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SOME WONDERFUL FORMULAE ...
FOOTNOTES TO APERY'S PROOF OF THE IRRATIONALITY OF $\zeta(3)$

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1. In his notorious lecture at Marseille [1], APÉRY reminded his audience of two curious formulae :

$$\zeta(2) = \frac{\pi^2}{6} = \sum_1^\infty \frac{1}{n^2} = 3 \sum_1^\infty \frac{1}{n^2\binom{2n}{n}} \tag{1}$$

$$\zeta(3) = \sum_1^\infty \frac{1}{n^3} = \frac{5}{2} \sum_1^\infty \frac{(-1)^{n-1}}{n^3\binom{2n}{n}} . \tag{2}$$

These expressions can be proved in a quite straightforward way. In fact, writing

$$X_{n,k} = \frac{(-1)^{k-1}(k-1)!^2\, n(n-k-1)!}{(n+k)!} , \quad k < n ,$$

one has

$$X_{n,k} = D_{n,k-1} - D_{n,k} \quad \text{with} \quad D_{n,k} = \frac{(-1)^k\, k!^2(n-k-1)!}{n(n+k)!}$$

$$(-1)^{n-1} X_{n,k} = F_{n,k} - F_{n-1,k} \quad \text{with} \quad F_{n,k} = \frac{1}{2}(-1)^{n+k} \frac{(k-1)!^2(n-k)!}{(n+k)!} = \frac{1}{2}(-1)^{n+k} \frac{1}{k^2\binom{n+k}{k}\binom{n}{k}}$$

$$\frac{1}{n} X_{n,k} = E_{n,k} - E_{n-1,k} \quad \text{with} \quad E_{n,k} = \frac{1}{2}(-1)^k \frac{(k-1)!^2(n-k)!}{k(n+k)!} = \frac{1}{2}(-1)^k \frac{1}{k^3\binom{n+k}{k}\binom{n}{k}}$$

and the formulae readily follow. For details see for example my report on Apéry's proof [5] or the companion lecture to the present one [6]. There remains the question of whether the formulae (1), (2) are isolated curiosities. Numerical experimentation quickly reveals that

$$\zeta(4) = \frac{\pi^4}{90} = \sum_1^\infty \frac{1}{n^4} = C \sum_1^\infty \frac{1}{n^4\binom{2n}{n}} \tag{3}$$

with $C = 2{,}1176470588 \ldots$ and indeed

$$C = 2 + \cfrac{1}{8 + \cfrac{1}{2}} = \frac{36}{17}$$

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whilst in no other case does one obtain a constant that is so apparently a rational number.

Thanks to SHANKS I found in COMTET [3] at p. 89 (only in the English translation, not the French original!) an exercise : show that

$$\sum_1^\infty \frac{1}{\binom{2n}{n}} = \frac{1}{3} + \frac{2\pi\sqrt{3}}{27}, \quad \sum_1^\infty \frac{1}{n\binom{2n}{n}} = \frac{\pi\sqrt{3}}{9}$$

$$\sum_1^\infty \frac{1}{n^2\binom{2n}{n}} = \frac{\pi^2}{18}, \quad \sum_1^\infty \frac{1}{n^4\binom{2n}{n}} = \frac{17\pi^4}{3240}.$$

No hint of a proof seemed to be provided. However it is well known that

$$2(\sin^{-1} x)^2 = \sum_1^\infty \frac{(2x)^{2n}}{n^2\binom{2n}{n}}$$

so the first three formulae follow easily. Moreover one has equally easily the companion formulae

$$\sum_1^\infty \frac{(-1)^{n-1}}{\binom{2n}{n}} = \frac{1}{5} + \frac{4\sqrt{5}}{25}\log\omega, \quad \sum_1^\infty \frac{(-1)^{n-1}}{n\binom{2n}{n}} = \frac{2\sqrt{5}}{5}\log\omega$$

