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NOTES ON AN EXTENSION OF KRULL'S PRINCIPAL IDEAL THEOREM

by David EISENBUD (*)

In this note, we will propose a generalization of Krull's principal ideal theorem which we can prove to be correct, for example, for rings containing a field. We will then sketch an application to the theory of determinantal ideals ; roughly speaking, our theorem implies a more precise version of the theorems of MACAULAY and EAGON on the heights of determinantal ideals. We also mention a speculative connection between our conjecture and the intersection conjectures of SERRE and PESKINE-SZPIRO. Details will appear elsewhere.

1. The generalized principal ideal theorem.

Throughout this paper, all rings will be assumed commutative and noetherian.

Krull's principal ideal theorem [5] states that an element a in the maximal ideal of a local ring R generates an ideal of height at most one (the apparently sharper statement that the minimal primes all have height at most one follows trivially by localization). Regarding a as a homomorphism $R \longrightarrow R$, and noting that the rank of R , as an R-module, is 1 , one might be lead to conjecture that something "similar" can be said about homomorphisms from an arbitrary module into R . To be more precise, one needs first the right notion of the rank of a module. Since we wish to work with homomorphisms to the ring, it is not unreasonable to require that an R-module M should have rank 0 if, and only if, $M^* = \mathrm{Hom}(M,R) = 0$. As with all notions of rank, a module M should have rank $\leqslant k$ if, and only if, its $(k+1)^{\text{th}}$-exterior power has rank 0 . These conditions uniquely specify a notion of the rank of a module, which can be more simply put as follows :

Definition. - Let $\mathfrak{U}$ be the set of nonzerodivisors of R . The rank of a finitely generated R-module M is the minimal number of generators of $M_{\mathfrak{U}}$ as an R-module.

Of course, if R is a domain, this is the usual notion of rank. We can now state our conjecture :

Generalized principal ideal conjecture. - Let R be a local ring with maximal ideal J , and let M be a finitely generated R-module of rank n . Let

$$M^* = \mathrm{Hom}(M , R) ,$$

and let φ be an element of JM^* . Then the height of the ideal $\varphi(M)$ is at most n .

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It is easy to see that, if $M = R^n$, this conjecture becomes the version of the Krull principal ideal theorem which states that the height of a proper n-generator ideal is at most n.

We can prove a weakned form of the conjecture, in which "height" is replaced by "depth", and we can prove the conjecture itself in many cases.

Recall that if $I \subseteq R$ is an ideal, and N is a finitely generated R-module, then depth (I, N) is the length of a maximal N-sequence in I.

THEOREM 1. - *Let* R *be a noetherian local ring with maximal ideal* J. *Let* M *be a finitely generated* R-*module, and let* $\varphi \in JM^*$.

(a) *If* N *is a finitely generated* R-*module, then* $\operatorname{depth}(\varphi(M), N) \leqslant \operatorname{rank} M$.

(b) *If* R *has (possibly not finitely generated) Cohen-Macaulay modules in the sense of HOCHSTER* [4], *then*

$$\operatorname{height} \varphi(M) \leqslant \operatorname{rank} M .$$

The extra hypothesis of (b) is known to be fulfilled if R contains a field, and is conjectured to be true in general [4].

It is perhaps amusing to note that our conjecture can be reformulated as giving a condition, in terms of the punctered spectrum, for an element of a module to be part of a minimal system of generators :

Conjecture (second version). - Let (R, J) be a local ring of dimension d, and let M be a module of rank $< d$. Suppose that a is an element of M such that, for every prime ideal $P \neq J$, a generates a free summand of M_P. Then a is part of a minimal system of generators for M.

2. Determinantal ideals.

One of the earliest generalizations of the principal ideal theorem was the theorem of MACAULAY [6] that (for polynomial rings) the height of the ideal of $p \times p$ minors of a $p \times q$ matrix, if the ideal is proper, is at most $q - p + 1$. This was generalized by EAGON in 1960, who showed (for a general noetherian ring) that the height of the ideal of $k \times k$ minors of a $p \times q$ matrix, if the ideal is proper, is at most $(p - k + 1)(q - k + 1)$ (There is a very elegant proof of this in [3]). On the basis of the conjecture made in the last section, we can extend this result to say something about what happens to the ideal of $k \times k$ minors when an extra column is added to the matrix.

