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VALUES OF MEROMORPHIC FUNCTIONS OF ORDER 2

by Gregory V. CHOODNOVSKY (*)

Résumé. - Soit $f(z)$ une fonction méromorphe sur $\mathbb{C}^n$, transcendante et d'ordre fini ρ. On étudie l'ensemble $S_{\bar{\mathbb{Q}}}$ des points de $\bar{\mathbb{Q}}^n$, où f, ainsi que toutes ses dérivées, prennent des valeurs entières. Il est contenu dans une hypersurface. Si $\rho < 2$ (ou si $n = 1$), on obtient de bonnes majorations pour le degré de cette hypersurface.

0. Introduction.

We are here interested in the arithmetic nature of the values of meromorphic functions and their derivatives, for functions of arbitrary finite order. Functions of order < 2 play a special role.

Given an arbitrary function f, analytic in the neighbourhood of a point $w \in \mathbb{C}$, we call the point w

algebraic, if $w \in \bar{\mathbb{Q}}$, and all the derivatives $f^{(k)}(w)$ are algebraic, $f^{(k)}(w) \in \bar{\mathbb{Q}}$, for all $k \geq 0$,

algebraic in the weak sense (or rational), if $w \in \bar{\mathbb{Q}}$ and $f^{(k)}(w) \in \bar{\mathbb{Q}}$, for all $k \geq 0$,

algebraic in the weakest sense (or integral), if $w \in \bar{\mathbb{Q}}$ and $f^{(k)}(w) \in \mathbb{Z}$, for all $k \geq 0$.

A general problem for arbitrary meromorphic functions of finite order can be formulated as follows.

PROBLEM 0. - *Suppose f is a transcendental function, meromorphic of finite order $\leq \rho$. Is the set of rational points associated to f finite ? If so, can one obtain a upper bound for its cardinality only in term of ρ ?*

There are important conjectures of BOMBIERI and WALDSCHMIDT on the number of algebraic points associated to f. According to the conjecture in [12], there are at most ρ such points. Unfortunately, this is not the case in general, as many examples show for $\rho < 1$. Simply consider

$$f(z) = \sum_{n=0}^{+\infty} \frac{a_n}{n!} z^n ,$$

with $a_n \in \mathbb{Q}$, $|a_n| \leq (n!)^{\rho-1}$, $\rho < 1$.

In this paper, we shall first describe the situation for algebraic points, and

(*) Cet exposé, fait dans le cadre du Séminaire Delange-Pisot-Poitou, a été prononcé à Orsay, en octobre 1977.

prove some results on integral points both in one dimensional and multidimensional situations. We shall also mention diophantine estimates concerning the approximation of integral points of functions of order < 2 by algebraic numbers.

Nothing is known on the algebraic values of f itself, except in the special case when f satisfies on algebraic differential equation, where we have, as a consequence of the SCHNEIDER-LANG theorem, the following proposition.

PROPOSITION 1 ([5], [8]). - Let $f_1 , \dots , f_n$ be meromorphic functions such that the derivative operator d/dz maps $\mathbb{Z}[f_1 , \dots , f_n]$ into itself. If f_1 is a transcendental function of order $\leq \rho$, and K denotes a number field, there are at most $[K : \mathbb{Q}]\rho$ points w in K such that $f_i(w) \in K$, for $i = 1 , \dots , n$.

There is one particular case in transcendence theory where it is known that an entire function admits only one algebraic point. It is the case of E-functions (their order is ≤ 1) which satisfy a linear differential equation : 0 is their only algebraic point.

Below, we generalize some of SIEGEL's results on E-functions by showing that a transcendental function, which is meromorphic of order < 2 , has at most one integral point. Moreover, if this function satisfies a polynomial differential equation, the same conclusion holds for its rational points.

THEOREM 2. - Let f denote a meromorphic transcendental function of order < 2 . There is at most one point w in $\overline{\mathbb{Q}}$ such that $f^{(k)}(w) \in \mathbb{Z}$, for all $k \in \mathbb{N}$.

In fact, we could even suppose, as we whall see later on, that, for all integers k , $f^{(k)}(w)$ is a rational number whose denominator divides C^k , for some integer C .

THEOREM 3. - Let f denote a meromorphic transcendental function of order < 2 . Suppose further that f satisfies a polynomial differential equation. Then, there is at most one point w in $\overline{\mathbb{Q}}$ such that $f^{(k)}(w) \in \mathbb{Q}$, for all $k \in \mathbb{N}$.

We now study the set of complex numbers which are simultaneously algebraic points for two algebraically independant functions. By the SCHNEIDER-LANG theorem, we have the following proposition.

PROPOSITION 4 ([5], [8]). - Let f_1 , f_2 be two algebraically independant meromorphic functions, of order $\leq \rho_1$, ρ_2 respectively. Suppose they satisfy a system of algebraic differential equations over $\mathbb{Q}$

$$P_i(f_i' , f_1 , f_2) = 0 \quad (i = 1 , 2) .$$

If K denotes a number field, the number of complex numbers w such that $f_i^{(k)}(w) \in K$, for all $k \geq 0$, $i = 1 , 2$, is bounded by $[K : \mathbb{Q}](\rho_1 + \rho_2)$.

If one of the functions satisfy a law of addition, the method of conjugate functions described below enables one to prove the following stronger statement.

THEOREM 5. - Suppose all the assumptions of proposition 4 are satisfied, and, further, that f_1 satisfies a law of addition over $\mathbb{Q}$. Then, there are at most $\rho_1 + \rho_2$ complex numbers w such that $f_1^{(k)}(w) \in \tilde{\mathbb{Q}}$, $f_2^{(k)}(w) \in \mathbb{Z}$, for all $k \geq 0$.

Consequently, if f is a meromorphic function of order $\leq \rho$, such that $f(z)$ and $\exp z$ are algebraically independent, there are at most $\rho + 1$ algebraic numbers α such that $f^{(k)}(\log \alpha) \in \mathbb{Z}$, for all $k \geq 0$. Similarly, if $\wp(z)$ denotes a Weierstrass elliptic function with rational invariants, and if f and $\wp$ are algebraically independent, there are at most $\rho + 2$ points u such that $\wp(u) \in \bar{\mathbb{Q}}$ and $f^{(k)}(u) \in \mathbb{Z}$, for all $k \geq 0$.

