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RIESZ VECTOR SPACES AND RIESZ ALGEBRAS

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1. Introduction.

In 1940, F. RIESZ investigated the bounded linear functionals on real function spaces S, and showed that they form a vector lattice whenever S is assumed to possess the following interpolation property.

(A) Riesz interpolation property. - If f_1, f_2, g_1, g_2 are functions in S such that $f_i \leqslant g_j$ for $i = 1, 2$ and $j = 1, 2$, then there is some $h \in S$ such that

$$f_i \leqslant h \leqslant g_j \quad \text{for all } i, j .$$

Clearly, if S is a lattice then it has the Riesz interpolation property (choose e. g. $h = f_1 \vee f_2$ or $g_1 \wedge g_2$), but there exist a number of important function spaces which are not lattice-ordered and have the Riesz interpolation property. For instance :

(1) The real polynomials in the unit interval $[0, 1]$ where we put $f \geqslant 0$ for a polynomial f if and only if $f(x) \geqslant 0$ for all $x \in [0, 1]$ (this is the so-called pointwise ordering).

(2) The rational functions with real coefficients, defined everywhere in $[0, 1]$, under the pointwise ordering.

(3) For a positive integer k, the k-times differentiable real functions in $[0, 1]$, under the pointwise ordering.

(4) The real analytic functions in $[0, 1]$, under the pointwise ordering.

(5) The real trigonometric functions in $[0, 2\pi]$, the ordering is pointwise.

(6) The real-valued continuous functions in the unit interval (or more generally, on a compact Hausdorff space) if $f > 0$ has the meaning $f(x) > 0$ for all x (this ordering will be called the ordering by strict inequalities).

(7) The function spaces mentioned in (1)-(5) under the ordering by strict inequalities.

(8) The cartesian product of function spaces with the Riesz interpolation property has again the same property.

As a further example of a partially ordered vector space with the Riesz interpolation property, we may mention the complex numbers (considered as a vector space over the reals) where $x + iy > 0$ means $x > 0$ and $y > 0$, or in another ordering, $x + iy > 0$ means $x > 0$ (x, y are reals).

These examples show that it might be of some interest to investigate systematically the partially ordered vector spaces with the Riesz interpolation property. Our present aim is to give a brief survey of the results ; some of the theorems can be formulated more generally for partially ordered groups. For the proofs, we refer to our papers [1]-[4].

2. Basic definitions.

By a vector space V, we shall mean one over the field $\mathbb{R}$ of real numbers. V is said to be partially ordered if it is partially ordered under a binary relation $\leqslant$ such that $a \leqslant b$ in V implies $a + c \leqslant b + c$ for all $c \in V$ and $\lambda a < \lambda b$ for all positive real λ. The set P of all $a \in V$, satisfying $a \geqslant 0$, is called the positivity domain of V, and the elements of P are called positive. P completely determines the partial order of V since $a \leqslant b$ is equivalent to $b - a \in P$.

V is called directed if, to each pair $a , b \in V$, there exists some $c \in V$ such that $a \leqslant c$ and $b \leqslant c$. V is directed if and only if every element is the difference of two positive elements.

If for all $a , b \in V$, either $a \leqslant b$ or $b \leqslant a$ in V, then V is totally ordered. If V is a lattice under its partial order, then it is called a vector lattice. We shall write $a \wedge b$, $a \vee b$ for the g. l. b. (intersection) and the l. u. b. (union) of the elements $a , b \in V$ which might exist even if V is not lattice-ordered. A directed partially ordered vector space with the Riesz interpolation property will be called a <u>Riesz vector space</u>. Note that the Riesz interpolation property (A) is equivalent to either one of the following conditions :

(B) <u>Decomposition property</u>. - If $0 \leqslant a \leqslant b_1 + b_2$ for a, $b_i \in V$, $b_i \geqslant 0$, then there exist $a_1 , a_2 \in V$ such that $a = a_1 + a_2$ where

$$0 \leqslant a_i \leqslant b_i .$$

(C) <u>Refinement property</u>. - If $a_1 + a_2 = b_1 + b_2$ for positive elements a_i, b_j of V, then there exist elements $c_{ij} \geqslant 0$ in V $(i = 1 , 2 ; \ j = 1 , 2)$ such that

$$a_i = c_{i1} + c_{i2} \quad (i = 1 , 2) \quad \text{and} \quad b_j = c_{1j} + c_{2j} \quad (j = 1 , 2) .$$

A subspace W of a partially ordered vector space V is said to be convex if $a, b \in W$, $x \in V$ and $a \leqslant x \leqslant b$ imply $x \in W$. If a subspace is trivially ordered (i. e. it contains no two elements a, b such that $a < b$), then it is necessarily convex. An o-ideal is defined as a directed convex subspace. If V has no o-ideals other than the trivial ones, it is called o-simple.

