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Séminaire Dubreil. Algèbre et théorie des nombres, tome 23, nº 2 (1969-1970), exp. nº DG 16, p. DG1-DG5
<http://www.numdam.org/item?id=SD_1969-1970__23_2_A13_0>

Séminaire DUBREIL-PISOT
(Algèbre et Théorie des nombres)
23e année 1969/70
Demi-groupes, n° 16, 5 p.

σ-REFLEXIVE SEMIGROUP AND RINGS

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σ-reflexive semigroups generalize hamiltonian groups and lend themselves to a precise study in the subdirectly irreducible case. A σ-reflexive semigroup S, which is the multiplicative semigroup of a ring, is shown to be commutative.

We shall call a semigroup S, a σ-reflexive semigroup, if any subsemigroup H in S is reflexive (i. e. for all $a, b \in S$, $ab \in H$ implies $ba \in H$ ([2], [4])). It can be verified that any group G is a σ-reflexive semigroup if, and only if, any subgroup of G is normal. In this paper, we characterize subdirectly irreducible σ-reflexive semigroups. We derive the following commutativity result : Any generalized commutative ring R ([1]), in which the integers $n = n(x, y)$ in the equation $(xy)^n = (yx)^m$ can be taken equal to 1, for all $x, y \in R$, must be a commutative ring.

Conventions. - If $S(R)$ is a semigroup (ring), then the multiplicative subsemigroup that is generated by a given element x is written $[x]$. A polynomial $f(t) \in Z[t]$ (the ring of integral polynomials) having the form

$$f = f(t) = t^k + r_{k+1} t^{k+1} + \ldots + r_{k+m} t^{k+m} \qquad (k \geqslant 1)$$

is termed lower monic polynomial of co-degree k. Henceforth, all polynomials $f(t) \in Z[t]$ are assumed to be without constant term.

I

In this part, S is a multiplicative semigroup. Our aim is to characterize subdirectly irreducible σ-reflexive semigroups S. The following proposition is evident.

PROPOSITION 1. - Any semigroup S is σ-reflexive if, and only if, it satisfies the following condition :

$$\forall a, b \in S, \ \exists m = m(a, b) \geqslant 1 ; \ ab = (ba)^m$$

(☆) This research has been supported by Grant A 4807 of the National Research Council of Canada.

(☆☆) This research has been supported by Grant A 4096 of the National Research council of Canada.

From proposition 1 follows proposition 2.

PROPOSITION 2. - Let a, b be any two non-commuting elements of a σ-reflexive semigroup S. Then for some $m > 1$, $(ab)^m = ab$.

Proof. - There exists $r \geq 1$ such that $ba = (ab)^r$. As $ab \neq ba$, $r > 1$. As $ba \in [(ab)^r]$, we have $ab \in [(ab)^r]$. Therefore, for some $s \geq 1$, $(ab)^{rs} = ab$ with $rs > 1$.

Proposition 2 is elementary, and is an important tool for the present considerations. We can now prove our first theorem.

THEOREM 1. - Any group G is σ-reflexive if, and only if, every subgroup of G is normal.

Proof. - The "only if" is evident. To prove the "if" it suffices to show that for any a, $b \in G$, if $ab \neq ba$, then $[ab]$ coincides with the cyclic subgroup that is generated by ab. But this is evident from proposition 2 and from the structure of finite cyclic semigroups.

THEOREM 2.

(1) Any σ-reflexive semigroup S is a central idempotent semigroup.

(2) Any σ-reflexive semigroup S without central idempotents is commutative.

Proof.

(1) Let e be an idempotent in S. Let $x \in S$. There are r, $s \geq 1$ such that

$$ex = (xe)^r, \quad xe = (ex)^s \qquad \text{(Prop. 1)}.$$

Then

$$exe = (xe)^r e = (xe)^r = ex \quad \text{and} \quad exe = e(ex)^s = (ex)^s = xe .$$

(2) By (1), S does not have idempotents. By proposition 2, no elements a, $b \in S$ do not commute pairwise.

The following proposition is evident.

PROPOSITION 3. - Any σ-reflexive semigroup is a subdirect product of subdirectly irreducible σ-reflexive semigroups.

We are now in a position to show our main result.

THEOREM 3. - Let S be a non commutative σ-reflexive semigroup which is subdirectly irreducible. Then S satisfies the following conditions :

(1) S has an identity, and $G = \{x \mid x \in s, \ y \in S, \ xy = 1\}$ is a σ-reflexive group which is noncommutative (hamiltonian group).

(2) <u>If</u> $D = S - G$ <u>is non empty, then</u> S <u>is a semigroup with zero</u> $0 \in D$, D <u>is the maximum ideal of</u> S , <u>and</u> D <u>is contained in the center of</u> S .

Proof. – In view of theorem 2, S must contain at least one central idempotent. Since S is subdirectly irreducible , an idempotent element of S is the zero of S , or the identity element 1 ([5]) .

