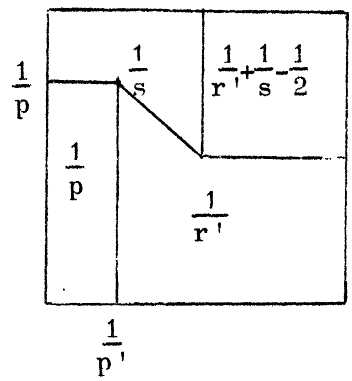




$\underline{\lambda_I(r,s,\pi_1)}$

$1<p<2$





The limit orders $\lambda_I(r,s,\pi_p)$ and $\lambda_I(r,s,\nu_{p'})$ with $2<p<\infty$

Since by theorem3 and 3* for $1 \le s \le 2$ we have

$$\lambda_I(r,s,\pi_p) = \lambda_I(r,s,\pi_2) \quad \text{and} \quad \lambda_I(s,r,\nu_{p'}) = \lambda_I(s,r,\nu_2)$$

in the following we need only consider the case $2 < s \le \infty$.
Since

$$\lambda_I(r,s,\pi_p) \le \lambda_I(r,s,\nu_p)$$

by (3*) we obtain

$$(4) \qquad \lambda_I(r,s,\pi_p) \le \begin{cases} 1/r' & \text{if} \quad 1 \le r \le p', \ p \le s \le \infty, \\ 1/p & \text{if} \quad p' \le r \le \infty, \ p \le s \le \infty, \\ 1/r' & \text{if} \quad 1 \le r \le s', \ 2 \le s \le p, \\ 1/s & \text{if} \quad s' \le r \le \infty, \ 2 \le s \le p \ . \end{cases}$$

It follows from lemma 6 and 10 that

$$\lambda_I(s,r,\nu_{p'}) \le \lambda_I(s,\infty,\|.\|) + \lambda_I(\infty,p',\nu_p) + \lambda_I(p',r,\|.\|)$$

$$\le 0 + 1/p' + \begin{cases} 1/r - 1/p' & \text{if} \quad p' \ge r, \\ 0 & \text{if} \quad p' \le r . \end{cases}$$

On the other hand lemma 6 and 8 imply that

$$\lambda_I(s,r,\nu_{p'}) \le \lambda_I(s,1,\|.\|) + \lambda_I(1,2,\nu_{p'}) + \lambda_I(2,r,\|.\|)$$

$$\le 1/s' + 0 + 0 \quad \text{if} \quad 2 \le r .$$

In each case, choosing the best result we obtain

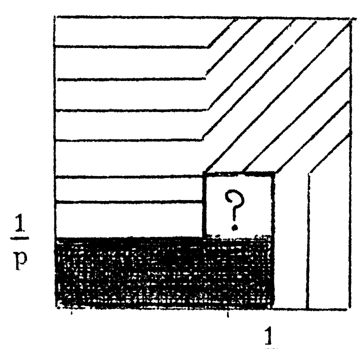
$$(4^*) \qquad \lambda_I(s,r,\nu_{p'}) \le \begin{cases} 1/r & \text{if} \quad 1 \le r \le p', \ p \le s \le \infty, \\ 1/p' & \text{if} \quad p' \le r \le \infty, \ p \le s \le \infty, \\ 1/r & \text{if} \quad 1 \le r \le p', \ 2 \le s \le p, \\ 1/s' & \text{if} \quad 2 \le r \le \infty, \ 2 \le s \le p \end{cases}$$

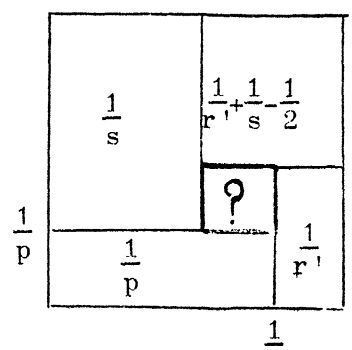
Because the square

$$Q_{I,p} := \{(1/r, 1/s) : p' < r < 2, \ 2 < s < p\}$$

does not appear in (4^*), we have the open problem wether identity holds for all r and s in (4).

$$\frac{\lambda_I(r,s,\pi_p)}{2<p<\infty}$$





§ 8. LITTELWOOD OPERATORS IN l_r^n-SPACES.

