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INVERSE AND PARTIALLY ORDERED SEMIGROUPS

by Liam O'CARROLL

(Dedicated to the memory of Mme Marie-Louise DUBREIL-JACOTIN)

We follow the notation and terminology of CLIFFORD and PRESTON [2].

Let S be an inverse semigroup with semilattice of idempotents E, and let ρ denote the minimum group congruence [5] on S. Then S is said to be *reduced* if $E\rho = E$ (SAITO [7] used the term *proper*), and a congruence τ on S is called *reduced* if S/τ is reduced.

THEOREM 1. - *Let* S *be an inverse semigroup. Then the congruence generated by* $\rho \cap \mathcal{R}$ *is the minimum reduced congruence on* S.

COROLLARY [7]. - *If* S *is a reduced inverse semigroup, then* $\rho \cap \mathcal{R}$ *is the identity congruence on* S.

The next result gives the structure of an arbitrary reduced inverse semigroup. The main idea behind the theorem is that each ρ-class of a reduced inverse semigroup is V-completed so as to build up an F-inverse semigroup ; the structure of the latter is known [4], and the structure of the reduced inverse semigroup is then recovered. First, we introduce some notation.

Let E be a semilattice ; then $M(E)$ denotes the semilattice

$$\{\square \subset H \subseteq E \mid EH = H\}$$

under the operation of set multiplication. The mapping $j : e \to Ee$ embeds E isomorphically in $M(E)$. Further, given a group G, a family $\Phi(G) \equiv \{\Phi_g \mid g \in G\}$ of endomorphisms of $M(E)$ is called *compatible* if it satisfies conditions (i), (ii) and (iii) of [4], theorem 4 for the semilattice $M(E)$, together with the further condition :

(iv) For each $F \in M(E)$ and $g \in G$, $F\Phi_g = \bigcup \{(Ef)\Phi_g \mid f \in F\}$.

Thus the family $\Phi(G)$ is specified by its action on Ej.

THEOREM 2. - *Let* E *be a semilattice,* G *a group, and* $\Phi(G)$ *a compatible family of endomorphisms of* $M(E)$. *Denote by* $[E ; G ; \Phi]$ *the set*

$$\{(Ee, g) \mid e \in E, g \in G, e \in E\Phi_g\}$$

under the operation

$$(Ee, g)(Ef, h) = (Ee.(Ef)\Phi_g, gh) .$$

Then $[E ; G ; \Phi]$ *is a reduced inverse semigroup, with semilattice of idempotents isomorphic to* E *, and maximal group homomorphic image* G .

Conversely, given a reduced inverse semigroup S *with semilattice of idempotents* E , $S \simeq [E ; S/\rho ; \Phi]$ *where for each* $H \in M(E)$ *and* $a \in S$, $H\Phi_{a\rho}$ *equals the set product* $a\rho.H.(a\rho)^{-1}$.

COROLLARY. - *An inverse semigroup* S *with semilattice of idempotents* E *is isomorphic to a semidirect product of a semilattice and a group if and only if*

$$E = \{xx^{-1} \mid x \in a\rho\}$$

for each $a \in S$ *and* S *is reduced ; equivalently, if and only if* $E = a\rho.(a\rho)^{-1}$ *for each* $a \in S$.

The theory has interesting specialisations to the semilattice of groups and bisimple inverse cases.

The V-completion of the ρ-classes is accomplished by applying a theorem in the theory of partially ordered semigroups ([6], theorem 3 with S a reduced inverse semigroup under the natural order, $\alpha = \rho^{\natural}$ and $D = S/\rho$ under the trivial order). For partially ordered semigroups, the following weaker result is obtained, generalising the main result of [3] :

THEOREM 3. - *Let* S *be a partially ordered semigroup. Then* S *is a strict* A-*nomal quasi residuated semigroup whose maximal elements form the group of units of* S *if and only if* S *is a semidirect product of* E *by* G , *where* E *is a quasi residuated semigroup with maximum element which is its identity element, and* G *is a trivially ordered group*.

In theorem 3, ρ is taken to be the zig-zag congruence [1] (see [8]), and S being strict means that each ρ-class has a maximum element and that S has an identity 1 which is the maximum element in 1ρ . In the semidirect product, the Cartesian ordering is employed, and the structural anti-homomorphism maps the G into the group of multiplicative, and order, automorphisms of E .

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