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THE STRUCTURAL COMPONENTS OF A COMPACT CONNECTED SEMIGROUP WITH 1

by Karl Heinrich HOFMANN

Definition. - A compact semigroup is a compact Hausdorffspace together with a jointly continuous multiplication satisfying $(ab)c = a(bc)$.

Paragraph 1. - Examples : Examples of objects in the category of compact semigroups are numerous :

1.1. - Compact groups.

1.2. - Totally ordered compact spaces under forming the minimum or maximum of two elements as product (the class of semigroups of this type is a subclass of the more general class of compact semilattices).

1.3. - Any compact space X (resp. Y) with the multiplication defined by $ab = a$ (resp. $ab = b$) and the direct product $X \times Y$ of two such semigroups (a socalled rectangular semigroup).

1.4. - In any locally compact normed topological ring the set $\{x : |x| \leqslant 1\}$ under multiplication. Specifically the real, complex, quaternion unit disc ; the contracting operators on finite dimensional real or complex Hilbert space.

The category of compact semigroups is closed under forming arbitrary products, quotients (modulo congruence relations $R = S \times S$ on a compact semigroup S such that R is a closed subspace of $S \times S$), and closed subgroups. There is a multitude of other methods to built up compact semigroups. The following examples are of a particular theoretical interest :

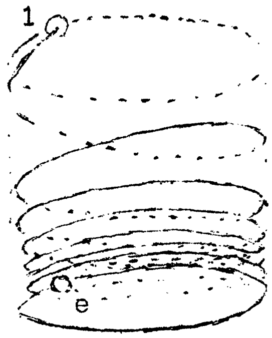
1.5. - Paragroups : Let X and Y be compact spaces and G a compact group. Suppose that $(y , x) \longrightarrow [y , x] : Y \times X \longrightarrow G$ is any continuous function. Then the product space $X \times G \times Y$ is a compact semigroup under the multiplication

$$(x , g , y)(x' , g' , y') = (x , g[y , x']g' , y') \quad .$$

(The rectangular semigroup is a special paragroup.)

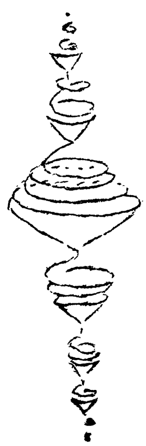
1.6. - Cylindrical semigroups : Let H be the complex unit circle under the ordinary multiplication and $[0 , 1]$ the real unit interval under ordinary multiplication. The compact subset $S \subset [0 , 1] \times H$, defined by

$$S = \{(e^{-r} , e^{2\pi iar}) : 0 \leqslant r\} \cup \{0\} \times H \quad ,$$



is a subsemigroup for any real number a. This construction can be generalized by taking any compact abelian group for H and replacing $r \longrightarrow e^{2\pi iar}$ by any morphism from the additive group of reals into H. Further one may multiply S directly by any compact group and forming a quotient of the product semigroup. The semigroups so arising are already fairly complicated although very well understood.

1.7. - Chains : In the previous example one can replace the unit circle H by the unit disc throughout ; then S will topologically be the union of an arc winding down on the periphery of a disc. The center of the disc will be the zero of the semigroup. It is possible to form chains with semigroups of this type as chain links. This process of linking cylindrical semigroups can be generalized to yield an important class of compact semigroups which turns out to be of utmost importance for the general theory of semigroups.



Paragraph 2. - For the purpose of this lecture, we describe the problem of a general theory of compact connected semigroups (as opposed to the structure theory of particular subclasses) as follows : "Find certain distinguished substructures of a compact semigroup which one might call structural components (such as e. g. the Sylow groups in finite or abelian groups, the maximal tori in compact connected Lie groups, the prime fields in fields, radicals, centers, etc.), study their structure and the nature of their embedding in the whole semigroup". In the theory of compact semigroups one has, roughly, three access routes to such structural components : (1) The algebra of semigroups ; (2) Algebra and topology combined ; (3) The predominant use of topology.