In the following n rangs over the set of all natural numbers $n = 2^k$ with $k = 1,2,\dots$. The symmetric Littlewood operators $A_n = (\alpha_{ik}^{(n)})$ are defined inductively by(cf. [20])

$$A_2 := \begin{pmatrix} 1, & 1 \\ 1, & -1 \end{pmatrix} \ ,\dots,\ A_{2n} := \begin{pmatrix} A_n, & A_n \\ A_n, & -A_n \end{pmatrix}, \dots .$$

Then

$$A_n^2 = n\, I_n \quad \text{and} \quad \alpha_{ik}^{(n)} = \pm 1 .$$

The limit orders $\lambda_A(r,s,\pi_p)$ and $\lambda_A(r,s,\nu_p)$ are introduced in the same way as in the case of identity operators.

Lemma 12 : If $\alpha + \beta \leq 2$,

$$\lambda_A(r,s,\pi_p) \leq \alpha \quad \text{and} \quad \lambda_A(s,r,\nu_{p'}) \leq \beta$$

then

$$\lambda_A(r,s,\pi_p) = \alpha \quad \text{and} \quad \lambda_A(s,r,\nu_{p'}) = \beta .$$

Proof : Since

$$n^2 = \operatorname{trace}(n\, I_n) \leq \pi_p(A_n : l_r^n \to l_s^n)\ \nu_{p'}(A_n : l_s^n \to l_r^n)$$

we have

$$2 \leq \lambda_A(r,s,\pi_p) + \lambda_A(s,r,\nu_{p'}) \leq \alpha+\beta \leq 2 .$$

Consequently, identity holds.

Lemma 13 : If $2 \leq s \leq \infty$ then

$$\lambda_A(r,s,\|.\|) \leq 1/s .$$

Proof : Since the operator $n^{-1/2}A_n$ is unitary we have

$$\|A_n : l_2^n \to l_2^n\| \leq n^{1/2} .$$

On the other hand, because $|\alpha_{ik}^{(n)}|=1$, it follows that

$$\|A_n : l_1^n \to l_\infty^n\| \le 1 \ .$$

Finally, if $2 \le s \le \infty$, the M. Riesz' connexity theorem implies

$$\|A_n : l_{s'}^n \to l_s^n\| \le n^{1/s} \ .$$

Lemma 14 :

$$\lambda_A(1,2,\nu_1) \le 1/2 \ .$$

Proof : The result follows from

$$\begin{aligned}\nu_1(A_n : l_1^n \to l_\infty^n) &= \pi_1(A_n : l_1^n \to l_\infty^n) \\ &= \pi_1(I_n : l_1^n \to l_2^n)\|A_n : l_2^n \to l_2^n\| \, \|I_n : l_2^n \to l_\infty^n\| \\ &\le c_L \ n^{1/2} \ .\end{aligned}$$

The limit orders $\lambda_A(r,s,\|.\|)$ and $\lambda_A(s,r,\nu_1)$

By lemma 6 and 13 we have

$$\lambda_A(r,s,\|.\|) \le \lambda_I(r,2,\|.\|) + \lambda_A(2,2,\|.\|) + \lambda_I(2,s,\|.\|)$$

$$\le \begin{cases} (1/2 - 1/r) + 1/2 + (1/s - 1/2) & \text{if} \quad r \ge 2, \ 2 \ge s, \\ \quad 0 \quad + 1/2 + (1/s - 1/2) & \text{if} \quad r \le 2, \ 2 \ge s, \\ (1/2 - 1/r) + 1/2 + \quad 0 & \text{if} \quad r \ge 2, \ 2 \le s \ . \end{cases}$$

On the other hand we obtain

$$\begin{aligned}\lambda_A(r,s,\|.\|) &\le \lambda_I(r,s',\|.\|) + \lambda_A(s',s,\|.\|) \\ &\le 0 + 1/s \quad \text{if} \quad r \le s' \quad \text{and} \quad 2 \le s,\end{aligned}$$

and

$$\begin{aligned}\lambda_A(r,s,\|.\|) &\le \lambda_A(r,r',\|.\|) + \lambda_I(r',s,\|.\|) \\ &\le 1/r' + 0 \quad \text{if} \quad r \le 2 \quad \text{and} \quad r' \le s \ .\end{aligned}$$