Of course, it is impossible to prove transcendence results for functions of order ≥ 2 , when we know the existence of only one algebraic point : Consider, for instance, the function $f(z) = \exp(z(z-1) \dots (z-n+1))$, which has order n , and n integral points. Thus, for functions of order $< n$, we have to assume the existence of $n-1$ integral points. The proof of the corresponding transcendence result does not follow from former methods, and new ideas are needed. Indeed, in the theorem of SCHNEIDER-LANG ([5], [8]), STRAUS [10] and others (see e. g. [1]), the dependence on the degree of integral points is essential.

We avoid this difficulty by introducing a new type of argument. From the usual auxiliary function

$$F(z) = P(z , f(z))$$

constructed by SIEGEL's lemma with zeroes of high multiplicity at some integral points $w_1 , \dots , w_h$, we pass to functions of the form

$$F_j(z) = P(z + \lambda_j , f(z)) ,$$

for some set $\{\lambda_j\}$ of elements of the number field $\mathbb{Q}(w_1 , \dots , w_h)$. These functions also have zeroes of high multiplicities at the points $w_1 , \dots , w_h$. A system of inequalities connecting the multiplicities of zeroes of the different auxiliary functions provides an upper bound for h .

We shall now prove a general result in this direction.

1. A general theorem on integral points.

THEOREM 6. - Let $f(z)$ be a meromorphic transcendental function of order $\leq \rho$. Then, there are at most ρ algebraic points $w \in \bar{\mathbb{Q}}$ such that $f(z)$ is analytic at $z = w$, and $f^{(k)}(w) \in \mathbb{Z}$, for all $k \geq 0$.

There is now a report of E. REYSSAT [7] devoted to the exposition of the proof of this result of mine. However I want to present another version of the proof. This variant can be considered as the refinement of the usual proof of the STRAUS-SCHNEIDER theorem.

Let $S = \{w_1 , \dots , w_n\}$ be a set of integral points of $f(z)$ and $n > \rho$. Then, for some Galois field K, containing S, we have

(1.1) $f(z)$ is analytic in a neighbourhood of w_i, $i = 1 , \dots , n$;

(1.2) K is a Galois field with Galois group G, and

$$[K : \underset{\sim}{Q}] = |G| = d ;$$

(1.3) $f^{(k)}(w_i) \in \underset{\sim}{Z}$, for any $k \geq 0$, $i = 1 , \dots , n$;

(1.4) $w_i \in K$, $i = 1 , \dots , n$.

As in the ordinary proof, we consider an auxiliary function of the form

$$F(z) = P(z , f(z)) ,$$

where $P(x , y) \in \underset{\sim}{Z}[x , y]$, $\deg_x(P) \leq L_1$, $\deg_y(P) \leq L_2$. We consider a parameter L sufficiently large with respect to $(n - \rho)^{-1}$, n, d, $\max H(w_i)$ ... Then by SIEGEL's lemma, we can find a non-zero polynomial $P(x , y) \in \underset{\sim}{Z}[x , y]$, $\deg_x(P) \leq L_1$, $\deg_y(P) \leq L_2$ such that

(1.5) $$L_1 = [L(\log L)^{-(1/4)}] , \quad L_2 = [(\log L)^{3/4}] .$$

Then

(1.6) $$H(P) \leq \exp(C_1 \, L(\log L_1)^{1/2}) ,$$

for $C_1 = C_1(d , n) > 0$, and, for the auxiliary function

(1.7) $$F(z) = P(z , f(z)) ,$$

we have

(1.8) $$F^{(k)}(w_i) = 0 ,$$

for any $k = 0 , 1 , \dots , L - 1$, and $i = 1 , \dots , n$.

Now together with the already constructed function $F(z)$, we shall consider a system of auxiliary functions of the form

$$F_j(z) = P(z + \lambda_j , f(z)) ,$$

for a special system $\{\lambda_j\}$ of algebraic numbers in K.

Let $\underset{\sim}{Z}[G]$ be a group ring of G, and $\underset{\sim}{Z}_0[G]$ be an ideal in $\underset{\sim}{Z}[G]$ of elements of zero trace

(1.9) $$\underset{\sim}{Z}_0[G] = \{\alpha = \textstyle\sum_{g\in G} n_g \, g : \sum_{g\in G} n_g = 0\} .$$

Our main objet is the ring $\mathfrak{J}_0 = \underset{\sim}{Z}_0[G]^n$. Let

(1.10) $$\mathfrak{J}_0 = \underset{\sim}{Z}_0[G]^n , \quad C = G \setminus \{1\} , \quad J = C \times \{1 , \dots , n\} = C \times n .$$

There exists an isomorphism between $\mathfrak{J}_0 = \underset{\sim}{Z}_0[G]^n$ and $\underset{\sim}{Z}^J = \underset{\sim}{Z}^{C\times n}$:

(1.11) $$v : \underset{\sim}{Z}_0[G]^n \longrightarrow Z^J .$$

We define v explicitly : If $\mathfrak{A} = (\theta_1 , \dots , \theta_n) \in \mathfrak{J}_0$, and

$$\theta_i = \sum_{g\in G} n(g\ ,\ i)g \in \underset{\sim}{Z}_0[G]\ ,\quad i = 1\ ,\ \dots\ ,\ n\ ,$$

then we set

(1.12) $$v(\mathfrak{A}) = (n(g\ ,\ i)\ ;\ \ g \in G \setminus \{1\}\ ,\ \ i = 1\ ,\ \dots\ ,\ n) = (n(g\ ,\ i)\ ;\ \ (g\ ,\ i) \in J) \in \underset{\sim}{Z}^J\ .$$

Now we define the set $\{\lambda_j\}$ of elements of K. Let $\mathfrak{A} = (\theta_1\ ,\ \dots\ ,\ \theta_n) \in \mathfrak{J}_0$, and $\theta_i = \sum_{g\in G} n(g\ ,\ i)g \in \underset{\sim}{Z}_0[G]$, $i = 1\ ,\ \dots\ ,\ n$. Then, we define