Let us suppose that S has no identity element 1 . Then S must have a zero element 0 . For some $a , b \in S$, we have $ab \neq ba$. Hence, by proposition 2, $(ab)^m = ab$ for some $m > 1$, and $(ab)^{m-1}$ is an idempotent. Therefore,

$$(ab)^{m-1} = 0 \ , \quad ab = 0 \quad \text{and} \quad ba = ab \ ,$$

which is a contradiction, and S has an identity follows. If $x \in G$ and $xy = 1$, then, since 1 is a subsemigroup of S , $yx = 1$. This shows that G is the group of invertible elements of S and that G is a σ-reflexive.

Assuming (2), it is evident that G is non commutative.

It remains to show (2). It is immediate that D is the maximum ideal of S . Let $x \in S$, $a \in D$. Suppose $ax \neq xa$. Then, for some $m > 1$ we have $(ax)^m = ax$ (Prop. 2). But $ax \neq 0$, and $(ax)^{m-1}$ is an idempotent $\neq 0$. Hence $(ax)^{m-1} = 1$ and $a \notin D$, a contradiction.

To see that S is a semigroup with zero, we proceed as follows. Let H be the intersection of all ideals of S containing more than one element. If D is reduced to one element z , then z is the zero of S . In the opposite case, $H \subseteq D$, and H is in the center of S . As S is subdirectly irreducible, H contains more than one element ([5]) . If for each $x \in H$, we have $Sx = xS = H$, then H is a group, hence contains is a non zero idempotent so H must be S , a contradiction. Therefore there exists at least one element $z \in H$ such that $Sz = \{z'\}$. As S has an identity element $z = z'$ follows and $0 = z$ is the zero of S .

II

In this part, R is a ring. In view of proposition 2, one can give the following generalization of σ-reflexive semigroups. A ring R is <u>Σ-reflexive</u> if, for any two elements $a , b \in R$, either $ab = ba$ or $ab = f(ba)$ for some integral polynomial $f(t)$ depending on a and b of degree $m \geq 2$.

Clearly, if the multiplicative semigroup of R is σ-reflexive, then R is Σ-reflexive. Our aim is to show that any Σ-reflexive ring is commutative. The analog of proposition 2 reads as follows :

PROPOSITION 4. - Let a , b be any two commuting elements of a Σ-reflexive ring. Then for some lower monic polynomial f of co-degree 1 , we have $f(ab) = 0$.

Proof. - There are $g(t)$ and $h(t)$ of degrees ≥ 2 such that $ab = g(ba)$, $ba = h(ab)$. Hence $ab = gh(ab)$ and $f(t) = t - gh(t)$ is the required polynomial.

PROPOSITION 5. - Any Σ-reflexive ring R is a central idempotent ring.

Proof. - Let e be an idempotent in R . Let $x \in R$. We can find two polynomials f , $g \in Z(t)$ of degree $m \geq 1$ such that $ex = f(xe)$, $xe = g(ex)$. Then

$$exe = f(xe)e = f(xe) = ex , \qquad exe = eg(ex) = g(ex) = xe .$$

THEOREM 4. - Any Σ-reflexive ring R is commutative.

Proof. - Our proof will go by reduction to the case where R is subdirectly irreducible. As a result of HERSTEIN ([3], theorem 17), all we will have to show is that for any $a \in R$ there is some lower monic polynomial f of co-degree 1 such that $f(a) \in C$, the center of R . Assume by contradiction that some a fails to satisfy the latter condition. Then $a \notin C$ and there must be some b such that $ab \neq ba$. By proposition 4, there is some lower monic polynomial $s(t)$ of co-degree 1 such that $s(ab) = 0$. Since the co-degree of $s(t)$ is 1 , we have for some r $ab = (ab)^2 r$ and $(ab)r = r(ab)$. Then $e = (ab)r$ is an idempotent. If $e = 0$, then $ab = 0$, and $ba = 0 = ab$, contrary to the hypothesis. Therefore e is non zero idempotent. Since R is subdirectly irreducible and since, by proposition 5, e is central, then e must be the identity of R . Therefore $(ab)r = r(ab) = 1$.

Repeating for ba , we see that b is invertible. Consider $b^{-1} a$ and C . If $(b^{-1} a)b = b(b^{-1} a)$, then $b^{-1}ab = a$ and $ab = ba$, contrary to the hypothesis. Therefore $b^{-1} a$ and b do not commute. By proposition 4 again, there is some lower monic polynomial $f(t)$ of co-degree 1 such that $f(b^{-1} ab) = 0$. As $f(b^{-1} ab) = b^{-1} f(a)b$, we have $b^{-1} f(a)b = 0$. Hence, $f(a) = 0$, and $f(a) \in C$, a contradiction. This establishes the theorem.

COROLLARY 1. - Any σ-reflexive semigroup which is the multiplicative semigroup of a ring is commutative.

COROLLARY 2. - Any generalized commutative ring R , in which the integers $n = n(x , y)$ in the equation $(xy)^n = (yx)^m$ can be taken equal to 1 , for all $x , y \in R$, is a commutative ring.

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(Texte reçu le 5 octobre 1970)

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