$$\sum_1^\infty \frac{(-1)^{n-1}}{n^2\binom{2n}{n}} = 2\log^2\omega$$

with $\omega = \frac{1}{2}(\sqrt{5}+1)$ (the "golden ratio"). Furthermore, one can write

$$\begin{aligned}\frac{1}{8}\sum_1^\infty \frac{1}{n^4\binom{2n}{n}} &= \int_0^{1/2}\left(\int_0^u (\sin^{-1} x)^2 \frac{dx}{x}\right)\frac{du}{u} \\ &= \left[\log 2u\left(\int_0^u (\sin^{-1} x)^2 \frac{dx}{x}\right)\right]_0^{1/2} - \int_0^{1/2} \log 2u(\sin^{-1} u)^2 \frac{du}{u} \\ &= \left[-\frac{1}{2}\log^2 2u(\sin^{-1} u)^2\right]_0^{1/2} + \int_0^{1/2} \log^2 2u \frac{\sin^{-1} u}{(1-u^2)^{1/2}}\, du .\end{aligned}$$

Thus we see that

$$\sum_1^\infty \frac{1}{n^4\binom{2n}{n}} = 2\int_0^{\pi/3} x \log^2(2\sin\tfrac{1}{2}x)dx = 2I .$$

From the preface to LEWIN's book [4], I knew that the definite integral I has the value

$$I = \frac{17\pi^4}{6480}$$

thereby verifying the experimental result (3).

2. Underlying all the results mentioned there appear the so-called polylogarithms, long neglected, vide LEWIN [4], but now enjoying something of a revival. One defines the dilogarithm by

$$\mathrm{Li}_2\, z = -\int_0^z \log(1-t)\frac{dt}{t}$$

and the higher polylogarithms by

$$\mathrm{Li}_n\, z = \int_0^z \mathrm{Li}_{n-1}\, t \frac{dt}{t}, \quad n > 2 .$$

The subject is a vast compendium of wonderful formulae ; for an introduction see my notes [6]. Here I mention only the following relevant examples :

Integrating

$$\mathrm{Li}_n\, z - \mathrm{Li}_n\, w = \int_w^z \mathrm{Li}_{n-1}\, t \frac{dt}{t}$$

by parts, one readily discovers that in particular

$$\frac{(-1)^{n-1}}{(n-1)!}\int_0^1 \log^{n-1} t \frac{dt}{1-t} = \mathrm{Li}_n\, 1 = \zeta(n) .$$

Furthermore for $0 \leq \theta < \pi$

$$(-1)^{n-1}\int_1^{e^{-i\theta}} \log^{n-1} t \frac{dt}{1-t} = \int_1^{e^{i\theta}} \log^{n-1} t \frac{dt}{1-t} + \int_1^{e^{i\theta}} \log^{n-1} t \frac{dt}{t}$$

which yields the real part, respectively the imaginary part, of the integral on the left according as n is even, respectively odd (of course this is just the observation that all the Bernoulli numbers B_{2n+1}, other than $B_1 = -\frac{1}{2}$, vanish).

On the other hand

$$\int_0^{1-e^{i\theta}} \log^{n-1} t \frac{dt}{1-t} = -i\int_0^{\theta}(\tfrac{1}{2}i(x-\pi) + \log|2\sin\tfrac{1}{2}x|)^{n-1}\, dx$$

and the right hand side yields linear combinations of the so-called logsine integrals

$$\mathrm{Ls}_{n,a}(\theta) = \int_0^{\theta} x^a \log^{n-a-1}|2\sin\tfrac{1}{2}x|\, dx ,$$

were we to choose $\theta = \frac{\pi}{3}$ then cleverly $e^{-i\theta} = 1 - e^{i\theta}$ and we obtain <u>inter alia</u> on taking appropriate real or imaginary parts

$$\int_0^{\pi/3}(-\tfrac{1}{4}(x-\pi)^2 + \log^2(2\sin\tfrac{1}{2}x))\, dx = \frac{1}{3!}(\frac{\pi}{3})^3 ,$$

$$\frac{1}{6}\int_0^{\pi/3}(\tfrac{1}{8}(x-\pi)^3 - \tfrac{3}{2}(x-\pi)\log^2(2\sin\tfrac{1}{2}x))\, dx = \frac{1}{2.4!}(\frac{\pi}{3})^4 + \frac{\pi^4}{90} .$$