2.1. - From the algebraic theory of semigroup one takes over a number of very useful equivalence relations on a semigroup which were first introduced by Green : The relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. E. g., in a semigroup with identity $\mathcal{L}$, $\mathcal{J}$ as defined by

$$x \mathcal{L} y \iff Sx = Sy \quad \text{and} \quad x \mathcal{J} y \iff SxS = SyS \ .$$

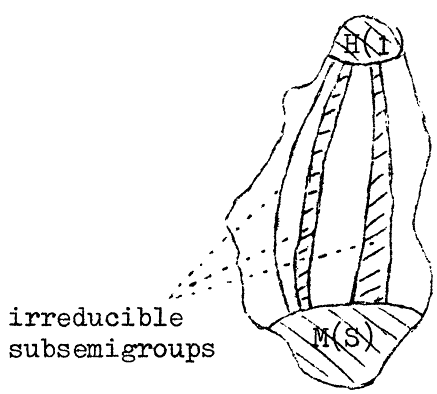
The set $S/\mathcal{J}$ has a natural partial order arising from the inclusion of ideals, and appropriate partial orderes may be defined on the quotient sets modulo the other relations. Of particular interest is the $\mathcal{H}$-class H(1) of the identity which is in fact the maximal group containing 1 and is called the group of units. Certainly this group may be considered as one of the structural components of S .

2.2. - From a combined use of algebra and topology one obtains in a compact semigroup the equality of $\mathcal{D}$ and $\mathcal{J}$ (WALLACE 1953, KOCH 1953, KOCH and WALLACE 1956) and the fact that

$$H(1) = L(1) = R(1) = D(1) \quad ,$$

i. e. that the maximal $\mathcal{D} = \mathcal{J}$-class is in fact the group of units (which is not the case in the absence of a compact semigroup topology). But also one finds a minimal $\mathcal{D} = \mathcal{J}$-class namely the minimal ideal which always exists in a compact semigroup. (This was in fact one of the first observations ever made in the theory of compact semigroups.) Again it is clear that the minimal ideal is an important structural component of a compact semigroup.

2.3. - By almost exclusively topological methods one obtains a structural component, which cuts clear across a compact connected semigroup joining the maximal and the minimal $\mathcal{D}$-class. Let $\mathcal{C}$ be the set of a compact connected subsemigroups in a compact connected semigroup with identity which contain the identity and meet the minimal ideal. This set is inductive relative to containment. Hence there are minimal such subsemigroups (which then do not contain any proper compact connected subsemigroup containing the identity and meeting its minimal ideal) ; they are called the irreducible semigroups. As opposed to the group of units and the minimal ideal a choice process was involved in finding these subsemigroups ; there are not uniquely determined. Example 1.7 is an irreducible semigroup.



Again for the purpose of this lecture, we will restrict our attention to the three structural components which we have described ; for the general theory, they seem to be the only ones to whose investigation substantial contributions have been made. The coarse structure of a compact connected semigroup with identity may be indicated in a diagramm which is supposed to reflect the $\mathcal{D}$-class partial order and thereby the ideal order of the semigroup.

Paragraph 3. - We now analyse the present state of knowledge about the three structural components in a compact connected semigroup with identity with the use of a table :

* \ ?	What is the structure of * ?	How is * embedded in S ?
The minimal ideal	Completely known (1st fundamental theorem). Tools : The algebraic structure theory of completely simple semigroups.	Completely known (1st fundamental theorem). Tools : Cohomology of compact spaces.
The group of units	Completely known (Classical theory of compact groups, unitary representation and Lie groups).	Rather well known although not in final form (Peripherality theorems). Tools : Cohomology, topology, transformation groups.
The irreducible semigroups	Completely known (2nd fundamental theorem). Tools : Transformation groups, general topological abelian compact groups, characters.	Very poorly understood.
S itself	Unreasonable question.	

Paragraph 4. - The first fundamental theorem :

(a) Let S be a compact semigroup. Then the minimal ideal is (isomorphic to) a paragroup (see 1.5). In particular, it is homeomorphic to a product space $X \times G \times Y$ with a compact group. If S is connected, all spaces X, G, Y are connected. If S has an identity and is connected, then X and Y have the cohomology of a point (they are, in fact, contractible under a homotopy whose parameter ranges through a compact connected space instead of the unit interval).