THEOREM 1. - The o-ideals of a Riesz vector space V form a distributive sublattice in the lattice of all subspaces of V.

The factor space V/W of a partially ordered vector space V mod its convex subspace W becomes a partially ordered vector space if a coset $a + W$ is defined to be positive whenever it contains a positive vector $a + w$ $(w \in W)$.

THEOREM 2. - If V is a Riesz vector space and I is an o-ideal of V, then both I and V/I are Riesz vector spaces.

A vector space homomorphism that preserves positivity is called an o-homomorphism. A vector space isomorphism that is order preserving in both directions is said to be an o-isomorphism.

3. Algebraic theory.

In order to investigate the Riesz vector spaces from the algebraic point of view, first we have to introduce the simplest and, at the same time, the most important type of Riesz vector spaces : the antilattices.

In a partially ordered vector space V, the elements a, b have always a g. l. b. $a \wedge b$ whenever $a \leqslant b$ or $b \leqslant a$. A Riesz vector space A, in which no other g. l. b. exist, i. e. the existence of $a \wedge b$ in A implies $a \wedge b = a$ or $a \wedge b = b$, is called an antilattice. Because of $a \vee b = -(-a \wedge -b)$, the existence of $a \vee b$ implies $a \vee b = a$ or $= b$ in an antilattice, thus an antilattice is a Riesz vector space in which only the trivial g. l. b. and l. u. b. exist.

It is easy to conclude that if a finite number of elements of an antilattice have a g. l. b., say $a_1 \wedge \ldots \wedge a_n = b$, then $a_j = b$ for some j.

Examples for antilattices are abundant ; see our examples (1)-(2), (4)-(7) in § 1. Also, all totally ordered vector spaces are antilattices. On the other hand, an antilattice which is lattice-ordered is a totally ordered vector space.

The following is the structure theorem on Riesz vector spaces. It shows that the antilattices play a similar role in the theory of Riesz vector spaces as the totally ordered vector spaces in the theory of vector lattices.

THEOREM 3. - <u>To every Riesz vector space</u> V <u>there exists a family of antilattices</u> A_α $(\alpha \in I)$ <u>and an</u> o-<u>isomorphism</u> φ <u>of</u> V <u>onto a subspace of the cartesian product</u> $\prod_{\leqslant} A_\alpha$ <u>such that</u> φ <u>preserves g. l. b. and l. u. b. too</u>.

Here the notation $\prod_{\leqslant}$ means that an element $\langle \dots , a_\alpha , \dots \rangle \in \prod A_\alpha$ is to be considered as $\geqslant 0$ if, and only if, $a_\alpha \geqslant 0$ for every $\alpha \in I$.

More information about the structure of antilattices may be obtained from the following two results.

THEOREM 4. - <u>If</u> A <u>is an antilattice and</u> C <u>is a maximal trivially ordered subspace of</u> A, <u>then</u> A/C <u>is totally ordered</u>.

Thus an antilattice is an extension of a trivially ordered vector space by a totally ordered vector space.

THEOREM 5. - <u>Assume that</u> A <u>is an antilattice such that the intersection</u> $\cap C$ <u>of all maximal trivially ordered subspaces</u> C <u>of</u> A <u>is</u> 0. <u>Then</u> A <u>can be embedded</u> o-<u>isomorphically in a cartesian product</u> $\prod_{<} B_\beta$ <u>of totally ordered vector spaces</u> B_β $(\beta \in J)$.

Here $\prod_{<}$ means that $\langle \dots , b_\beta , \dots \rangle > 0$ is defined if, and only if, $b_\beta > 0$ for all β.

If A is as in theorem 5 and is in addition o-simple, then all the B_β are o-isomorphic to the real numbers, and we get :

THEOREM 6. - <u>If</u> A <u>is an</u> o-<u>simple antilattice such that</u> $\cap C = 0$, <u>then</u> A <u>is</u> o-<u>isomorphic to a vector space of real-valued functions on some set</u> Ω <u>where the ordering is given by strict inequalities</u>.

Note that by choosing suitable isomorphisms $B_\beta \simeq \underset{\sim}{R}$, the functions corresponding to elements of A will be bounded.

4. Topological questions.

The continuous functions on a compact Hausdorff space Ω are usually furnished with the uniform topology. This topology may be regarded as the one obtained by ordering the continuous functions by strict inequalities and then taking as a subbase of neighbourhoods at 0 the open intervals $(-f , f)$ with functions $f > 0$. In accordance with this remark, we introduce the open-interval topology as follows.