(1.13) $$\lambda(\mathfrak{A}) = \sum_{i=1}^{n} \sum_{g\in G} n(g\ ,\ i)w_i^{(g)} \in K\ .$$

By the isomorphism v (1.11), we transfer the definition of $\lambda(\vec{n})$ to all the elements $\vec{n} \in \underset{\sim}{Z}^J$

(1.14) $$\lambda(\vec{n}) = \lambda(v^{-1}(\vec{n})) \in K\ ,\quad \vec{n} \in \underset{\sim}{Z}^J\ .$$

Now the following basic property of the sequence $\{\lambda(\mathfrak{A})\ ;\ \mathfrak{A} \in \mathfrak{J}_0\}$ can be easily shown. Any number conjugate to $w_i + \lambda(\mathfrak{A})$, and $\mathfrak{A} \in \mathfrak{J}_0$, $i = 1\ ,\ \dots\ ,\ n$ also, has the form $w_i + \lambda(\mathfrak{A}_1)$, for some $\mathfrak{A}_1 \in \mathfrak{J}_0$.

In fact, let $i = 1\ ,\ \dots\ ,\ n$, $g \in G$, and $\mathfrak{A} = (\theta_1\ ,\ \dots\ ,\ \theta_n) \in \mathfrak{J}_0$, $\theta_i \in \underset{\sim}{Z}_0[G]$ $(i = 1\ ,\dots,\ n)$. If $g = 1$, then $(w_i + \lambda(\mathfrak{A}))^{(g)} = w_i + \lambda(\mathfrak{A})$.

Suppose that $g \neq 1$, i. e. $g \in C$. We put

(1.15) $$\theta_j^1 = \theta_j\, g\ ,\quad j = 1\ ,\dots,\ n\ ,\quad j \neq i\ ;$$

(1.16) $$\theta_i^1 = \theta_i\, g + g - 1\ .$$

So, we have

(1.17) $$\theta_j^1 \in \underset{\sim}{Z}_0[G]\ ,\quad j = 1\ ,\ \dots\ ,\ n \quad \text{and} \quad \mathfrak{A}_1 = (\theta_1^1\ ,\ \dots\ ,\ \theta_n^1) \in \mathfrak{J}_0\ .$$

Then, as it can be easily verified from (1.13), (1.15) and (1.16), we have

(1.18) $$(w_i + \lambda(\mathfrak{A}))^{(g)} = w_i + \lambda(\mathfrak{A}_1)\ ,$$

where $\mathfrak{A}_1$ is defined in (1.15)-(1.17). If $\underset{\sim}{f}_i$ denotes a vector $(0\ ,\ \dots\ ,\ 0\ ,\ 1\ ,\ 0\ ,\ \dots\ ,\ 0)$ of the length n, having unit coordinate at the i-th place, then $\mathfrak{A}_1$ can be briefly rewritten in the form

$$\mathfrak{A}_1 = \mathfrak{A}g + \underset{\sim}{f}_i(g - 1) \in \mathfrak{J}_0\ ,$$

where $\mathfrak{A}g \in \mathfrak{J}_0$, and $\underset{\sim}{f}_i(g - 1) \in \mathfrak{J}_0$. Then (1.18) is replaced by

(1.19) $$(w_i + \lambda(\mathfrak{A}))^{(g)} = w_i + \lambda(\mathfrak{A}g + \underset{\sim}{f}_i(g - 1))\ .$$

Now, we consider a $(d - 1)n$-dimensional cube $C(M)$ in $\underset{\sim}{Z}^J = \underset{\sim}{Z}^{C\times n}$

$$C(M) = (\underset{\sim}{Z} \cap (-M\ ,\ M))^J\ ,$$

and the functions

(1.20) $$F_{\vec{n}}(z) = P(z + \lambda(\vec{n})\ ,\ f(z)\ ;\ \ \vec{n} \in C(N)\ ,\ \text{or}$$
$$F_{\mathfrak{A}}(z) = P(z + \lambda(\mathfrak{A})\ ,\ f(z))\ ;\ \ \mathfrak{A} \in \mathfrak{J}_0\ ,\ \ v(\mathfrak{A}) \in C(N)\ ,$$

for N sufficiently large with respect to d, n, $(n - \rho)^{-1}$, and L sufficiently large with respect to N.

For $\vec{n} \in \underset{\sim}{Z}^J$ or $\mathfrak{A} \in \mathfrak{J}_0$, we denote by $u_i^{\vec{n}}$ or $u_i^{\mathfrak{A}}$, the smallest $u \geq 0$ such that $F_{\vec{n}}^{(u)}(w_i) \neq 0$ or, correspondingly, $F_{\mathfrak{A}}^{(u)}(w_i) \neq 0$.

By the transcendence of $f(z)$, $u_i^{\vec{n}} < \infty$ and the construction of $P(x, y)$, we have

$$(1.21) \qquad u_i^{\vec{0}} \geq L, \text{ for } i = 1, \ldots, n.$$

Now applying JENSEN's formula, we obtain an analytic inequality : Let $\vec{n} \in C(N)$ and $i = 1, \ldots, n$. Then, for $c_2 \geq 0$,

$$(1.22) \quad \sum_{j=1}^{n} u_j^{\vec{n}} \log u_i^{\vec{n}} \leq \rho(c_2\, u_i^{\vec{n}}(\log u_i^{\vec{n}})^{3/4} + c_2\, L(\log L)^{3/4} + c_2\, L(\log L)^{-(1/4)} \log N + u_i^{\vec{n}} \log u_i^{\vec{n}} - \log|F_{\vec{n}}^{(u_i^{\vec{n}})}(w_i)|).$$