(b) If $i : H(e) \longrightarrow M(S)$ is the inclusion of a maximal group in the minimal ideal $(H(e) \cong G)$ and $j : M(S) \longrightarrow S$ is the inclusion map, then

$$H(i) : H^*(M(S)) \longrightarrow H^*(H(e)) \text{ and } H^*(j) : H^*(S) \longrightarrow H^*(M(S))$$

are isomorphisms for ALEXANDER-SPANIER-ČECH cohomology over any coefficient groups if S is connected and has an identity. Thus the minimal ideal, in fact the groups in the minimal ideal, carry all the topological cohomology of the whole semigroup (WALLACE, up to 1956).

Paragraph 5. - The second fundamental theorem.

All irreducible compact semigroups are abelian. Their identity is the only unit. The quotient semigroup $S/\mathcal{H}$ is a totally ordered compact connected semigroup with

1 and 0 as endpoints. The structure of an irreducible semigroup can in fact be very accurately described in terms of the chaining of cylindrical semigroups. The totally ordered lattice X of idempotents in S is isomorphic to that of $S/\mathcal{H}$; there is a unique irreducible compact semigroup $\mathrm{Irr}(X)$ depending only on X such that for every irreducible semigroup S whose lattice of idempotents is isomorphic to X there is a surjective morphism $\varphi : \mathrm{Irr}\, X \longrightarrow S$ such that the following diagramm commutes

$$\begin{array}{ccc} \mathrm{Irr}\, X & \longrightarrow & (\mathrm{Irr}\, X)/\mathcal{H} \\ \varphi \downarrow & & \downarrow \cong \\ S & \longrightarrow & S/\mathcal{H} \end{array}$$ (HOFMANN and MOSTERT 1965, BOREL 1965).

Paragraph 6. - Peripherality theorems.

In order to appreciate the peripherality theorems one has to understand the difficulty of defining the concept of peripherality. It is not at all obvious what it should mean that a point in a compact connected space be peripheral (i. e. a sort of boundary point) if the space is not embedded in the larger space. Here are two definitions :

6.1 (MOSTERT and HOFMANN 1965, MOSTERT and SHIELDS 1959). - A point s in a compact space S is called peripheral if the following condition is satisfied : For each neighborhood T of s there is a neighborhood $U \subset T$ of s such that for any continuous map $F : C \times X \longrightarrow U$ such that C is compact connected, X is compact, $s \notin F(\{a\} \times X)$, $F(\{b\} \times X) = \{u\} \subset U$ for some $a , b \in C$, there is a continuous map $F' : C' \times X \longrightarrow C$ such that C' is compact connected,

$$F'(a' , x) = F(a , x) \quad \text{for all} \quad x \in X$$

and

$F'(\{b'\} \times X) = \{t\} \subset T$ and $s \notin F'(C' \times X)$, for suitable points $a' , b' \in C'$.

It is clear that one has to sit down and draw a few pictures to vizualize what this definition means e. g. for a point on the boundary of a half plane. Similar concepts have been used in topology under the notation of "homotopical stability".

6.2 (WALLACE 1955). - A point s in a compact space is called peripheral if it has a basis of open neighborhoods U such that the inclusion $i : X \setminus U \longrightarrow X$ induces an isomorphism $H^*(i) : H^*(X) \longrightarrow H^*(X \setminus U)$.

This definition is cleaner in its formulation, but there are some doubts whether or not it is geometrically more intuitive than the other one. In any event, we have the following theorem :

6.3. - In a compact connected semigroup any unit is peripheral (relative to both definitions).

There are other concepts of peripherality which lead to the same result. Forerunners of 6.3 are theorems by WALLACE 1953, 1956, by MOSTERT and SHIELDS 1959. The present version is due to HOFMANN and MOSTERT 1965. For the peripherality in the sense of 6.2, the proof requires the second fundamental theorem. There are some consequences of the peripherality theorems :

(i) A compact connected semigroup with identity on a manifold is a Lie group. This result has two generalizations :

(ii) A compact connected finite dimensional homogeneous semigroup with identity is a group (HUDSON and MOSTERT 1963). This theorem is not proved by peripherality methods although it would seem feasible and desireable to find such a proof.