Let V be a Riesz vector space. For all $u > 0$ in V, take the open intervals $(-u, u)$ as a subbase of open neighbourhoods at 0. This gives rise to a topology on V which will be referred to as the <u>open-interval topology</u>. We have :

THEOREM 7. - <u>For the open-interval topology of a Riesz vector space</u> V <u>the following hold</u> :

(i) <u>It is Hausdorff if, and only if, the intersection</u> $\bigcap C$ <u>of all maximal trivially ordered subspaces</u> C <u>of</u> V <u>is</u> 0 ;

(ii) <u>If it is Hausdorff, then</u> V <u>is topological group in this topology</u> ;

(iii) V <u>is non-discrete in the open-interval topology if, and only if, it is an antilattice</u> ;

(iv) <u>If</u> V <u>is not non-discrete then every dense subspace of</u> V <u>is an antilattice</u>.

Notice that the hypothesis of theorem 5 is thus equivalent to a simple topological condition.

Let A be an antilattice that is a Hausdorff space in the open-interval topology. We know by (ii) that A is then a topological group, and so we may ask for the structure of its topological completion A^*. By standard results on topological groups, A^* is again a topological group which is, in addition, a vector space over $\mathbb{R}$. If A^* is constructed by means of Cauchy nets, and if we call a Cauchy net positive whenever it contains a net consisting of positive elements, then one readily checks that A^* is likewise a partially ordered vector space. If A is, in particular, the antilattice of all polynomials in $[0, 1]$ under the ordering by strict inequalities, then A^* is just the vector space of all continuous functions in $[0, 1]$, thus A^* is a vector lattice in this case. This is, however, not true in general, as is shown for instance by our example (1) in § 1.

We shall say that the antilattice A has the <u>approximation property</u>, or A is an <u>approximation antilattice</u>, if the following condition is satisfied : Given $a, b \in A$ and $u > 0$ in A, there exists $a, c \in A$ such that $c < a, b$, and if $x \in A$ satisfies $x < a, b$ then $x < c + u$. It is easy to see that the approximation property is necessary for A in order that A^* be a vector lattice.

THEOREM 8. - <u>Let</u> A <u>be an antilattice which is Hausdorff in its open interval topology. Then</u> :

(i) If A is metrizable (i. e. it has a countable system of neighbourhoods at 0), then its topological completion A^* is a Riesz vector space ;

(ii) A^* is a vector lattice if, and only if, A has the approximation property.

The canonical map of A into A^* is of course a continuous isomorphism, but it is not necessarily an o-isomorphism. The embedding $A \longrightarrow A^*$ preserves order relations, but the image of A in A^* contains more positive elements in general, namely, the positivity domain of the image of A in A^* is the least closed set $\overline{P}$ containing the image of the positivity domain P of A. For instance, if A is the vector space of polynomials in $(0 , 1)$ under the ordering by strict inequalities, then in A^* a polynomial f will be positive if it can be approximated by polynomials positive in A, i. e. in the image of A in A^* the ordering will be pointwise.

Let L be a vector lattice. We say that L can be approximated by the antilattice A, if :

(a) $\overline{A}$ is a vector subspace of L ($\overline{A}$ denotes the partially ordered vector space obtained from A by replacing its positivity domain P by its closure $\overline{P}$).

(b) The completion A^* of A is a vector lattice containing L as a vector sublattice :

$$\overline{A} \subseteq L \subseteq A^* .$$

THEOREM 9. - Let L be a vector lattice. There exists an antilattice A approximating L such that $\overline{A} = L$, and every dense subspace B of A is an antilattice approximating L. Conversely, if B is an antilattice approximating L, then there exists an antilattice A in which B is dense and which satisfies $\overline{A} = L$.

It is an open problem in which cases A has a proper dense subspace B.

5. Antilattices which are topological vector spaces.

Next we turn our attention to antilattices A, which are not only topological groups, but also topological vector spaces in the open-interval topology. That is to say, we also assume that the mapping

$$(\lambda , a) \longrightarrow \lambda a$$

of $\mathbb{R} \times A$ into A is continuous (where $\mathbb{R}$ carries its usual topology ; actually, this is its open-interval topology). It is easy to prove :

THEOREM 10. - An antilattice A is a topological vector space in its open-interval topology if, and only if, it is o-simple.

For the sake of brevity, we shall call an antilattice A that is a topological vector space topological.