From these results, we can disentangle the truly wonderful formulae :

$$\mathrm{Ls}_3(\frac{\pi}{3}) = \int_0^{\pi/3} \log^2(2\sin\tfrac{1}{2}x)\, dx = \frac{7\pi^3}{108}$$

$$\mathrm{Ls}_{4,1}(\frac{\pi}{3}) = \int_0^{\pi/3} x\log^2(2\sin\tfrac{1}{2}x)\, dx = \frac{17\pi^4}{6480} .$$

In the same spirit one sees for example that $2\mathrm{Ls}_5(\pi/3) - 3\mathrm{Ls}_{5,2}(\pi/3)$ is a rational multiple of π^5. However numerical experimentation (in which I was kindly assisted by Dennis PAYNE) has convinced me that in no cases, other than those cited, does one obtain a tidy simple closed expression for logsine integrals at $\pi/3$. This begins to explain why there are no further formulae of the shape of those cited in section 1 above.

In the manner described, we can similarly obtain a "logsinh" integral

$$\sum_1^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}} = -2\int_0^{\log\omega^2} x\log(2\sinh\tfrac{1}{2}x)\, dx = -2I .$$

A careful evaluation in which one eliminates dilogarithms by applying some straightforward functional equations, for details see [6], yields

$$I = -\frac{2}{3}\log^3 \omega + \frac{2\pi^2}{15}\log \omega + \mathrm{Li}_3\, \omega^{-2} - \zeta(3) .$$

On the other hand, LEWIN [4], p. 139, reports the celebrated identity

$$(4) \qquad 0 = \frac{2}{3}\log^3 \omega - \frac{2\pi^2}{15}\log \omega - \mathrm{Li}_3\, \omega^{-2} + \frac{4}{5}\zeta(3)$$

and we obtain the formula (2) as being equivalent to that identity. Thus

$$\int_0^{\log \omega^2} x \log(2 \sinh \tfrac{1}{2} x)\, dx = -\frac{1}{5}\zeta(3) .$$

LEWIN mentions the equivalent formula :

$$\zeta(3) = 10 \int_0^{\log \omega} x^2 \coth x \, dx .$$

I have not been able to find any ingenious method, analogous to that applicable to the logsine integrals, for evaluating the logsinh integrals. Incidentally, (4) is obtained by substituting $z = \omega^{-2}$ in the functional equation

$$(5) \qquad \begin{aligned} &\mathrm{Li}_3\left(-\frac{z}{1-z}\right) + \mathrm{Li}_3(1-z) + \mathrm{Li}_3(z) \\ &\quad = \frac{1}{6}\log^3(1-z) - \frac{1}{2}\log z \log^2(1-z) + \frac{\pi^2}{6}\log(1-z) + \mathrm{Li}_3\, 1 , \end{aligned}$$

noting that

$$\frac{1}{4}\mathrm{Li}_3\, z^2 = \mathrm{Li}_3\, z + \mathrm{Li}_3(-z) .$$

The absence of so simple a functional equation as (5) for higher polylogarithms, again begins to explain the absence of further formulae of the shape of those reported in section 1.

