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1. Introduction and statement of the main theorems.

The concept of a subdirect sum of integers is important in the study of abelian latticed-ordered groups (" ℓ-groups") since WEINBERG [12] has shown that a free abelian ℓ-group is a subdirect sum of integers and hence each abelian ℓ-group is a homomorphic image of a subdirect sum of integers. In this paper, those ℓ-groups which are subdirect sums of integers are characterized. We also characterize those ℓ-groups which are subdirect sums of subgroups of the naturally ordered additive group R of real numbers. TOPPING [10] has shown that each vector lattice is a homomorphic image of such an ℓ-group.

PAPPERT [9] has determined a necessary and sufficient condition for a vector lattice to be a subdirect sum of reals, and BERNAU [2] has shown that with a slight modification her theory applies to an arbitrary ℓ-group. Both of these authors use the fact that an archimedean ℓ-group can be represented by almost finite functions on a Stone space to obtain their results. Our condition is simplier, and the proof is elementary.

In [3], BERNAU characterizes those subdirect sums of integers which contain the small sum, and those which contain a dense subset of bounded elements. We can also characterize these classes of ℓ-groups. These and other special cases and corollaries of our two main theorems are contained in Section 3.

For each $\lambda \in \Lambda$, let G_λ be a totally ordered group (" o-group") that is o-isomorphic to a subgroup of R . Thus, each G_λ is an archimedean o-group, or equivalently an o-group without proper convex subgroups. $\prod G_\lambda$ will denote the large or unrestricted direct sum of the G_λ ordered pointwise, the large cardinal sum of the G_λ, and $\sum G_\lambda$ will denote the small cardinal sum of the G_λ. In particular, $\prod G_\lambda$ is an ℓ-group, and $\sum G_\lambda$ is an ℓ-ideal of $\prod G_\lambda$. If there exists an ℓ-isomorphism of an ℓ-group G onto a subdirect sum of $\prod G_\lambda$, then we say that G is a subdirect sum of reals. If, in addition, each G_λ is cyclic, then we say that G is a subdirect sum of integers.

Let G be an ℓ-group, $G^+ = \{g \in G \mid g > 0\}$, and let Z^+ be the set of all strictly positive integers. An element $x \in G^+$ will be called real, if there exists a map $y \twoheadrightarrow \overline{y}$ of G^+ into Z^+ such that :

(I) $(\bar{y}x - y) \wedge (\bar{z}x - z) \nleqslant 0$ for all $y , z \in G^+$.

If, in addition, for all $y \in G^+$ and all $n \in Z^+$:

(II) $\bar{y} = 1$ implies $\overline{ny} = 1$,

(III) $x \geqslant 2y$ implies $\bar{y} = 1$,

then x will be called an <u>integral</u> element of G .

THEOREM 1. - <u>An</u> ℓ-<u>group</u> G <u>is a subdirect sum of reals if, and only if, each</u> $y \in G^+$ <u>exceeds a real element</u>.

THEOREM 2. - <u>An</u> ℓ-<u>group</u> G <u>is a subdirect sum of integers if, and only if, each</u> $y \in G^+$ <u>exceeds an integral element</u>.

2. Proofs of theorems 1 and 2.

In all that follows, let $G \neq 0$ be an ℓ-group. A <u>convex</u> ℓ-<u>subgroup</u> M of G is a subgroup that satisfies

$$|x| \leqslant |a| \quad \text{for} \quad x \in G \quad \text{and} \quad a \in M \qquad \text{implies} \qquad x \in M \ ,$$

or equivalently M is a sublattice and a convex subset of G . In particular, the set of all right cosets of a convex ℓ-subgroup M is a distributive lattice such that, for all $a , b \in G$,

$$M + a \vee M + b = M + a \vee b \ ,$$

and dually, where, by definition, $M + a \geqslant M + b$ if $x + a \geqslant b$ for some $x \in M$. A <u>prime</u> subgroup of G is a convex ℓ-subgroup for which the lattice of right cosets is totally ordered. For a convex ℓ-subgroup M of G , the following properties are equivalent :

(a) M is prime ;

(b) The set of convex ℓ-subgroups that contain M is a chain with respect to inclusion ;

(c) If $a , b \in G^+ \setminus M$, then $a \wedge b \in G^+ \setminus M$.