We can connect the numbers $F_{\vec{n}}^{(u_i^{\vec{n}})}(w_i)$ and their conjugates using relation (1.19). From the definition of $F_{\mathfrak{A}}(z)$, it follows that

$$(1.23) \quad (F_{\mathfrak{A}}^{(k)}(w_i))^{(g)} = F_{\mathfrak{A}g+\underset{\sim}{f}_i(g-1)}^{(k)}(w_i), \quad k \geq 0,\ 1, \ldots, n,\ \mathfrak{A} \in \mathfrak{J}_0,$$

and also

$$(1.24) \qquad u_i^{\mathfrak{A}} = u_i^{\mathfrak{A}g+\underset{\sim}{f}_i(g-1)}, \quad i = 1, \ldots, n,\ \mathfrak{A} \in \mathfrak{J}_0.$$

From the product formula, we obtain the following system of inequalities, for $\mathfrak{A} \in \mathfrak{J}_0$, $v(\mathfrak{A}) \in C(N)$, $i = 1, \ldots, n$:

$$(1.25) \quad d(\rho - 1)u_i^{\mathfrak{A}} + c_3\, u_i^{\mathfrak{A}}(\log u_i^{\mathfrak{A}})^{-(1/4)} + \frac{c_3}{\log u_i^{\mathfrak{A}}}(L(\log L)^{3/4} + L(\log L)^{-(1/4)}(\log N)) \geq \sum_{g\in G} \sum_{j=1, j\neq i}^{n} u_j^{\mathfrak{A}g+\underset{\sim}{f}_i(g-1)},$$

where $u_i^{\vec{0}} \geq L$, $i = 1, \ldots, n$. Because $u_j^{\mathfrak{A}g+\underset{\sim}{f}_i(g-1)} = u_j^{\mathfrak{A}+\underset{\sim}{f}_i-\underset{\sim}{f}_j-\underset{\sim}{f}_i g^{-1}+\underset{\sim}{f}_j g^{-1}}$, we have, instead of (1.25),

$$(1.26) \quad d(\rho - 1)u_i^{\mathfrak{A}} + c_3\, u_i^{\mathfrak{A}}(\log u_i^{\mathfrak{A}})^{-(1/4)} + \frac{c_3}{\log u_i^{\mathfrak{A}}}(L(\log L)^{3/4} + L(\log L)^{-(1/4)}(\log N)) \geq \sum_{g\in G} \sum_{j=1, j\neq i}^{n} u_j^{\mathfrak{A}+\underset{\sim}{f}_i-\underset{\sim}{f}_j-\underset{\sim}{f}_i g+\underset{\sim}{f}_j g},$$

for $\mathfrak{A} \in \mathfrak{J}_0$, $v(\mathfrak{A}) \in C(N)$. Now, we apply the mapping v to $\mathfrak{J}_0$, $v : \mathfrak{J} \longrightarrow \underset{\sim}{Z}^J$, and consider the basis vector $\vec{e}(\vec{j}) = \vec{e}(g, i)$ of $\underset{\sim}{Z}^J$, $\vec{j} = (g, i) \in J$. Then

$$v(\mathfrak{A} + \underset{\sim}{f}_i - \underset{\sim}{f}_j - \underset{\sim}{f}_i\, g + \underset{\sim}{f}_j\, g) = v(\mathfrak{A}) - \vec{e}(g, i) + \vec{e}(g, j), \text{ for } g \neq 1,$$

and, instead of (1.26), we obtain a new system, for $i = 1, \ldots, n$, and $\vec{n} \in C(N)$,

$$(1.27)\quad d(\rho - 1)u_i^{\vec{n}} + c_3\, u_i^{\vec{n}}(\log u_i^{\vec{n}})^{-(1/4)} + \frac{c_3}{\log u_i^{\vec{n}}}\,(L(\log L)^{3/4} + L(\log L)^{-(1/4)} \log N) \geqslant \sum_{g\in C} \sum_{j=1, j\neq i}^{n} u_j^{\vec{n}+\vec{e}(g,j)-\vec{e}(g,i)} + \sum_{j=1, j\neq i}^{n} u_j^{\vec{n}} .$$

Because $u_i^{\vec{0}} \geqslant L$, $i = 1, \ldots, n$, we can show, using (1.27), that for

$$(\log L)^{1/4} \geqslant (6c_3\, n^2\, d^2)^{2ndN} ,$$

we have

$$(1.28)\qquad u_i^{\vec{n}} \geqslant (6c_3\, n^2\, d^2)^{-2ndN}\, L$$

provided $\vec{n} \in C(N)$.

Thus from (1.28), we obtain the main system (W) , connecting different $u_k^{\vec{m}}$. We have, for $i = 1, \ldots, n$, $\vec{n} \in C(N)$ and $\log\log L \geqslant c_6 N$:

$$(W)\qquad d(\rho - 1)u_i^{\vec{n}} + c_4\, u_i^{\vec{n}}(\log u_i^{\vec{n}})^{-(1/4)} \geqslant \sum_{j=1, j\neq i}^{n} u_j^{\vec{n}} + \sum_{j=1, j\neq i}^{n} \sum_{g\in C} u_j^{\vec{n}+\vec{e}(g,j)-\vec{e}(g,i)} ,$$

where $u_i^{\vec{n}} \geqslant c_5^{-N} L$, and c_4 , c_5 , c_6 depend only on n , d . In this system (W) , the numbers $u_k^{\vec{m}}$ are always $\geqslant 0$.

The system (W) for positive $u_i^{\vec{n}}$, $i = 1, \ldots, n$, $\vec{n} \in C(N)$ is inconsistent for N , sufficiently large with respect to n , d , $(n - \rho)^{-1}$. The inconsistency of (W) is a consequence of the fact that, in the left side of (W) , we have a constant factor $d(\rho - 1) \leqslant \chi < d(n - 1)$, while, in the right side of (W) , we have $d(n - 1)$ summands.

Thus the system (W) is much more restrictive than the usual scheme of random walk in $\underset{\sim}{Z}^J$. This allows us to use, for instance, the method of generating functions (see [4]) to show the following lemma.

LEMMA 7. - <u>System</u> (W) <u>is inconsistent for non-negative</u> $u_i^{\vec{n}}$, $i = 1, \ldots, n$, $\vec{n} \in C(N)$, <u>when</u> $N \geqslant N_0(n, d, (n - \rho)^{-1})$.

