Séminaire Dubreil.
Algèbre et théorie
des nombres

Robert M. Fossum

Minimal injective resolutions

Séminaire Dubreil. Algèbre et théorie des nombres, tome 28, nº 1 (1974-1975), exp. nº 16, p. 1-5
<http://www.numdam.org/item?id=SD_1974-1975__28_1_A10_0>

Séminaire P. DUBREIL
(Algèbre)
28e année, 1974/75, n° 16, 5 p.

7 avril 1975

MINIMAL INJECTIVE RESOLUTIONS

by Robert M. FOSSUM

0. Introduction.

The many results about commutative noetherian rings which have a non -zero module of finite type with finite injective dimension seem to indicate that the minimal injective resolution of a module of finite type should contain a great amount of information about the module. See for example PESKINE and SZIRO's paper [5]. In this report, I will outline a proof of a result due principally to FOXBY and GRIFFITH (and proved independently by P. ROBERTS using differents methods) which states :

If A is a noetherian local ring with maximal ideal $\mathfrak{m}$ and if M is an A-module of finite type, then $\mathrm{Ext}_A^j(A/\mathfrak{m}\ ,\ M) \neq 0$ for all j in the range

$$\mathrm{depth}\ M \leqslant j \leqslant \mathrm{id}_A\ M\ .$$

This can be interpreted as a rigidity result. It also gives information about the minimal injective resolution of M. For if

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I' \longrightarrow \cdots$$

is a minimal injective resolution of M, then the result states that the injective envelope of the residue class field is a direct summand of I^j for those integers j in the range depth $M \leqslant j \leqslant$ id M. An interesting aspect of the proof is that is uses HOCHSTER's result establishing the existence of a maximal Cohen-Macaulay module (not necessarily of finite type) for a local ring of characteristic p (see HOCHSTER [4]), while the result itself is independent of characteristic.

Complete details can be found in a paper by FOSSUM, FOXBY, GRIFFITH and REITEN [2].

1. Preliminary results.

Let A be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue class field $k = A/\mathfrak{m}$. Let M be an A-module of finite type. It is standard that

$$\mathrm{depth}_A\ M = \inf\{i\ ;\ \ \mathrm{Ext}_A^i(k\ ,\ M) \neq 0\}$$

and

$$\mathrm{id}_A\ M = \sup\{i\ ;\ \ \mathrm{Ext}_A^i(k\ ,\ M) \neq 0\}\ .$$

So the question is : what happens to $\mathrm{Ext}_A^j(k\ ,\ M)$ for j in the interval between $\mathrm{depth}_A\ M$ and $\mathrm{id}_A\ M$?

BASS reported two results [1].

PROPOSITION 1. - <u>If</u> $id_A\ M < \infty$, <u>then</u> $id_A\ M = depth_A\ A$.

PROPOSITION 2. - <u>If</u> $id_A\ M = \infty$, <u>then</u> $Ext_A^j(k\ ,\ M) \neq 0$ <u>for all</u> j <u>with</u> $j \geqslant \dim A$.

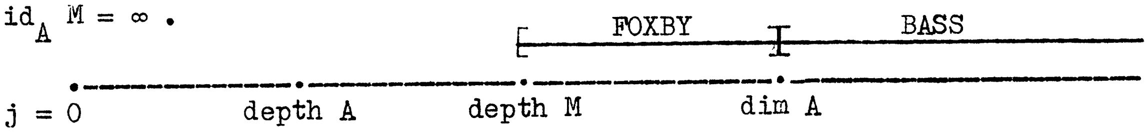
Much later FOXBY [3] extended the range in which $Ext_A^j(k\ ,\ M) \neq 0$ for very special modules.

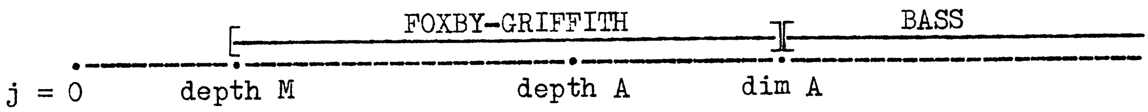
PROPOSITION 3. - <u>If</u> depth $A \leqslant$ depth M , <u>then</u> $Ext_A^j(k\ ,\ M) \neq 0$ <u>for those</u> j <u>with</u>

$$\text{depth } M \leqslant j \leqslant \text{id } M \ .$$

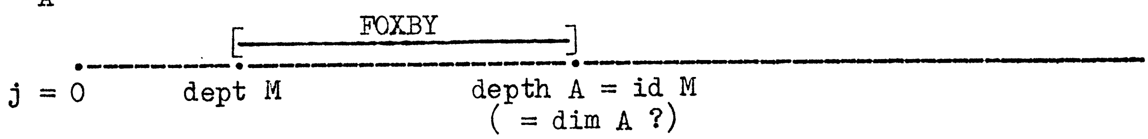
A diagram explains these results. The intervals with solid lines indicate the range of j where $Ext_A^j(k\ ,\ M) \neq 0$.