3. We now realise that APÉRY's proof of the irrationality of $\zeta(3)$ is no more than the observation that

$$(6) \qquad \zeta(3) = 6 \sum_1^\infty \frac{1}{n^3\, b_n\, b_{n-1}}$$

with

$$b_n = \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2 ,$$

together with

$$(7) \qquad 6 \sum_{m=1}^n \frac{1}{m^3\, b_m\, b_{m-1}} = \frac{p_n}{q_n}$$

for integers p_n , q_n with $q_n = [1 , 2 , \ldots , n]^3\, b_n$. Then the p_n/q_n approximate $\zeta(3)$ too well for it to be rational. BEUKERS [2] has very elegantly noted that (with $P_n(x) = \frac{1}{n!}\frac{d^n}{dx^n} x^n(1-x)^n$, the Legendre polynomials on $(0 , 1)$),

$$I_n = -\frac{1}{2}\int_0^1\int_0^1 \frac{P_n(x)\, P_n(y)\log xy}{1-xy}\, dx\, dy = [1 , 2 , \ldots , n]^{-3}\,(q_n\,\zeta(3) - p_n)$$

and since he can show directly that $|I| < \zeta(3)(1-\sqrt{2})^{-4n}$ this entirely bypasses the difficulties involved in finding the exact formulae (6), (7). These exact formulae arise from the discovery that the sequence $\{b_n\}$ satisfies the recursion

$$(8)\qquad (n+1)^3 u_{n+1} + n^3 u_{n-1} = (34n^3 + 51n^2 + 27n + 5)u_n$$

and that an independent solution $\{a_n\}$, $a_0 = 0$, $a_1 = 6$, is such that $a_n/b_n = p_n/q_n$.

Proving that (8) is satisfied is quite non-trivial (see the ingenious argument of ZAGIER reported in [5]). However ASKEY has remarked to me that b_n is a balanced hypergeometric polynomial evaluated at 1 ,

$$b_n = {}_4F_3\left(\begin{matrix} n+1 , & -n , & n+1 , & -n \\ 1 , & 1 , & 1 & \end{matrix} ; 1\right) ;$$

here, if $(\alpha)_m = \alpha(\alpha+1) \ldots (\alpha+m-1)$, we write

$${}_pF_q\left(\begin{matrix} \alpha_1 , \ldots , \alpha_p \\ \beta_1 , \ldots , \beta_q \end{matrix} ; z\right) = \sum_{m\geq 0} \frac{(\alpha_1)_m \ldots (\alpha_p)_m}{(\beta_1)_m \ldots (\beta_q)_m} \times \frac{z^m}{m!} .$$

This viewpoint encourages one to consider the three-term linear relations that connect contiguous expressions

$${}_4F_3\left(\begin{matrix} a ; & b ; & c ; & d \\ e ; & f ; & g ; & \end{matrix} ; 1\right) \quad \text{with} \quad a + b + c + d = e + f + g - 1$$

as two of the parameters change by 1 in such a manner as to maintain the balancing condition. The recursion (8) now appears as a very degenerate case of a strategic combination of these contiguity relations. This is a nice instance of generalisation providing a simplification. In this same spirit one should attack the problem of proving that BEUKERS' integral satisfies (8) by considering the integrals

$$\int_0^1 \int_0^1 \frac{P_n(x)\, P_m(y) \log xy}{1 - xy}\, dx\, dy$$

together with the known recurrence relations for the Legendre polynomials. I conclude these rather sparse remarks by mentioning that details of the ${}_4F_3$ relations are given by WILSON in his thesis [7].

4. I now turn to what I believe to be the real mathematics underlying APÉRY's proof. Consider the differential equation

$$(x^4 - 34x^3 + x^2)\frac{d^4 y}{dx^4} + (10x^3 - 255x^2 + 5x)\frac{d^3 y}{dx^3}$$
$$+ (25x^2 - 418x + 4)\frac{d^2 y}{dx^2} + (15x - 117)\frac{dy}{dx} + y = 0 .$$

From general theory (just consider the leading coefficient) one knows there are two independent solutions $a(x)$, $b(x)$ regular at the origin, and these we may normalise so that, say

$$a(x) = 6x + a_2 x^2 + a_3 x^3 + \ldots , \quad b(x) = 1 + 5x + b_2 x^2 + \ldots$$