Summarizing the results we have

$$(5)\qquad \lambda_A(r,s,\|.\|) \le \begin{cases} 1/r' + 1/s - 1/2 & \text{if} \quad 2 \le r \le \infty,\ 1 \le s \le 2, \\ 1/s & \text{if} \quad 1 \le r \le 2,\ 1 \le s \le r', \\ 1/r' & \text{if} \quad s' \le r \le \infty,\ 2 \le s \le \infty . \end{cases}$$

With the known values of $\lambda_I(s,r,\nu_1)$ we obtain

$$\lambda_A(s,r,\nu_1) \le \lambda_I(s,1,\nu_1) + \lambda_A(1,r,\|.\| \le 1 + 1/r,$$

and

$$\lambda_A(s,r,\nu_1) \le \lambda_A(s,\infty,\|.\|) + \lambda_I(\infty,r,\nu_1) \le 1/s' + 1 \quad .$$

On the other hand it follows from lemma 14 that

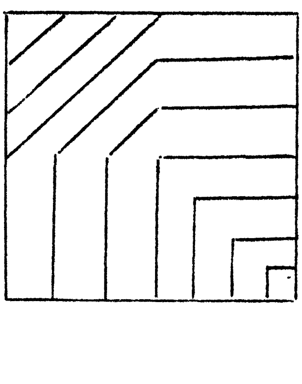
$$\begin{aligned}\lambda_A(s,r,\nu_1) &\le \lambda_I(s,1,\|.\|) + \lambda_A(1,\infty,\nu_1) + \lambda_I(\infty,r,\|.\|) \\ &\le 1/s' + 1/2 + 1/r \quad .\end{aligned}$$

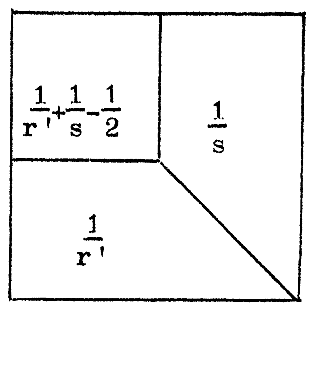
In each case, choosing the best result we obtain,

$$(5^*)\qquad \lambda_A(s,r,\nu_1) \le \begin{cases} 1/r + 1/s' + 1/2 & \text{if} \quad 2 \le r \le \infty,\ 1 \le s \le 2, \\ 1/s' + 1 & \text{if} \quad 1 \le r \le 2,\ 1 \le s \le r', \\ 1/r + 1 & \text{if} \quad s' \le r \le \infty,\ 2 \le s \le \infty . \end{cases}$$

Finally, lemma 12 implies that identity holds in (5) and (5*).

<u>$\lambda_A(r,s,\|.\|)$</u>





<u>The limit orders</u> $\lambda_A(r,s,\pi_2)$ <u>and</u> $\lambda_A(s,r,\nu_2)$

Since

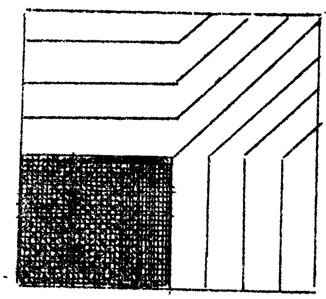
$$\lambda_A(r,s,\pi_2) \le \lambda_I(r,2,\pi_2) + \lambda_A(2,s,\|.\|)$$

we obtain, using the known values of $\lambda_I(r,2,\pi_1)$ and $\lambda_A(2,s,\|.\|)$,

$$(6) \qquad \lambda_A(r,s,\pi_2) \le \begin{cases} 1/r' + 1/s & \text{if} \quad 1 \le r \le 2, \quad 1 \le s \le 2, \\ 1/2 + 1/s & \text{if} \quad 2 \le r \le \infty, \quad 1 \le s \le 2, \\ 1/r' + 1/2 & \text{if} \quad 1 \le r \le 2, \quad 2 \le s \le \infty \\ 1/2 + 1/2 & \text{if} \quad 2 \le r \le \infty, \quad 2 \le s \le \infty \,. \end{cases}$$