Let $\mathcal{M}$ be the set of all maximal prime subgroups of G . If $M \in \mathcal{M}$ and $M \triangleleft G$, then G/M is o-isomorphic to a subgroup of R (notation $G/M \prec R$). For proofs of the above, see [6].

We shall consider the following properties of $x \in G^+$:

(1) There exists $M \in \mathcal{M}$ such that $M + x$ covers M and, for each $y \in G^+$, $M + nx > M + y$ for some $n \in Z^+$;

(2) x is an integral element of G ;

(3) x is a real element of G ;

(4) There exists $M \in \mathfrak{M}$ such that, for each $y \in G^+$, $M + nx > M + y$ for some $n \in Z^+$.

LEMMA. - (1) ==> (2) ==> (3) <==> (4) , _and if each_ $M \in \mathfrak{M}$ _is normal in_ G , _then_ (2) ==> (1) .

Proof. - It follows from the definition of real and integral elements that (2) ==> (3) .

(4) ==> (3) : For each $y \in G^+$, let $\bar{y}$ be the least element in Z^+ such that $M + \bar{y}x > M + y$. Then, for all $y , z \in G^+$,

$$M + (\bar{y}x - y) \wedge (\bar{z}x - z) = M + (\bar{y}x - y) \wedge M + (\bar{z}x - z) > M \quad .$$

Thus $(\bar{y}x - y) \wedge (\bar{z}x - z) \not\leq 0$, and so x is real.

(1) ==> (2) : Define $\bar{y}$ as above. Since $M + x$ covers M , for $y \in G^+$ and $n \in Z^+$, the following are equivalent :

$$\bar{y} = 1 , \qquad y \in M , \qquad ny \in M \qquad \text{and} \quad \overline{ny} = 1 \quad .$$

If $y \in G^+$ and $x \geqslant 2y$, then $y \in M$, and so $\bar{y} = 1$. For if $y \notin M$, then $M + x \geqslant M + 2y > M + y > M$, but this contradicts the fact that $M + x$ covers M . Therefore x is an integral element in G .

(3) ==> (4) : For $y , z \in G^+$,

$$[(\bar{y}x - y) \vee 0] \wedge [(\bar{z}x - z) \vee 0] = [(\bar{y}x - y) \wedge (\bar{z}x - z)] \vee 0 \in G^+ \quad .$$

Thus, $Q_x = \{(\bar{y}x - y) \vee 0 \mid y \in G^+\}$ is contained in an ultrafilter K of G^+ . That is, $0 < a \wedge b \in K$ for all $a , b \in K$, and K is maximal with respect to this property. It follows that

$$N = \bigcup_{k \in K} k'$$

is a minimal prime subgroup of G , and $K = G^+ \setminus N$, where

$$k' = \{g \in G \mid |g| \wedge k = 0\}$$

is the polar of k . This is theorem 5.1 in [7], and this result is also implicit in [1] and [8].

(A) $$N + \bar{y}x > N + y , \qquad \text{for each } y \in G^+ \quad .$$

For $(\bar{y}x - y) \vee 0 \in K = G^+ \setminus N$, and hence $N + (\bar{y}x - y) \vee 0 > N$, and so

$$N + \bar{y}x - y > N \quad .$$

Since the convex ℓ-subgroups of G that contain N form a chain, there is a unique convex ℓ-subgroup $M \supseteq N$ that is maximal, with respect to $x \notin M$.

(B) $$M \in \mathfrak{M} \quad .$$

For if $y \in G^+$, then $N + \bar{y}x > N + y$, and hence $a + \bar{y}x > y > 0$ for some $a \in N$. But clearly, $a + \bar{y}x$ is contained in any convex ℓ-subgroup that properly contains M . Therefore, G covers M , and hence $M \in \mathfrak{M}$. It follows from (A) that

$$M + (\bar{y} + 1)x > M + \bar{y}x \geqslant M + y \quad .$$

Therefore (4) is satisfied.