The complete proof of the lemma 7, with good estimates for $N_0(n, d, (n - \rho)^{-1})$ is contained in my preprint [3] with another version of proof of theorem 6. Of course, the paper of E. REYSSAT [7] gives a shorter proof of lemma 7.

Because all $u_i^{\vec{n}}$ are $\geqslant 0$ (as multiplicities of zeroes), it follows from lemma 7 that $n \leqslant \rho$. Thus theorem 6 is proved. Theorems 2 and 3 follow from theorem 6.

2. Various generalizations for one-and multidimensional situations.

In fact, the method of proof of theorem 6 can be easily generalized to the case of two functions $f_1(z)$ and $f_2(z)$, when one of these functions (say $f_1(z)$) satisfies an algebraic law of addition over $\underset{\sim}{Q}$, i. e. $f_1(z) = z$, $\exp z$ or $\wp(z)$, where $\wp(z)$ has rational invariants, see theorem 5 of § 0. The method of conjugate functions, used in the proof of theorem 6, gives also the possibility to generalize all previous results (see [11], [5], [8], [12]) in the case when one of the functions satisfies an algebraic law of addition.

We shall give here one interesting generalization of beautiful results of D. BERTRAND [1] and M. WALDSCHMIDT [11] (only in one-dimensional situation).

In order to state these results, we use the following definition of D. BERTRAND [1] of well-behaved points.

Let $f_1(z)$, $f_2(z)$ be algebraically independent meromorphic functions of orders $\leqslant \rho_1$, ρ_2 . We consider algebraic points w of $\{f_1(z) , f_2(z)\}$ with the following properties :

(i) All the numbers $f_1^{(k)}(w)$, $f_2^{(k)}(w)$, for $k \geqslant 0$, lie in a fixed algebraic number field K_w ; $\delta_w = [K_w : \underset{\sim}{Q}]$;

(ii) The sizes of these algebraic numbers satisfy

$$\limsup_{m\to+\infty} \frac{\log|\overline{f_i^{(m)}(w)}|}{m \log m} \leqslant c_w , \quad i = 1 , 2 ;$$

(iii) For the denominators of these algebraic numbers, we have :

$d_w^{m+1}[d'_w m)!]^{d''_w} f_i^{(m)}(w)$ ($m \in \mathbb{N}$, $i = 1 , 2$ fixed d_w , d'_w , d''_w) are algebraic integers.

If $f_1(z)$ satisfies a law of addition over $\underset{\sim}{Q}$, then we can replace assumption (i) by

(i') $f_1^{(k)}(w)$ are algebraic numbers, $k \geqslant 0$, and the field

$$L_w = \underset{\sim}{Q}(f_2(w) , f_2'(w) , \dots)$$

has finite degree $[L_w : \underset{\sim}{Q}] = \lambda_w$.

DEFINITION 8.

(a) *When properties* (i),(ii),(iii) *are satisfied for an algebraic point* w *of* $\{f_1(z) , f_2(z)\}$, *then* w *is called a well-behaved point of* $\{f_1(z) , f_2(z)\}$.

(b) *If* $f_1(z)$ *satisfies an algebraic law of addition over* $\underset{\sim}{Q}$, *an algebraic point* w *of* $\{f_1(z) , f_2(z)\}$ *satisfying* (i'), (ii), (iii) *is called a well-behaved point of* $\{f_1(z) , f_2(z)\}$.

The one-dimensional result of D. BERTRAND [1] (see M. WALDSCHMIDT [11] for a multidimensional generalization) can be formulated as follows.

THEOREM 9 [1]. - *For algebraically independent* $f_1(z)$, $f_2(z)$, *the following sum* $\sum_w$ *over all well-behaved algebraic points* w *of* $\{f_1(z) , f_2(z)\}$,

$$\sum_w([K_w : \underset{\sim}{Q}]d'_w d''_w + 1 + ([K_w : \underset{\sim}{Q}] - 1)c_w)^{-1} ,$$

converges to a limit not exceeding $\rho_1 + \rho_2$.

However, if we suppose that $f_1(z)$ admits a law of addition over $\mathbb{Q}$ (i. e. $f_1(z) = z$, $\exp z$, $\wp(z)$, where $\wp(z)$ has rational invariants), and we consider well-behaved points in the sense of (i'), (ii), (iii), then we obtain the following theorem.

THEOREM 10. - Assume that the hypotheses of theorem 9 are satisfied, and that $f_1(z)$ satisfies a law of addition over $\mathbb{Q}$. Then as w ranges over all well-behaved points of $\{f_1(z) , f_2(z)\}$, the following sum converges to a limit not exceeding $\rho_1 + \rho_2$

$$\sum_w (\lambda_w d'_w d''_w + 1 + (\lambda_w - 1)c_w)^{-1} \leq \rho_1 + \rho_2 ,$$

where $\lambda_w = [\mathbb{Q}(f_2(w) , f'_2(w) , f''_2(w) , \ldots) : \mathbb{Q}]$.

The proof of theorem 10 differs only in a few points from the proof of theorem 6. For all the details of such kind of proofs, see expositions in ([11], [1]).

Now we shall consider some possible generalizations of theorem 6 for meromorphic functions in $\mathbb{C}^n$ of order < 2 .

The situation in $\mathbb{C}^n$ differs from that of $\mathbb{C}^1$. First, the algebraic points may form not a discrete set, but some subvariety of codimension 1 : For

$$f(z_1 , z_2) = \exp(z_1 - z_2) ,$$

the line $z_1 = z_2$ gives the set of integral points of $f(z_1 , z_2)$.

Nice results of E. BOMBIERI [2] and M. WALDSCHMIDT ([11], [12]) give us estimates for degree of hypersurfaces in $\mathbb{C}^n$, containing all algebraic points of meromorphic transcendental function $f(\underset{\sim}{z})$ in $\mathbb{C}^N$.

We mention in particular the following theorem.

THEOREM 11 ([11], [12]). - Let $f(\underset{\sim}{z})$ be a transcendental meromorphic function of order $\leq \rho$ in $\mathbb{C}^n$, and let K be an algebraic number field. The set S_K of points $\underset{\sim}{w} \in K^n$ such that

$$\partial^{\underset{\sim}{k}} f(\underset{\sim}{w}) \in \mathbb{Z} , \quad \underset{\sim}{k} \in \mathbb{N}^n$$

is contained in an algebraic hypersurface of degree $\leq n\rho[K : \mathbb{Q}]$.