One now notices that it happens to happen that the b_n all are integers and that $[1, 2, \ldots, n]^3 a_n$ is always integral! Denote by $\alpha' = (1 - \sqrt{2})^4$ the smaller zero of $x^2 - 34x + 1$. Then the general theory implies that the limit

$$\lim_{x\to\alpha'} \frac{a(x)}{b(x)} = \lambda$$

indeed exists. Moreover, the series

$$c(x) = \sum c_n x^n = a(x) - \lambda b(x)$$

is regular for $|x| \leqslant \alpha'$ and thus converges for x with $|x| < \alpha = (1 + \sqrt{2})^4$. It follows that

$$|a_n - \lambda b_n|^{1/n} \to (1 + \sqrt{2})^{-4} ,$$

so the a_n/b_n $(= p_n/q_n)$ are excellent rational approximations to λ. Finally, it happens that $\lambda = \zeta(3)$, which is therefore irrational.

A simpler example (shown to me by BEUKERS, independently of APÉRY's proof) is the case

$$b(x) = \sum b_n x^n = (1 - 6x + x^2)^{-1/2}$$

(which has $b_n = \sum_k \binom{n}{k}\binom{n+k}{k}$). Writing

$$a(x) = \sum a_n x^n = 2(1 - 6x + x^2)^{-1/2} \int_0^x (1 - 6t + t^2)^{-1/2}\, dt$$

one notices that there is a constant λ, namely

$$\lambda = 2 \int_0^{\alpha'} (1 - 6t + t^2)^{-1/2}\, dt = \log 2 \qquad (\alpha' = (1 - \sqrt{2})^2)$$

so that $a(x) - \lambda b(x)$ is regular for $|x| < (1 + \sqrt{2})^2$; this yields good irrationality measures for $\log 2$, and the idea is readily generalised to do the same thing for $\log(1 + \frac{1}{m})$, $m = 1, 2, \ldots$. One sees that the linear homogeneous equation that gives rise to $\log 2$ is

$$(x^2 - 6x + 1) \frac{d^2 y}{dx^2} + (3x - 9) \frac{dy}{dx} + y = 0 .$$

It is plain that the approach just sketched is readily generalised ; one may obtain irrationality measures for the numbers λ that so arise in suitable circumstances. In particular, it is important that the related solutions $a(x)$ and $b(x)$ of the differential equation be G-functions.

5. DWORK has made the following remark : Consider a linear homogeneous differential equation

(9) $$a_s(x) \frac{d^s y}{dx^s} + \ldots + a_0(x)\, y = 0 , \quad a_i(x) \in \mathbb{Z}[X] ,$$

and suppose that the equation has a complete set of solutions regular at $x = 0$. Let $c(x) = \sum c_n x^n$ be such a solution with rational coefficients c_n, and denote by d_n the lowest common multiple of the denominators of the rational numbers $c_0, \ldots, c_n$. Then

$$A \leqslant \limsup \frac{\log d_n}{n} \leqslant A + s - 1$$

where

$$A = \sum_p \max(0 , - \log r_p) ,$$

with r_p the p-adic radius of convergence of the series $c(x)$, and the sum is over all primes p . We have G-function solutions exactly when A is finite.

I should add that this observation very likely is relevant to the mild controversy that surrounds the appropriate definition of G- and E-functions. I join LANG in suggesting the following : suppose we have a solution $c(x)$ of (9) with

$$c(x) = \sum c_n x^n \quad \text{or} \quad c(x) = \sum c_n \frac{x^n}{n!}$$

and that the common denominator d_n is such that $d_n = O(n^{\varepsilon n})$ all $\varepsilon > 0$; then there is already an A' such that $|d_n|^{1/n} < A'$; one should be able to make similar remarks about the size of the c_n themselves ; moreover everything should generalise the case when things are defined over an algebraic number field.

The point of view of the last two sections is being studied by my student Greg GRIFFITHS.

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