To complete the proof, we need to show that (2) ==> (1) , provided that each $M \in \mathfrak{M}$ is normal in G . Let x be an integral element, and let M and N be as above. Suppose (by way of contradiction) that $M + x > M + y > M$ for some $y \in G$. Then, since

$$M + y \vee 0 = M + y \vee M = M + y \qquad \text{and} \qquad M + x \wedge y = M + x \wedge M + y = M + y \quad ,$$

we may assume that $x > y > 0$. Now, $x = x - y + y$, and since $x - y$, $y \in G^+ \setminus M$, and M is prime, $d = (x - y) \wedge y \in G^+ \setminus M$. Clearly, $x \geqslant 2d$, and hence $\bar{d} = 1$ and $\overline{nd} = 1$ for all $n \in Z^+$. Thus,

$$M + x = M + \overline{nd}x \geqslant M + nd \geqslant M + d > M \, , \qquad \text{for all} \quad n \in Z^+ \quad ,$$

but this is impossible, because $G/M \prec R$.

COROLLARY. - <u>Suppose that each</u> $M \in \mathfrak{M}$ <u>is normal in</u> G , <u>and consider</u> $x \in G^+$.

(a) x <u>is a real element of</u> G <u>if, and only if,</u> $x \in G \setminus M$ <u>for some</u> $M \in \mathfrak{M}$.

(b) x <u>is an integral element of</u> G <u>if, and only if,</u> $M + x$ <u>covers</u> M <u>for some</u> $M \in \mathfrak{M}$.

<u>Proof</u>. - This is an immediate consequence of the lemma and the fact that $G/M \prec R$ is an archimedean o-group for each $M \in \mathfrak{M}$.

BYRD [4] has shown that G is a subdirect sum of o-groups if, and only if, for each prime subgroup M and each $g \in G$, $-g + M + g \subseteq M$ or $-g + M + g \supseteq M$. Thus, for this class of ℓ-groups, each $M \in \mathfrak{M}$ is normal.

<u>Proof of theorem</u> 1. - Suppose that G is a sublattice and a subdirect sum of $\prod R_\lambda$ $(\lambda \in \Lambda)$, where each $R_\lambda \subseteq R$. If $x \in G^+$, then $x_\lambda > 0$ for some $\lambda \in \Lambda$. Let $M = \{g \in G \mid g_\lambda = 0\}$. Then $M \in \mathfrak{M}$ and $x \in G \setminus M$. Thus, by the corollary, x is real, and so <u>each</u> $x \in G^+$ <u>is real</u>.

Conversely, suppose that each element in G^+ exceeds a real element, and consider y , $z \in G^+$. There exists a real element $x \leqslant z$. Thus $\bar{y}x \nleqslant y$, and hence $\bar{y}z \nleqslant y$. Therefore G is archimedean, and hence abelian. By the corollary, $x \in G \setminus M$ for some $M \in \mathfrak{M}$, and hence $z \in G \setminus M$. Therefore, $0 = \bigcap \{M \mid M \in \mathfrak{M}\}$, and so G is a subdirect sum of reals.

Proof of theorem 2. - Suppose that G is a sublattice and a subdirect sum of $\prod Z_\lambda$ $(\lambda \in \Lambda)$, where each $Z_\lambda = Z$. If $g \in G^+$, then $g \geqslant x > 0$ for some $x \in G$, where $x_\lambda = 1$ for some $\lambda \in \Lambda$. Let $M = \{g \in G \mid g_\lambda = 0\}$. Then $M \in \mathfrak{M}$, and $M + x$ covers M , and hence, by the corollary, x is integral. Therefore each element in G^+ exceeds an integral element.

Conversely, suppose that each element in G^+ exceeds an integral element. Then, as in the proof of theorem 1, G is abelian. Let $\mathfrak{I} = \{M \in \mathfrak{M} \mid G/M$ is cyclic$\}$. Then, by the corollary, $\bigcap \{M \mid M \in \mathfrak{I}\}$ must be zero, since it contains no integral element. Therefore G is a subdirect sum of integers.