1° $id_A\ M = \infty$.





2° $id_A\ M < \infty$



2. Main theorem.

The main theorem, which is stated in the local case in the introduction, follows :

THEOREM 1. - <u>Let</u> A <u>be a noetherian ring and</u> M <u>an</u> A-<u>module of finite type</u>. <u>Let</u>

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

<u>be a minimal injective resolution of</u> M . <u>If</u> j <u>is an integer, if</u> $p \in \operatorname{spec} A$, <u>and if</u> $depth_{A_p}\ M_p \leqslant j \leqslant id_{A_p}\ M_p$, <u>then the injective envelope of</u> A/p <u>is a direct summand of</u> I^j .

It is clear that we may assume that A is local and even complete, if necessary. Furthermore we can assume that depth $M <$ depth A by FOXBY's result.

<u>Reduction</u>. - Let $f_1\ ,\ \dots\ ,\ f_r$ be a set of elements in A that forms a regular M-sequence and a regular A-sequence. Let $\mathfrak{f}$ denote the ideal generated by these elements. Since

$$Ext_A^j(k\ ,\ M) \cong Ext_{A/\mathfrak{f}}^{j-r}(k\ ,\ M/\mathfrak{f}\, M)\ ,$$

it may be assumed that depth $M = 0$.

3. The main lemma.

The principal lemma that connects the maximal Cohen-Macaulay modules with the problem under consideration follows. Let $E(k)$ denote the injective envelope of k as an A-module. The functor, that to M associates $\operatorname{Hom}_A(M , E(k))$ is denoted by $M^{\vee}$.

LEMMA 1. – *Suppose* N *is an* A-*module* (*not necessarily of finite type*), *suppose* $x_1 , \ldots , x_n$, *is a regular* N-*sequence such that the annihilator*

$$\operatorname{Ann}(N/(x_1 , \ldots , x_n)N)$$

is proper and $\mathfrak{m}$-*primary*. *If* M *is an* A-*module of finite type with* depth $M = 0$, *then*

$$\operatorname{Ext}^i_A(N/(x_1 , \ldots , x_j)N , M) \neq 0$$

for all i *and* j *with* $0 \leqslant i \leqslant j$.

Proof. – The proof goes by induction on j . Suppose $j = 0$. Since

$$\operatorname{Hom}_A(N , k) \simeq \operatorname{Hom}_A(N , \operatorname{Hom}_A(k , E(k)) \cong \operatorname{Hom}_A(N \otimes_A k , E(k)) ,$$

it is sufficient to show that $N \otimes_A k \neq 0$. But it is assumed that

$$\operatorname{Ann}(N/(x_1 , \ldots , x_n)N)$$

is $\mathfrak{m}$-primary and therefore $\mathfrak{m}(N/(x_1 , \ldots , x_n)N) \neq N/(x_1 , \ldots , x_n)N$. Hence $N/\mathfrak{m}N \neq 0$. The assumption depth $M = 0$ is equivalent to the assumption that k is isomorphic to a submodule of M . Therefore $\operatorname{Hom}_A(N , k) \neq 0$ implies $\operatorname{Hom}_A(N,M) \neq 0$.

The induction step uses the isomorphisms

$$\operatorname{Ext}^i_A(N/(x_1 , \ldots , x_j)N , M) \cong (\operatorname{Tor}^i_A(N/(x_1 , \ldots , x_j) , M^{\vee})^{\vee}$$

and the exact sequences

$$0 \longrightarrow N/(x_1 ,\ldots, x_{j-1})N \xrightarrow{.x_j} N/(x_1 ,\ldots, x_{j-1})N \longrightarrow N/(x_1 ,\ldots, x_j)N \longrightarrow 0$$

to show that $\operatorname{Tor}^A_i(N/(x_1 , \ldots , x_j)N , M^{\vee}) \neq 0$ and therefore

$$\operatorname{Ext}^i_A(N/(x_1 , \ldots , x_j)N , M) \neq 0 .$$

LEMMA 2. – *If* $\operatorname{Ext}^j_A(k , M) = 0$, *then* $\operatorname{Ext}^j_A(T , M) = 0$ *for all* A-*modules* T *with* $\operatorname{Supp} T \leqslant \{\mathfrak{m}\}$.

