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THE ORDERABILITY OF IDEMPOTENT SEMIGROUPS

by Toru SAITÔ

The orderability condition for idempotent semigroups are studied by some authors : M.-L. DUBREIL-JACOTIN [3] for idempotent semigroups with identity, T. MERLIER [4] for finite idempotent semigroups, and T. SAITÔ [6] for commutative idempotent semigroups. The purpose of this note is to give orderability conditions for general idempotent semigroups. The detailed version [7] will appear elsewhere.

The terminologies of CLIFFORD and PRESTON [2] are used throughout. Let E be a semilattice with respect to a partial order \leq . E is called a tree semilattice, if, for every $\alpha \in E$, the set $\{\xi \in E ; \xi \leq \alpha\}$ is a simply ordered set. Let α be an element of a tree semilattice E . We define a binary relation \sim on the set $\bar{U}_\alpha = \{\xi \in E ; \alpha < \xi\}$ by

$$\text{for } \xi, \eta \in \bar{U}_\alpha, \xi \sim \eta \text{ if and only if } \alpha < \xi\eta.$$

Then \sim is an equivalence relation on \bar{U}_α . Each \sim -equivalence class is called a branch at α .

Let S be an idempotent semigroup. Then S is a semilattice of rectangular bands $\{D_\alpha ; \alpha \in S^*\}$, and the decomposition of S into $\{D_\alpha ; \alpha \in S^*\}$ coincides with the decomposition of S into \mathcal{Q} -classes. The semilattice S^* is called the associated semilattice of S .

Let S be an idempotent semigroup such that the associated semilattice S^* is a tree semilattice and let D be a \mathcal{Q} -class of S . Let \mathfrak{B} be a branch of S^* at $D \in S^*$. Then, the subset $B = \{x \in S ; D_x \in \mathfrak{B}\}$ of S is called the component-branch at D associated with the branch \mathfrak{B} .

If a \mathcal{Q} -class D of an idempotent semigroup S consists of one \mathcal{L} -class, then D is called of L-type, while if D consists of one \mathcal{R} -class, then D is called of R-type.

By an ordered semigroup S , we mean a semigroup S with a simple order \leq satisfying the condition

$$\text{for } x, y, z \in S, x \leq y \text{ implies } xz \leq yz \text{ and } zx \leq zy.$$

A semigroup S is called orderable if there exists a simple order \leq on S such that the system $S(., \leq)$ is an ordered semigroup.

Here we refer to some preliminary lemmas :

LEMMA 1 ([5], theorem 3). - The associated semilattice S^* of an ordered idem-

potent semigroup S is a tree semilattice.

LEMMA 2 ([5], theorem 1). - In an ordered idempotent semigroup S , each \mathcal{O} -class consists of either one \mathcal{L} -class or one \mathcal{R} -class.

LEMMA 3 [1]. - Let S be a set with a ternary relation β satisfying the conditions :

- (a) $(x, y, z)\beta$ implies $(z, y, x)\beta$;
- (b) $(x, y, x)\beta$ implies $x = y$;
- (c) $(x, y, z)\beta$, $(y, z, u)\beta$ and $y \neq z$ imply $(x, y, u)\beta$;
- (d) For every $x, y, z \in S$, either $(x, y, z)\beta$ or $(y, z, x)\beta$ or $(z, x, y)\beta$;
- (e) $(x, y, z)\beta$ and $(x, z, u)\beta$ imply $(y, z, u)\beta$.

Then, there exists a simple order \leq on S such that $(x, y, z)\beta$ if and only if either $x \leq y \leq z$ or $z \leq y \leq x$:

I

THEOREM A. - An idempotent semigroup S is orderable if and only if it satisfies the following conditions :