It is natural to propose the following conjecture.

CONJECTURE 12. - In theorem 11, the bound $n\rho[K : \mathbb{Q}]$, for the degree of the hypersurface containing S_K , aan be sharpened to ρ .

Below we give a partial answer to conjecture 12 only for function of order of growth < 1 . Conjecture 12 is unclear for arbitrary ρ . Probably, the most difficult part will be the removing of factor n in product $n\rho[K : \mathbb{Q}]$.

First of all, we reaall M. WALDSCHMIDT's result [12], for $\rho = 1$.

PROPOSITION 13 ([12], [9]). - Let $f(\underset{\sim}{z})$ be an entire transcendental function of order 1 in $\underset{\sim}{C}^n$. Then the set $S_{\underset{\sim}{\bar{Q}}}$ of points $\underset{\sim}{w} \in \underset{\sim}{\bar{Q}}^n$ such that

$$\partial^{\underset{\sim}{k}} f(\underset{\sim}{w}) \in \underset{\sim}{Z} \ \text{for all} \ \underset{\sim}{k} \in \underset{\sim}{N}^n$$

is contained in an algebraic hypersurface in $\underset{\sim}{C}^n$ of degree $\leq n$.

We can considerably improve proposition 13.

THEOREM 14. - Let $f(\underset{\sim}{z})$ be an entire transcendental function of order 1 in $\underset{\sim}{C}^n$. Then, the set $S_{\underset{\sim}{\bar{Q}}}$ of points $\underset{\sim}{w} \in \underset{\sim}{\bar{Q}}^n$ such that

$$\partial^{\underset{\sim}{k}} f(\underset{\sim}{w}) \in \underset{\sim}{Z} \ \text{for all} \ \underset{\sim}{k} \in \underset{\sim}{N}^n$$

is containing in an algebraic hyperplane $\mathfrak{L}$ (of degree 1) in $\underset{\sim}{C}^n$.

Before giving the proof of this result, we shall present some auxiliary results on algebraic functions of one variable having several algebraic (well-behaved) points.

We shall use the following remark : The method of proof of theorems 2 and 10 can be used in the reverse direction. Instead of considering transcendental functions and, then, obtaining the bounds for the number of algebraic points, we can consider meromorphic function having a lot of algebraic points and, then prove that this function is algebraic (and obtain bounds for its degree).

Let's consider e. g. the situation in theorem 9, with $f_1(z) = z$. Let $f(z)$ be an arbitrary meromorphic function of order of growth $\leq \rho$, and W be a finite set, $W \subset \underset{\sim}{\bar{Q}}$, of well-behaved points of $\{z , f(z)\}$ such that

$$\sum_{w\in W} (\delta_w d'_w d''_w + 1 + (\delta_w - 1)c_w)^{-1} > \rho .$$

Then $f(z)$ is algebraic, i. e. satisfy an equation $P(z , f(z)) \equiv 0$. Moreover $\deg(P)$ depends only on W and on the constants of growth of $f(z)$. If $f(z) = \frac{h(z)}{g(z)}$ where $h(z)$, $g(z)$ are entire functions, then

$$\log|h|_R \leq aR^\rho + b , \quad \log|g|_R \leq cR^\rho + d ,$$

where a, b, c, d are the constants of growth of $f(z)$. We shall give a precise result only for entire functions.

PROPOSITION 15. - Let $f(z)$ be an entire function in $\underset{\sim}{C}$ of order of growth $\leq \rho$, and let W be a finite set, $W \subset \underset{\sim}{\bar{Q}}$, of well-behaved points of $\{z , f(z)\}$ in the sense of (i) and (iii), such that

$$\sum_{w\in W} (\delta_w d'_w d''_w + 1 + (\delta_w - 1)c_W)^{-1} > \rho .$$

Then, $f(z)$ is a polynomial. Moreover, if

$$\log|f|_R \leq aR^\rho + b \ \text{for any} \ R > 0 ,$$

and

$$L \log L > C_0(a , W)b ,$$

<u>where</u> $C_0(a , W) > 0$ <u>is a constant depending only on</u> a <u>and</u> W , <u>then</u>

$$\deg(f) \leq L .$$

<u>Proof</u>. - We consider the usual scheme of proof of theorem 9 (see [11], [1] or proof of theorem 6). We take the auxiliary function $F(z)$ in the form

$$F(z) = P(z , f(z)) , \quad P(x , y) \in \mathbb{Z}[x , y] , \quad \deg_x(P) \leq L_1 , \quad \deg_y(P) \leq L_2$$

and

$$L_1 = [LL_2^{-(1/2)}] , \quad L_2 = [L_2] ,$$

L is sufficiently big number and L_2 is a constant depending only on a and W .

By SIEGEL's lemma [8], there exists $P(x , y) \in \mathbb{Z}[x , y]$, $P(x , y) \not\equiv 0$, such that, for $F(z) = P(z , f(z))$, we have

$$F^{(k)}(w) = 0 , \quad k = 0 , 1 , \ldots , L - 1 \quad \text{and} \quad w \in W ,$$

where $\log H(P) \leq L_2^{-(1/3)} L \log L$.

Now we choose L_1 (or L) so that

$$L_1 \log L_1 > c_1 L_2 b , \quad \log L_1 > c_2(a , W) ,$$

for $c_1 = c_1(W) > 0$, depending only on W . Then SCHWARZ lemma together with considerations of D. BERTRAND (see [1]) shows that $F(z) \equiv 0$ or $P(z , f(z)) \equiv 0$.

Thus, $f(z)$ is polynomial and

$$\deg(f).\deg_y(P) \leq \deg_x(P) .$$

In particular,

$$\deg(f) \leq L_1 .$$

Proposition 15 can also be formulated as follows.