Proof. – By induction on length, it is clear that $\operatorname{Ext}^j_A(T , M) = 0$ for all A-modules T of finite length. Otherwise write $T = \varinjlim T_\alpha$ where each T_α has finite length. Then

$$\operatorname{Ext}^j_A(T , M) = \varprojlim \operatorname{Ext}^j_A(T_\alpha , M) .$$

4. Proof of the theorem.

We assume, which we may, that A is a complete local ring and that $\operatorname{depth} M = 0$. Suppose j is an integer in the range $0 < j < \dim A$. Let $d = \dim A$.

Suppose p is the characteristic of the residue class field. Let $R = A/pA$. We now quote a result due to HOCHSTER [4].

THEOREM 2. - *If R is an equi-characteristic local ring of dimension t, then there is an R-module T (not necessarily of finite type) such that if $x_1, \ldots, x_t$ is a system of parameters of R, then $(x_1, \ldots, x_t)T \neq T$ and these elements form a regular T-sequence. Such a T is called a maximal Cohen-Macaulay module.*

Apply this theorem to the ring R above. If $\dim R = d - 1$, pick elements $x_1, \ldots, x_{d-1}$ in A that form a system of parameters in R and if $\dim R = \dim A$, pick $x_1, \ldots, x_d$ in A forming a system of parameters in R. Let T be a maximal Cohen-Macaulay module for R. Set $N = T/x_d T$ (where $x_d = 0$ in case $\dim R = -1 + \dim A$). Then $x_1, \ldots, x_{d-1}$ is a regular N-sequence and $\operatorname{Ann} N/(x_1, \ldots, x_{d-1})N$ is $\mathfrak{m}$-primary. Apply lemma 1 to get

$$\operatorname{Ext}_A^r(N/(x_1, \ldots, x_{d-1})N, M) \neq 0 \quad \text{for} \quad 0 \leqslant r \leqslant d-1 .$$

Apply lemma 2 to get $\operatorname{Ext}_A^r(k, M) \neq 0$ for $0 \leqslant r \leqslant d - 1$. This proves the theorem.

COROLLARY 1. - *If j is an integer with $j > \operatorname{depth} M$, then $\operatorname{Ext}_A^j(k, M) = 0$ if, and only if, $\operatorname{id}_A M < j$.*

COROLLARY 2. - *If $0 \to M \to I^0 \to I^1 \to \ldots$ is an injective resolution of M, then $E(k)$ is a direct summand of I^j if, and only if, $\operatorname{depth} M \leqslant j \leqslant \operatorname{id} M$.*

Remark 1. - It does not follow, nor as examples show is it even true, that the local cohomology modules $H_{\mathfrak{m}}^j(M) \neq 0$.

Remark 2. - If M is a nonzero module of finite type and finite injective dimension, then $\operatorname{id} M = \operatorname{depth} A$. If $\operatorname{depth} A \leqslant \operatorname{depth} M$, then A is Cohen-Macaulay. If $\operatorname{depth} M < \operatorname{depth} A$, then it is clear from the last paragraph of the proof that $\dim A - \operatorname{depth} A \leqslant 1$. In particular, if $\dim A = \dim(A/pA)$, then the proof shows that A is Cohen-Macaulay. But this also is easily obtained from [5]

REFERENCES

[1] BASS (H.). - On the ubiquity of Gorenstein rings, Math. Z., t. 82, 1963, p. 8-28.

[2] FOSSUM (R.), FOXBY (H.-B.), GRIFFITH (P.) and REITEN (I.). - Minimal injective resolutions with applications to dualizing modules and Gorenstein modules, University of Illinois, 1974.

[3] FOXBY (H.-B.). - On the μ^i in a minimal injective resolution, Math. Scand., t. 29, 1971, p. 175-186.

[4] HOCHSTER (M.). - Deep local rings, Aarhus Universitets Matematiske Institut, Preprint Series n° 8, 1973/74.

[5] PESKINE (C.) and SZPIRO (L.). - Dimension projective finie et cohomologie locale. - Paris, Presses universitaires de France, 1973 (Institut des Hautes Etudes Scientifiques. Publications mathématiques, 42, p. 47-119).

(Texte reçu le 8 avril 1975)

Robert M. FOSSUM
Dept of Mathematics
University of Illinois
URBANA, Illinois 61801
(Etats-Unis)