Finally, it follows from lemma 12 and $\lambda_A(s,r,\nu_2) = \lambda_A(s,r,\pi_2)$ that identity holds in (6)

<u>$\lambda_A(r,s,\pi_2)$</u>



$\frac{1}{2}+\frac{1}{s}$	$\frac{1}{r'}+\frac{1}{s}$
1	$\frac{1}{r'}+\frac{1}{2}$

<u>The limit orders</u> $\lambda_A(r,s,\pi_p)$ <u>and</u> $\lambda_A(s,r,\nu_{p'})$ <u>with $1 \le p < 2$</u>

Since by theorem 2 and 2* for $1 \le r \le 2$, we have

$$\lambda_A(r,s,\pi_p) = \lambda_A(r,s,\pi_2) \quad \text{and} \quad \lambda_A(s,r,\nu_{p'}) = \lambda_A(s,r,\nu_2)$$

in the following we need only consider the case $2 < r \le \infty$.

Since

$$\lambda_A(r,s,\pi_p) \le \lambda_I(r,p,\pi_p) + \lambda_A(p,s,\|.\|)$$

we obtain, using the known values of $\lambda_I(r,p,\pi_p)$ and $\lambda_A(p,s,\|.\|)$,

$$\lambda_A(r,s,\pi_p) \le 1/p + \begin{cases} 1/p' & \text{if } s \ge p', \\ 1/s & \text{if } s \le p'. \end{cases}$$

On the other hand it follows from

$$\lambda_A(r,s,\pi_p) \le \lambda_I(r,r',\pi_p) + \lambda_A(r',s,\|.\|)$$

that

$$\lambda_A(r,s,\pi_p) \le 1/r' + \begin{cases} 1/r & \text{if } s \ge r, \\ 1/s & \text{if } s \le r. \end{cases}$$

In each case, choosing the best result we obtain

$$(7) \qquad \lambda_A(r,s,\pi_p) \le \begin{cases} 1 & \text{if } p' \le r \le \infty, \quad p' \le s \le \infty, \\ 1/p + 1/s & \text{if } p' \le r \le \infty, \quad 1 \le s \le p', \\ 1 & \text{if } 2 \le r \le p', \quad r \le s \le \infty, \\ 1/r' + 1/s & \text{if } 2 \le r \le p', \quad 1 \le s \le r. \end{cases}$$

Moreover,

$$\begin{aligned} \lambda_A(s,r,\nu_{p'}) &\le \lambda_A(s,\infty,\|.\|) + \lambda_I(\infty,r,\nu_{p'}) \\ &\le 1/s + \begin{cases} 1/r & \text{if } r \le p', \\ 1/p' & \text{if } r \ge p', \end{cases} \end{aligned}$$

and

$$\begin{aligned} \lambda_A(s,r,\nu_{p'}) &\le \lambda_A(s,2,\nu_{p'}) + \lambda_I(2,r,\|.\|) \\ &\le \lambda_A(s,2,\nu_2) \le 1 \quad \text{if } 2 \le s \le \infty. \end{aligned}$$

Consequently,

$$(7^*) \qquad \lambda_A(s,r,\nu_{p'}) \le \begin{cases} 1 & \text{if } p' \le r \le \infty, \quad p' \le s \le \infty, \\ 1/p' + 1/s' & \text{if } p' \le r \le \infty, \quad 1 \le s \le p', \\ 1 & \text{if } 2 \le r \le p', \quad r \le s \le \infty, \\ 1/r + 1/s' & \text{if } 2 \le r \le p', \quad 1 \le s \le r. \end{cases}$$

Finally, lemma 12 implies that identity holds in (7) and (7*).

$\underline{\lambda_A(r,s,\pi_1)}$

$\underline{\lambda_A(r,s,\pi_p)}$
$1 < p < 2$

<u>The limit orders</u> $\lambda_A(r,s,\pi_p)$ <u>and</u> $\lambda_A(s,r,\nu_{p'})$ <u>with</u> $2 < p < \infty$

Since by theorem 3 and 3* for $1 \le s \le 2$ we have

$$\lambda_A(r,s,\pi_p) = \lambda_A(r,s,\pi_2) \quad \text{and} \quad \lambda_A(s,r,\nu_{p'}) = \lambda_A(s,r,\nu_2)$$

in the following we need only consider the case $2 < s \le \infty$.