Before stating our result, we remark on a result that can be proved by the technique of [3].

PROPOSITION. - *Let* φ *be a* $p \times q$ *matrix over a ring* R, *and suppose that the* $(\ell + 1) \times (\ell + 1)$ *minors of* φ *are all* 0. *Then the ideal generated by the* $\ell \times \ell$

<u>minors of</u> φ <u>has height at most</u>

$$p + q - 2\ell + 1 .$$

Now suppose that R is local and that we adjoin a new column, with entries in the maximal ideal, to a $p \times q$ matrix φ, obtaining a $p \times (q + 1)$ matrix φ'. Suppose that the $k \times k$ minors of φ are all 0. What can the height h of the ideal of $k \times k$ minors of φ' be ? Of course, it is contained in the ideal of $(k - 1) \times (k - 1)$ minors of φ, and also in the ideal generated by the p entries of the new column, so one obtains a bound from the proposition :

$$h \leqslant \min(p , p + q - 2k + 3) .$$

The next theorem shows that one can do better (at least much of the time !) :

THEOREM. - <u>Suppose that</u> R <u>is a local ring satisfying the generalized principal ideal conjecture.</u> (<u>For example</u>, <u>suppose that</u> R <u>contains a field.</u>) <u>Let</u> φ <u>be a</u> $p \times q$ <u>matrix over</u> R <u>whose</u> $k \times k$ <u>minors are all</u> 0, <u>and let</u> φ' <u>be a matrix obtained from</u> φ <u>by adjoining a column whose entries are in the maximal ideal</u>. <u>Then the height of the ideal of</u> $k \times k$ <u>minors of</u> φ' <u>is at most</u> $p - k + 1$.

As a consequence of the theorem and the proposition, we can prove a result which generalizes a "rigidity" theorem of BUCHSBAUM and RIM [1], which, in turn, generalized the result that if n elements $f_1 , \ldots , f_n$ of a local ring generate an ideal of height n, then any k of them generate an ideal of height k :

COROLLARY. - <u>Suppose that</u> R <u>is a local ring satisfying the generalized principal ideal conjecture</u>, <u>and that</u> φ <u>is a</u> $p \times q$ <u>matrix over</u> R, <u>with coefficients in the maximal ideal</u>, <u>such that the ideal of</u> $k \times k$ <u>minors of</u> φ <u>has height</u> $(p - k + 1)(q - k + 1)$, <u>the largest possible value</u>. <u>Then for every integer</u> $\ell \geqslant k$, <u>and every</u> $s \times t$ <u>submatrix</u> $\overline{\varphi}$ <u>of</u> φ, <u>the height of the ideal of</u> $\ell \times \ell$ <u>minors of</u> $\overline{\varphi}$ <u>is</u>

$$(s - \ell + 1)(t - \ell + 1) ,$$

<u>again the largest possible value</u>.

In particular, no $\ell \times \ell$ minor of φ is 0.

To see that the hypothesis about coefficients being in the maximal ideal are necessary for this, consider the following matrix over $F[x , y]$, when F is a field :

$$\begin{pmatrix} 0 & 0 & 1 \\ x & y & 0 \end{pmatrix} .$$

Here the height of the ideal of 2×2 minors is $2 (= (3 - 2 + 1)$, but the first 2×2 minor is 0.

3. A remark on intersection theory.

The remark is easy and speculative : Suppose that M and N are modules of ranks m and n over a local ring (R, J) (containing a field, say). Suppose that $\varphi \in JM^*$ and $\psi \in JN^*$, and write

$$X = \varphi(M)$$
$$Y = \psi(N) .$$

Then the ideal $X + Y$ can be written as $(\varphi, \psi)(M \oplus N)$, so $X + Y$ has height at most $m + n$. This gives some hold on the "intersection theory" of ideals of the form $\varphi(M)$. For example, if one could prove that for every prime ideal P of a regular local ring R, J' there exists a module M with rank $M = \text{ht } P$ and an element $\varphi \in JM^*$ with $\varphi(M) = P$, then one could deduce Serre's intersection theorem ([7], ch. V, theorem 3).

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