- (A) The associated semilattice S^* of S is a tree semilattice ;
- (B) Each \mathcal{O} -class of S consists of either one \mathcal{L} -class or one \mathcal{R} -class ;
- (C) If D is a \mathcal{O} -class of S and $a \in S$ such that $D < D_a$ in the associated semilattice S^* , then either aD or Da consists of at most two elements of S ;
- (D) If D is a \mathcal{O} -class of S of L-type [R-type] and if $a, b, x \in S$ such that $D < D_{ab}$ in the associated semilattice S^* and $x \in D$, then $ax=bx$ [$xa=xb$] ;
- (E) If D is a \mathcal{O} -class of S of L-type [R-type] and if $a, b, x \in S$ such that $a, b \notin D$, $ab \in D$, $x \in D$ and $ab \neq ax$ [$ba \neq xa$], then $ba=bx$ [$ab=xb$] ;
- (F) If D is a \mathcal{O} -class of S of L-type [R-type] and if $a, b, c \in S$ such that $a, b, c \notin D$, $ab \in D$, $ab = ac$ and $bc = ba$ [$ba = ca$ and $cb = ab$], then $ca \neq cb$ [$ac \neq bc$] ;
- (G) If D is a \mathcal{O} -class of S of L-type [R-type] and if $a, b, x \in S$ such that $a, b \notin D$, $ab \in D$, $x \in D$ and $ax = bx$ [$xa = xb$], then $x=ab$ [$x=ba$] .

Here we give only a brief survey of the proof of the "if" part by steps.

Let S be an idempotent semigroup satisfying the conditions given in the theorem and let D be a \mathcal{O} -class of S . We denote by $\{B_\lambda ; \lambda \in \Lambda\}$ the set of all component-branches at D .

- (1) If $B_\lambda \neq B_\mu$, then $B_\lambda B_\mu$ is a one-element subset of S .

We define the ternary relation β on $\{B_\lambda; \lambda \in \Lambda\}$ by:

If D is of L-type [R-type], then $(B_\lambda, B_\mu, B_\nu)\beta$ if and only if either $B_\lambda = B_\mu$ or $B_\mu = B_\nu$ or $B_\lambda \neq B_\mu$, $B_\mu \neq B_\nu$ and $B_\mu B_\lambda \neq B_\mu B_\nu$ [$B_\lambda B_\mu \neq B_\nu B_\mu$].

- (2) β satisfies the conditions in lemma 3.

For each $\lambda \in \Lambda$, we define a subset L_λ of D as follows:

(i) The case when Λ contains at least two elements.

(ia) If there exists $\mu \in \Lambda$ such that $B_\mu < B_\lambda$ and if D is of L-type [R-type] then put

$$L_\lambda = \{x \in D; B_\lambda x = B_\lambda B_\mu\} \quad [L_\lambda = \{x \in D; xB_\lambda = B_\mu B_\lambda\}].$$

(ib) If there exists $\nu \in \Lambda$ such that $B_\lambda < B_\nu$ and if D is of L-type [R-type], then put

$$L_\lambda = \{x \in D; B_\lambda x \neq B_\lambda B_\nu\} \quad [L_\lambda = \{x \in D; xB_\lambda \neq B_\nu B_\lambda\}].$$

(ii) The case when Λ consists of one and only one element λ . We take $x_0 \in D$ arbitrarily and fix it. Then put

$$L_\lambda = \{x \in D; B_\lambda x = B_\lambda x_0\} \quad [L_\lambda = \{x \in D; xB_\lambda = x_0 B_\lambda\}].$$

Further, we define the binary relation γ on D by

$$x \gamma y \text{ if and only if } \lambda \in \Lambda \text{ and } y \in L_\lambda \text{ implies } x \in L_\lambda.$$

(3) γ is a reflexive and transitive relation on D . Moreover, for each pair of elements x and y of D , we have either $x \gamma y$ or $y \gamma x$.

Hence, if we define

$$x \delta y \stackrel{\text{def}}{\iff} x \gamma y \text{ and } y \gamma x,$$

then δ is an equivalence relation on D and the quotient set D/δ is a simply ordered set with respect to the relation \leq defined by:

$$\text{for } K_1, K_2 \in D/\delta, \quad K_1 \leq K_2 \stackrel{\text{def}}{\iff} x \gamma y \text{ for some } x \in K_1, \text{ and } y \in K_2.$$

We denote the quotient set D/δ by \mathfrak{R}_D and call an element of \mathfrak{R}_D a component of D .

(4) Let D be of L-type [R-type].