From (7*) and

$$\lambda_A(r,s,\pi_p) \le \lambda_A(r,s,\nu_p)$$

we obtain

$$\text{(8)} \qquad \lambda_A(r,s,\pi_p) \le \begin{cases} 1/r' + 1/s & \text{if} \quad 1 \le r \le s, \quad 2 \le s \le p, \\ 1 & \text{if} \quad s \le r \le \infty, \quad 2 \le s \le p, \\ 1/r' + 1/p & \text{if} \quad 1 \le r \le p, \quad p \le s \le \infty, \\ 1 & \text{if} \quad p \le r \le \infty, \quad p \le s \le \infty . \end{cases}$$

On the other hand, we have

$$\lambda_A(s,r,\nu_{p'}) \leq \lambda_I(s,p',\nu_{p'}) + \lambda_A(p',r,\|.\|)$$

$$\leq 1/p' + \begin{cases} 1/p & \text{if} \quad p \leq r, \\ 1/r & \text{if} \quad p \geq r, \end{cases}$$

and

$$\lambda_A(s,r,\nu_{p'}) \leq \lambda_I(s,2,\|.\|) + \lambda_A(2,2,\nu_{p'}) + \lambda_I(2,r,\|.\|)$$

$$\leq (1/2 - 1/s) + 1 + (1/r - 1/2) \quad \text{if} \quad 1 \leq r \leq 2 \text{ and } 2 \leq s \leq \infty .$$

In each case, choosing the best result we obtain

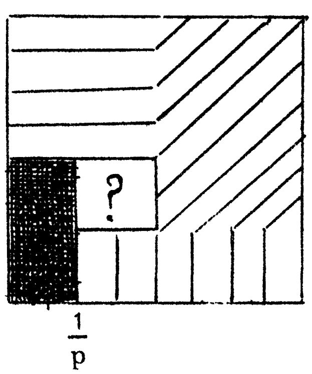
$$\text{(8*)} \qquad \lambda_A(s,r,\nu_{p'}) \leq \begin{cases} 1/r + 1/s' & \text{if} \quad 1 \leq r \leq 2, \ 2 \leq s \leq p, \\ 1 & \text{if} \quad p \leq r \leq \infty, \ 2 \leq s \leq p, \\ 1/r + 1/p' & \text{if} \quad 1 \leq r \leq p, \ p \leq s \leq \infty, \\ 1 & \text{if} \quad p \leq r \leq \infty, \ p \leq s \leq \infty . \end{cases}$$

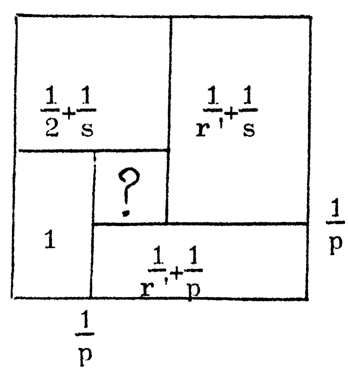
Because the square

$$Q_{A,p} := \{(1/r', 1/s): \ 2 < r < p, \ 2 < s < p\}$$

does not appear in (8*), we have the open problem wether identity holds for all r and s in (8).

$$\frac{\lambda_A(r,s,\pi_p)}{2<p<\infty}$$





Final remark (Cf. end of part I)

Let L_r and L_s be infinite dimensional. Then $\mathcal{P}_p(L_r,L_s)$ is strictly increasing

1) if $2 \leq r \leq \infty$, $1 \leq s \leq 2$, and $r' \leq p \leq 2$ since $\lambda_A(r,s,\pi_p) = 1/p + 1/s$,
2) if $1 \leq r \leq 2$, $2 \leq s \leq \infty$, and $2 \leq p \leq s$ since $\lambda_A(r,s,\pi_p) = 1/p + 1/r'$,
3) if $2 \leq r \leq \infty$, $2 \leq s \leq \infty$, and $r' \leq p \leq s$ since $\lambda_I(r,s,\pi_p) = 1/p$.

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