3. Special cases of theorems 1 and 2.

An element $s \in G^+$ is called basic, if $\{g \in G \mid 0 \leqslant g \leqslant s\}$ is totally ordered.

PROPOSITION A. - *For an ℓ-group G , the following properties are equivalent* :

(1) G *is a subdirect sum of reals that contains the small sum* ;

(2) *Each element in G^+ exceeds a real element that is also basic* ;

(3) G *is archimedean, and each element in G^+ exceeds a basic element.*

Proof. - It is shown in [5] that (1) <==> (3) . If each element in G^+ exceeds a real element, then G is archimedean, and hence (2) ==> (3) . If (1) holds, then each element in G^+ is real, and hence (1) and (3) imply (2).

There are many other equivalent conditions proven in the literature (see for example [11]).

An element $a \in G^+$ is an atom, if it covers 0 . It is shown in [5] that x is a basic element in an archimedean ℓ-group G if, and only if, $x'' < R$, and G is the cardinal sum of x'' and x' . Thus a basic element x is integral if, and only if, x'' is cyclic, and hence if, and only if, x is an atom.

PROPOSITION B. - *For an ℓ-group G , the following properties are equivalent* :

(1) G *is a subdirect sum of integers that contains the small sum* ;

(2) *Each element in G^+ exceeds an integral element that is also basic* ;

(3) G <u>is archimedean, and each element in</u> G^+ <u>exceeds an atom</u>.

<u>Proof</u>. - Clearly (1) ==> (2) ==> (3) .

(3) ==> (1) : Since each atom is a basic element, it follows from proposition A that G is a subdirect sum of reals that contains the small sum. Thus, without loss of generality,

$$\sum R_\lambda \subseteq G \subseteq \prod R_\lambda \quad ,$$

where $R_\lambda \subseteq R$ for each $\lambda \in \Lambda$. If R_λ is not cyclic, then there exists an element in $R_\lambda^+ \subseteq G^+$ that does not exceed an atom, a contradiction. Therefore (1) holds.

An element $s \in G^+$ is called <u>singular</u>, if $a \wedge (s - a) = 0$ for each $0 \leqslant a \leqslant s$.

PROPOSITION C. - <u>For an ℓ-group</u> G , <u>the following properties are equivalent</u> :

(1) G <u>is a subdirect sum of integers, and each element in</u> G^+ <u>exceeds a bounded element</u> ;

(2) <u>Each element in</u> G^+ <u>exceeds an integral element that is also singular</u> ;

(3) G <u>is a subdirect sum of reals, and each element in</u> G^+ <u>exceeds a singular element</u>.

<u>Proof</u>. - In [7], it is shown that (1) <==> (3) , and clearly (2) ==> (3) . Suppose that (1) and (3) hold. Then, without loss of generality, $G \subseteq \prod Z_\lambda$, where for each $\lambda \in \Lambda$, $Z_\lambda = Z$, and in [7], it is shown that if $s \in G$ is singular, then $s_\lambda = 1$ or 0 . Thus, it follows that s is integral, and hence we have (2).

BERNAU [3] has established (1) <==> (3) in proposition B, and has derived a condition that is equivalent to (1) in proposition C.

Suppose that $x \in G^+$ is real, and let A_x be the set of all maps $\pi :\ G^+ \to Z^+$, such that for all $y , z \in G^+$,

$$((y\pi)x - y) \wedge ((z\pi)x - z) \nleqslant 0 \quad .$$

For $\alpha , \beta \in A_x$, define $\alpha \leqslant \beta$ if $y\alpha \leqslant y\beta$ for all $y \in G^+$. Then $(A , \leqslant)$ is a po-set, and each element in A_x exceeds a minimal element in A_x . For if

$$\{\alpha_\lambda \mid \lambda \in \Lambda\}$$

is a chain in A_x , then for each $y \in G^+$, define

$$y\pi = \min\{y\alpha_\lambda \mid \lambda \in \Lambda\} \quad .$$

If $y , z \in G^+$, then there exists $\lambda \in \Lambda$ such that $y\alpha_\lambda$ and $z\alpha_\lambda$ are minimal, and so

$$((y\pi)x - y) \wedge ((z\pi) - z) = ((y\alpha_\lambda)x - y) \wedge ((z\alpha_\lambda)x - z) \nleq 0 \quad .$$

Therefore $\pi \in A_x$, and hence, by Zorn's lemma, each map in A_x exceeds a minimal map.