(a) If $L_\lambda \neq \emptyset$, then $B_\lambda L_\lambda$ [$L_\lambda B_\lambda$] consists of one and only one element ℓ_λ of D .

(b) If $D \setminus L_\lambda \neq \emptyset$, then $B_\lambda(D \setminus L_\lambda)$ [$(D \setminus L_\lambda)B_\lambda$] consists of one and only one element u_λ of D .

The element ℓ_λ is called the lower distinguished element of D corresponding to λ and the element u_λ is called the upper distinguished element of D corresponding to λ .

(5) (a) $\ell_\lambda \vee \ell_\mu$ if and only if $B_\lambda \leq B_\mu$,

(b) $u_\lambda \vee u_\mu$ if and only if $B_\lambda \leq B_\mu$.

(6) If $\ell_\lambda = u_\mu$, then the component K containing the element ℓ_λ consists of one and only one element.

Hence, by the well-ordering principle, we can take a simple order in such a way that, if K contains a lower distinguished element ℓ_λ , then ℓ_λ is the greatest element of K and, if K contains an upper distinguished element u_μ , then u_μ is the least element of K . Now, we define the simple order on D as the ordinal sum of these simply ordered components.

Finally, we define, for $x, y \in S$, $x < y$ if and only if either one of the following conditions is satisfied :

(a) $D_{xy} < D_x$, $D_{xy} < D_y$, B_λ is the component-branch at D_{xy} containing x , B_μ is the component-branch at D_{xy} containing y , and $B_\lambda < B_\mu$;

(b) $D_x = D_{xy} < D_y$, B_λ is the component-branch at D_{xy} containing y , and $x \leq \ell_\lambda$ in D_{xy} ;

(c) $D_x > D_{xy} = D_y$, B_μ is the component-branch at D_{xy} containing x , and $u_\mu \leq y$ in D_{xy} ;

(d) $D_x = D_y = D_{xy}$ and $x < y$ in D_{xy} .

(7) The relation $<$ on S defines a simple order which is compatible with the semigroup operation.

II

Let S be an idempotent semigroup.

We divide the condition (D) into following three conditions :

(D1) If D is a \mathcal{Q} -class of S of L-type [R-type] and if $a, b \in S$, $ab=ba=b$ and $D < D_{ab}$, then $ax = bx$ [$xa = xb$] for every $x \in D$;

(D2) If D is a \mathcal{Q} -class of S of L-type [R-type] and if $a, b \in S$, $a \mathcal{L} b$ [$a \mathcal{R} b$], and $D < D_{ab}$, then $ax = bx$ [$xa = xb$] for every $x \in D$;

(D3) If D is a \mathcal{Q} -class of S of L-type [R-type] and if $a, b \in S$, $a \mathcal{R} b$ [$a \mathcal{L} b$], and $D < D_{ab}$, then $ax = bx$ [$xa = xb$] for every $x \in D$.

Now, we have :

(a) S satisfies condition (B) $\Leftrightarrow S$ does not contain a subsemigroup isomorphic to the semigroup S_1 :

$$S_1 :$$

	x	y	z	u
x	x	z	z	x
y	u	y	y	u
z	x	z	z	x
u	u	y	y	u

(b) S satisfies condition (D1) $\Leftrightarrow S$ does not contain a subsemigroup isomorphic to either one of the two semigroups :

$$S_2 :$$

	a	b	x	y
a	a	b	x	y
b	b	b	y	y
x	x	x	x	x
y	y	y	y	y

$$S_2^* :$$

	a	b	x	y
a	a	b	x	y
b	b	b	x	y
x	x	y	x	x
y	y	y	x	y

(c) Let S satisfy (B) and (D1). Then, S satisfies condition (A) $\Leftrightarrow S$ does not contain a subsemigroup isomorphic to the semigroup S_3 :

$$S_3 :$$

	a	b	c	x
a	a	b	c	x
b	b	b	x	x
c	c	x	c	x
x	x	x	x	x

(d) Let S satisfy (B). Then, S satisfies condition (C) $\Leftrightarrow S$ does not contain a subsemigroup isomorphic to either one of the two semigroups :