PROPOSITION 15'. - <u>Let</u> $f(z)$ <u>be an entire function in</u> $\mathbb{C}$ <u>of order</u> $\leq \rho$ <u>and</u> W , $W \subset \overline{\mathbb{Q}}$, <u>be finite set of well-behaved points of</u> $\{z , f(z)\}$ <u>such that</u>

$$\sum_{w \in W} (\delta_w d'_w d''_w + 1 + (\delta_w - 1)c_w)^{-1} > \rho \geq 1$$

<u>(for definition 8(a)), or</u>

$$\sum_{w \in W} (\lambda_w d'_w d''_w + 1 + (\lambda_w - 1)c_w)^{-1} > \rho \geq 1$$

<u>(for definition 8(b)).</u>

<u>Then</u> $f(z)$ <u>is polynomial. If</u> $\log|f|_R \leq R^\rho$, <u>for</u> $R \geq R_0$, <u>and</u> $L \log L > C_0(W)R_0^\rho$ <u>then</u>

$$\deg(f) \leq L .$$

We shall use proposition 15 or proposition 15' only for well-behaved points in the sense of definition 8(a).

Proof of theorem 14. - Let $f(\underset{\sim}{z})$ be an entire transcendental function in $\underset{\sim}{C}^n$ of order 1, and suppose $S_{\underset{\sim}{Q}} = S_{\underset{\sim}{Q}}(f)$ is not contained in an hyperplane in $\underset{\sim}{C}^n$. Then, there exist $\underset{\sim}{w}_1, \dots, \underset{\sim}{w}_{n+1} \in S_{\underset{\sim}{Q}}(f)$ such that the vectors $\overrightarrow{w_1 w_2}, \dots, \overrightarrow{w_1 w_{n+1}}$ are linearly independent over $\underset{\sim}{C}$.

For any $i, j = 1, \dots, n+1$, $i \neq j$, we consider the line $L_{i,j}$ in $\underset{\sim}{C}^n$, connecting $\underset{\sim}{w}_i$ and $\underset{\sim}{w}_j$. Because $\underset{\sim}{w}_i \in \overline{\underset{\sim}{Q}}^n$, the equation of $L_{i,j}$ can be written in the form

$$L_{i,j} : z_1 = \alpha^1_{i,j} t + \beta^1_{i,j}, \dots, z_n = \alpha^n_{i,j} t + \beta^n_{i,j},$$

where $\alpha^k_{i,j}$, $\beta^k_{i,j}$ are algebraic integers, $k = 1, \dots, n$ and $i, j = 1, \dots, n+1$, $i \neq j$.

We consider the restriction of derivatives $\partial^{\underset{\sim}{k}} f(\underset{\sim}{z})$ of $f(\underset{\sim}{z})$ on the line $L_{i,j}$

$$g^{i,j}_{\underset{\sim}{k}}(t) = \partial^{\underset{\sim}{k}} f(\alpha^1_{ij} t + \beta^1_{ij}, \dots, \alpha^n_{ij} t + \beta^n_{ij}),$$

for $\underset{\sim}{k} \in \underset{\sim}{N}^n$, $i, j = 1, \dots, n+1$, $i \neq j$. The coordinates $t = 0$ and $t = 1$ correspond to $\underset{\sim}{z} = \underset{\sim}{w}_i$ and $\underset{\sim}{z} = \underset{\sim}{w}_j$ on the line $L_{i,j}$, $i, j = 1, \dots, n+1$, $i \neq j$.

LEMMA 16. - <u>For any</u> $\underset{\sim}{k} \in \underset{\sim}{N}^n$, $i, j = 1, \dots, n+1$, $i \neq j$, $g^{i,j}_{\underset{\sim}{k}}(t)$ <u>is a polynomial in</u> t <u>of degree</u> $\leq c_2 |\underset{\sim}{k}|$, <u>for</u> $c_2 = c_2(\underset{\sim}{w}_i, \underset{\sim}{w}_j) > 0$, <u>having algebraic coefficients.</u>

Proof of lemma 16. - For any $\underset{\sim}{k} \in \underset{\sim}{N}^n$, $i, j = 1, \dots, n+1$, $i \neq j$, $g^{i,j}_{\underset{\sim}{k}}(t)$ is an entire function of order 1. If K is a number field such that $\underset{\sim}{w}_i \in K^n$, α^k_{ij}, $\beta^k_{ij} \in K$, $i, j = 1, \dots, n+1$, $k = 1, \dots, n$, then, by definition of $L_{i,j}$ and $g^{i,j}_{\underset{\sim}{k}}(t)$, we have

$$g^{i,j}_{\underset{\sim}{k}}(t)^{(L)} \in K \quad \text{for any } L \geq 0 \text{ at } t = 0, 1,$$

when $\underset{\sim}{k} \in \underset{\sim}{N}^n$, $i, j = 1, \dots, n+1$. Moreover, each of the points $t = o, 1$ is the well-behaved point of $g^{i,j}_{\underset{\sim}{k}}(t)$ such that

$$d' = d'' = 0, \quad \delta = [K : \underset{\sim}{Q}], \quad C = 0,$$

because

$$\partial^{\underset{\sim}{\ell}} f(\underset{\sim}{w}_i) \in \underset{\sim}{Z}, \quad \underset{\sim}{\ell} \in \underset{\sim}{N}^n, \quad i = 1, \dots, n+1,$$

and $f(\underset{\sim}{z})$ has order 1 (see [11], lemma 3). Thus by theorem 9, $g^{i,j}_{\underset{\sim}{k}}(t)$ is a polynomial for any $\underset{\sim}{k} \in \underset{\sim}{N}^n$, $i, j = 1, \dots, n+1$, $i \neq j$. In order to obtain the bound for the degree of $g^{i,j}_{\underset{\sim}{k}}(t)$, we use propositions 15 and 15'.

From Cauchy's integral formula for $f(\underset{\sim}{z})$, we obtain an estimate for $\log|g^{i,j}_{\underset{\sim}{k}}|_R$

$$\log|g^{i,j}_{\underset{\sim}{k}}|_R \ll |\underset{\sim}{k}| \log|\underset{\sim}{k}| + |f|_R .$$

Then propositions 15 and 15' give us $d(g^{i,j}_{\underset{\sim}{k}}) \leq c_2 |\underset{\sim}{k}|$, for $\underset{\sim}{k} \in \underset{\sim}{N}^n$. Finally, since $g^{i,j}_{\underset{\sim}{k}}(t)$ has algebraic derivatives (in K) at $t = 0$, then

$$g_{\underset{\sim}{k}}^{i,j}(t) \in K[t] .$$

This proves lemma 16.