Definition. - A real element $x \in G^+$ for which there exists a minimal map $y \to \bar{y}$ in A_x that also satisfies (II), will be called a ☆-element.

PROPOSITION D. - For an ℓ-group, the following properties are equivalent :

(1) Each element in G^+ exceeds a ☆-element ;

(2) G is (ℓ-isomorphic to) a subdirect sum of $\prod Z_\lambda$, where for each $\lambda \in \Lambda$, $Z_\lambda = Z$, and $G_\lambda = \{g \in G \mid g_\lambda = 0\}$ is both a maximal and a minimal prime subgroup of G .

Proof.

(1) $\Longrightarrow$ (2) : Since each ☆-element is real, it follows from theorem 1 that G is abelian. Let x be a ☆-element in G , and let $y \to \bar{y}$ be a minimal map in A_x that also satisfies (II). Construct M and N as in the proof of (3) $\Longrightarrow$ (4) in the lemma. Since $N + \bar{y}x > N + y$ for all $y \in G^+$, and the map $y \to \bar{y}$ is minimal, it follows that $\bar{y}$ is the least element in Z^+ for which $N + \bar{y}x > N + y$. Suppose (by way of contradiction) that $M \supset N$, and pick $0 < z \in M \setminus N$, and let $y = -(x \wedge z) + x$. Then,

$$M + x = M + y \qquad \text{and} \qquad N + x > N + y \quad .$$

Therefore $\bar{y} = 1$, and hence $\overline{2y} = 1$, but clearly $N + \overline{2y}x = N + x < N + 2y$, that is a contradiction. Thus, $N = M$ is both maximal and minimal. If $M + x > M + y$, then $\bar{y} = 1$, and hence $M + x = M + \overline{ny}x \geqslant M + ny$ for all $n \in Z^+$. Thus, since $G/M < R$, it follows that $y \in M$, and so G/M is cyclic.

(2) $\Longrightarrow$ (1) : We may assume that $G \subseteq \prod Z_\lambda$. If $z \in G^+$, then $z \geqslant x \in G^+$, where $x_\lambda = 1$ for some $\lambda \in \Lambda$. For $y \in G^+$, define $\bar{y}$ to be the least element in Z^+ such that $\bar{y}x_\lambda > y_\lambda$. Then, the map $y \to \bar{y}$ satisfies (I), (II) and (III). It remains to be shown that this map is minimal in A_x . Suppose that $y \to \tilde{y}$ is a map in A_x , and $\tilde{y} \leqslant \bar{y}$ for all $y \in G^+$. Construct M and N as above, using the map $y \to \tilde{y}$. In particular, $N + \tilde{y}x > N + y$ and $M + \tilde{y} \geqslant M + y$ for all $y \in G^+$.

If $M \neq G_\lambda$, then there exists $y \in G^+$ such that $y_\lambda = 0$ and $y \notin M$. Since $y_\lambda = 0$, $\bar{y} = 1$, and so $\overline{ny} = \widetilde{ny} = 1$ for all $n \in Z^+$, but this means that

$M + x \geqslant M + \widetilde{nyx} \geqslant M + ny$ for all $n \in Z^+$, and this contradicts the fact that $G/M < R$.

If $M = G_\lambda$, then, since G_λ is a minimal prime, $M = N$, and so $M + \tilde{y}x > M + y$ for all $y \in G^+$, and it follows that $\bar{y} = \tilde{y}$ for all $y \in G^+$. Therefore x is a ☆-element, and hence (1) is satisfied.

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