$$S_4 :$$

	a	x	y	z
a	a	x	y	z
x	x	x	x	x
y	y	y	y	y
z	z	z	z	z

$$S_4^* :$$

	a	x	y	z
a	a	x	y	z
x	x	x	y	z
y	y	x	y	z
z	z	x	y	z

(e) S satisfies condition (D2) $\Leftrightarrow S$ does not contain a subsemigroup isomorphic to either one of the two semigroups :

$$S_5 :$$

	a	b	x	y
a	a	a	x	x
b	b	b	y	y
x	x	x	x	x
y	y	y	y	y

$$S_5^* :$$

	a	b	x	y
a	a	b	x	y
b	a	b	x	y
x	x	y	x	y
y	x	y	x	y

(f) S satisfies condition (G) $\Leftrightarrow S$ does not contain a subsemigroup isomorphic to either one of the two semigroups :

		a	b	x	y
	a	a	y	y	y
S_6	:	b	y	b	y
	x	x	x	x	x
	y	y	y	y	y

		a	b	x	y
	a	a	y	x	y
S_6^*	:	b	y	b	x
	x	y	y	x	y
	y	y	y	x	y

(g) Let S satisfy (G). Then, S satisfies condition (F) \Leftrightarrow S does not contain a subsemigroup isomorphic to either one of the three semigroups :

		a	b	c	x
	a	a	x	x	x
S_7	:	b	x	b	x
	c	x	x	c	x
	x	x	x	x	x

		a	b	c	x	y	z
	a	a	x	x	x	x	x
	b	y	b	y	y	y	y
S_8	:	c	z	z	c	z	z
	x	x	x	x	x	x	x
	y	y	y	y	y	y	y
	z	z	z	z	z	z	z

		a	b	c	x	y	z
	a	a	y	z	x	y	z
	b	x	b	z	x	y	z
S_8^*	:	c	x	y	c	x	y
	x	x	y	z	x	y	z
	y	x	y	z	x	y	z
	z	x	y	z	x	y	z

(h) S satisfies condition (D3) \Leftrightarrow S does not contain a subsemigroup isomorphic to either one of the two semigroups :

		a	b	x	y	z
	a	a	b	x	y	x
S_9	:	b	a	b	x	y
	x	x	x	x	x	x
	y	y	y	y	y	y
	z	z	z	z	z	z

		a	b	x	y	z
	a	a	a	x	y	z
S_9^*	:	b	b	b	x	y
	x	x	x	x	y	z
	y	y	y	x	y	z
	z	x	y	x	y	z

(i) S satisfies condition (E) \Leftrightarrow S does not contain a subsemigroup isomorphic to either one of the four semigroups :

		a	b	x	y	z	u
	a	a	u	y	y	u	u
	b	u	b	z	u	z	u
S_{10}	:	x	x	x	x	x	x
	y	y	y	y	y	y	y
	z	z	z	z	z	z	z
	u	u	u	u	u	u	u

		a	b	x	y	z	u
	a	a	u	x	y	z	u
	b	u	b	x	y	z	u
S_{10}^*	:	x	y	z	x	y	z
	y	y	u	x	y	z	u
	z	u	z	x	y	z	u
	u	u	u	x	y	z	u

		a	b	x	y	z	u	v				a	b	x	y	z	u	v	
	a	a	u	y	y	u	u	u		a	a	v	x	y	z	u	v		
	b	v	b	z	v	z	v	v		b	u	b	x	y	z	u	v		
	x	x	x	x	x	x	x	x		x	y	z	x	y	z	u	v		
S_{11}	:	y	y	y	y	y	y	y		S_{11}^*	:	y	y	v	x	y	z	u	v
	z	z	z	z	z	z	z	z			z	u	z	x	y	z	u	v	
	u	u	u	u	u	u	u	u			u	u	v	x	y	z	u	v	
	v	v	v	v	v	v	v	v			v	u	v	x	y	z	u	v	

THEOREM B. - An idempotent semigroup S is orderable if and only if it does not contain a subsemigroup isomorphic to either one of semigroups $S_1 - S_{11}^*$ given above.

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