Let us fix some element $u > 0$ in a topological antilattice A, and call $U = (-u, u)$ the unit ball of A. This is justified in view of the fact that, if for an arbitrary $a \in A$, we set

$$\|a\| = \inf \{\lambda \mid \lambda^{-1} a \in U \text{ for real } \lambda > 0\}$$

(which is the well-known Minkowski functional), then we obtain :

THEOREM 11. - *A topological antilattice A is a normed vector space under the norm $\|\ldots\|$, and the dual space of A is an abstract Lebesgue space.*

Recall that an abstract Lebesgue space is a normed vector lattice which is a real Banach space such that

$$a \wedge b = 0 \quad \text{implies} \quad \|a + b\| = \|a - b\|$$

and

$$a, b \geqslant 0 \quad \text{implies} \quad \|a + b\| = \|a\| + \|b\| .$$

The following result gives a fairly good characterisation of topological antilattices.

THEOREM 12. - *For every topological antilattice A, there exists a topological o-isomorphism Φ of A, with a subspace of all real-valued continuous functions on a compact Hausdorff space Ω, where the functions are ordered by strict inequalities. Moreover, Φ can be chosen so as to be an isometry as well.*

Here Ω is the space of all maximal trivially ordered subspaces of A, furnished with the Gelfand topology.

The following theorem of Stone-Weierstrass type may be mentioned.

THEOREM 13. - *If A is a topological antilattice satisfying the approximation property, then it is o-isomorphic and isometric to a dense subspace of the space of all real continuous functions on a compact Hausdorff space Ω, ordered by strict inequalities, and the topological completion A^* of A is the vector lattice $C(\Omega)$ of all continuous functions on Ω under the pointwise ordering.*

6. Riesz algebras.

An algebra over the real numbers is called a partially ordered algebra if is a partially ordered vector space with the property that its possitivity domain is closed under algebra multiplication. The last property is equivalent to the condition that inequalities may be multiplied from the left and from the right by positive elements.

A partially ordered algebra X is called a Riesz algebra if its underlying vector space is a Riesz vector space. Our examples (1)-(8) in § 1 are Riesz algebras.

A partially ordered algebra, that is an antilattice, is said to be an antilattice-algebra.

THEOREM 14. - Every Riesz algebra X can be embedded as an o-ideal in a Riesz algebra Y with identity e. There is an embedding such that if $x \wedge y = 0$ for some $x, y \in Y$ with $x, y > 0$ then either $x \in X$ or $y \in X$, and if, e. g., $x \in X$ then some $z \in X$, $z > 0$ satisfies $x \wedge z = 0$.

Here we have meant by an o-ideal of an algebra X a convex directed algebra ideal. As a corollary to theorem 14, one obtains that an antilattice-algebra is o-isomorphic with an o-ideal of an antilattice-algebra with identity.

7. Antilattice-algebras.

If we introduce the open-interval topology in antilattice algebras, then ring multiplication need not be continuous in this topology. We are going to give a criterion for the continuity of ring multiplication.

Call the partially ordered algebra X m-bounded if every $a \in X$ satisfies : to each $u > 0$ in X, there is some $v > 0$ in X such that

$$av < u \quad \text{and} \quad va < u .$$

THEOREM 15. - In an antilattice-algebra X that is a Hausdorff space in the open-interval topology multiplication is continuous if, and only if, X is m-bounded.

Note that if the additive group of an antilattice-algebra is o-simple, then it is m-bounded.

For m-bounded antilattice-algebras the analogue of theorem 8 holds.

Let us turn finally to antilattice-algebras which are topological algebras, i. e. not only topological rings in the open-interval topology, but at the same time topological vector spaces.

THEOREM 16. - An antilattice algebra X which is Hausdorff in its open-interval topology is a topological algebra if, and only if, its underlying vertor space is o-simple.

The main result is the following representation theorem.

THEOREM 17. - Let X be an antilattice algebra with identity e and assume it is a Hausdorff space in the open-interval topology. If X is o-simple as a vector space, then it is o-isomorphic to an algebra of continuous functions on a compact Hausdorff space Ω where the functions are ordered by strict inequalities.

In this case, X is a normed algebra, and the o-isomorphism may be assumed to be an isometry as well.

Note that we have not a priori supposed that X was associative or commutative, but our result indicates that it must be both associative and commutative.

If X is as in theorem 17, then its topological completion is a commutative Banach algebra which is a Riesz algebra too. If, in addition, X enjoys the approximation property, then its completion is the algebra of all continuous functions on Ω where the functions are ordered pointwise.

If X is the real polynomials in $[0 , 1]$, ordered by strict inequalities, then it is easy to see that Ω will be homeomorphic to $[0 , 1]$.

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