(iii) A compact connected semigroup with identity on a limit $\lim X_i$ of manifolds is a group if for arbitrarily large indices i the map $\pi_i : X \longrightarrow X_i$ induces a non-zero homomorphism

$$H^n(\pi_i) : H^n(X) \longrightarrow H^n(X_i) \text{ with } n = \dim X_i \ .$$

All spaces underlying a compact connected group are such limit manifolds.

PROBLEM. - If S is homeomorphic to the quotient space of a compact group modulo a closed subgroup, and is a connected semigroup with identity, is it a group ?

A special case of (iii) has been announced by SELDEN 1961.

The following theorem also belongs to the context if the peripheral location of the group of units ; example 1.6 shows how utterly it fails for the minimal ideal in place of the group of units. The result for separable semigroups is due to ANDERSON and HUNTER (1963).

6.4. - The wedge theorem. - If S is a compact connected semigroup with identity then there is a non-degenerate continuum T with $T \cap H(1) = 1$.

By the second fundamental theorem one can take T to be a semigroup.

Paragraph 7. - The question about the nature of the embedding of irreducible semigroups is still wide open. The following list of problems indicates what sort of issues are at stake.

7.1. The general centralizing problem. - The whole second fundamental theorem rests on the fact that for any compact connected abelian group $G \subset H(1)$ of units the connected component of 1 in the centralizer

$$\{s \in S : sg = gs \text{ for all } g \in G\}$$

meets the minimal ideal in a compact connected semigroup.

Is it true that the centralizer of $H(1)$ has an identity component meeting $M(S)$?

7.2. The special centralizing problem. - Suppose that, in 7.1, S has only two idempotents 0 and 1. Is the conjecture true in that case ? Since one knows that all irreducible subsemigroups in that case are one parameter semigroups (i. e. homomorphic images of the real unit interval under ordinary multiplication) one may just as well ask : Is there a non-trivial one parameter semigroup in the centralizer of $H(1)$?

This is true if $S/\mathcal{D}$ is totally ordered (which is still very hard to prove in that case) ; it is also true if a whole neighborhood of $H(1)$ can be continuously and bijectively mapped onto a subset of a Lie group under preservation of multiplication (which is moderately hard to prove). Other special cases are known in which the conjecture is true. But how bad our knowledge of the embedding issue is may be indicated by the following problem.

7.3. The regular $\mathcal{D}$-class bumping problem. - Let S be a compact connected semigroup with identity and 0, and suppose that $S/\mathcal{D}$ is totally ordered. Suppose that there is a one parameter semigroup from 1 to 0. Are there other idempotents except 1 and 0, in other words can a one parameter semigroup bump through a regular $\mathcal{D}$-class without getting caught ? This is impossible if the regular $\mathcal{D}$-class is a paragroup. But otherwise the answer is not known.

7.4. The $\mathcal{H}$-class absorbing problem. - Let S be a compact connected semigroup with identity. Is there an irreducible semigroup such that the union of all $\mathcal{H}$-classes which it meets is a subsemigroup ?

The structure of such a subsemigroup would be well known and closely related to the structure of the irreducible semigroup itself. If the answer to 7.1 were positive, the answer to 7.4 should be yes.

7.5. The $\mathcal{D}$-class absorbing problem. - In 7.4 replace $\mathcal{H}$ by $\mathcal{D}$.

The poor state of a general global theory is indicated by the following two test problems :

7.6. The idempotent skin problem. - Is the following theorem true : If S is a semigroup on an n-cell such that the bounding sphere is exactly the set of idempotents then $n = 1$. One does not know whether the case $n = 2$ is possible or not.

7.7. The 2-sphere problem. - If S is a semigroup on the 2-sphere such that $SS = S$, then either $xy = x$ or $xy = y$ for all $x , y \in S$. Is that so ?

Although the consequence of an answer to these two problems would hardly be earth shaking it is annoying to see that the general theory is insufficient to answer these questions.

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