LEMMA 17. - Let $\underset{\sim}{k} \in \underset{\sim}{N}^n$, $i , j = 1 , \ldots , n+1$, $i \neq j$. Then any algebraic point $\underset{\sim}{z}_0 \in \bar{\underset{\sim}{Q}}^n$ on $L_{i,j}$ is a well-behaved point of $f(\underset{\sim}{z})$ in the sense that, for the algebraic number field $K_{\underset{\sim}{z}_0} = K(\underset{\sim}{z}_0)$, we have

1° $\underset{\sim}{z}_0 \in K^n_{\underset{\sim}{z}_0}$, $\partial^{\underset{\sim}{m}} f(\underset{\sim}{z}_0) \in K_{\underset{\sim}{z}_0}$, for all $\underset{\sim}{m} \in \underset{\sim}{N}^n$;

2° $\lim_{|m|\to\infty} (\log|\partial^{\underset{\sim}{m}} f(\underset{\sim}{z}_0)|)/(|\underset{\sim}{m}|\log|\underset{\sim}{m}|) = 0$;

3° There exists $d_1 = d_1(\underset{\sim}{z}_0)$ such that $d_1^{|\underset{\sim}{m}|}(c_3 |\underset{\sim}{m}|)!$ is a denominator of $\partial^{\underset{\sim}{m}} f(\underset{\sim}{z}_0)$ for any $\underset{\sim}{m} \in \underset{\sim}{N}^n$.

This lemma follows immediatly from the previous one and the lemmae of § 3 from [11].

According to theorem 1 (and lemma 7) from [11] for any given δ , the set $S[\delta]$ of points $\underset{\sim}{z}_0 \in \bar{\underset{\sim}{Q}}^n$, satisfying 1°, 2°, 3° with $[K(\underset{\sim}{z}_0) : \underset{\sim}{Q}] \leq \delta$ is contained in a hypersurface of degree $\leq n(c_2 \delta + 1)$. From lemma 17, it follows that, for any $L \supset K$, $[L : \underset{\sim}{Q}] \leq \delta$,

$$\bigcup_{i,j=1,\ldots,n+1,i\neq j} L_{i,j} \cap L^n \subset S[\delta] .$$

Now let $\underset{\approx}{\ell}$ be any line in $\underset{\sim}{C}^n$ connecting $\underset{\sim}{w}_i$ with any element from $S[\delta]$. Then along this line, the function $f(\underset{\sim}{z})$, as well as all its derivatives, is algebraic.

LEMMA 18. - Let $\underset{\sim}{\ell}$ be any line in $\underset{\sim}{C}^n$ containing $\underset{\sim}{w}_i$ and another point $\underset{\sim}{w}' \in S[\delta]$, for $\delta < \infty$, and denote its equation by

$$\underset{\sim}{\ell} : \quad z_i = \alpha_i t + \beta_i , \quad i = 1 , \ldots , n .$$

Then, for any $\underset{\sim}{k} \in \underset{\sim}{N}^n$, the function

$$g_{\underset{\sim}{k}}^{\underset{\sim}{\ell}}(t) = \partial^{\underset{\sim}{k}} f(\alpha_1 t + \beta_1 , \ldots , \alpha_n t + \beta_N)$$

is a polynomial of degree $\leq c_2 |\underset{\sim}{k}|$, having coefficients in the field

$$K[\underset{\sim}{w}' , \alpha_i , \beta_i , \quad i = 1 , \ldots , n] .$$

Proof of lemma 18. - For $\underset{\sim}{k} \in \underset{\sim}{N}^n$, the function $g_{\underset{\sim}{k}}^{\underset{\sim}{\ell}}(t)$ is an entire of order 1 . Because $\underset{\sim}{w}' \in S[\delta]$, i. e. satisfies 1° and 3° of lemma 17, the points $\underset{\sim}{w}_i$ and $\underset{\sim}{w}'$ correspond to well-behaved points of $\{t , g_{\underset{\sim}{k}}^{\underset{\sim}{\ell}}(t)\}$. Let the parameters $t = 0$ and $t = 1$ correspond to $\underset{\sim}{w}_i$ and $\underset{\sim}{w}'$, respectively. Then, we have

$$d_0' = d_0'' = 0 , \quad \delta_0 = [K : \underset{\sim}{Q}] , \quad c_0 = 0 ,$$

$$d_1' = c_2 , \quad d_1'' = 0 , \quad \delta_1 = \delta , \quad c_1 = 0 .$$

By theorem 9, the function $g_{\underset{\sim}{k}}^{\underset{\sim}{\ell}}(t)$ is algebraic. A bound for the degree of $g_{\underset{\sim}{k}}^{\underset{\sim}{\ell}}(t)$

follows immediately from propositions 15 and 15' and Cauchy's formula. It is clear that the coefficients of $g_{\underset{\sim}{k}}^{\ell}(t)$ lie in $K[\underset{\sim}{w}', \alpha_i, \beta_i, i = 1, \dots, n]$ because all the derivatives of $g_{\underset{\sim}{k}}^{\ell}(t)$ at $t = 0$ lie in this field. Lemma 18 is proved.

From lemma 18 and [11], we obtain (cf. lemma 17) the following corollary.

COROLLARY 19. – Let $\underset{\sim}{\ell}$ be any line in $\underset{\sim}{C}^n$, containing $\underset{\sim}{w}_i$ and another point $\underset{\sim}{w}' \in S[\delta]$, for any $\delta < \infty$. Then, any algebraic point $\underset{\sim}{z}_0 \in \bar{\underset{\sim}{Q}}^n$, lying on $\underset{\sim}{\ell}$ satisfy all the conditions 1°, 2°, 3° of lemma 17. In particular, for $L \supset K$, $[L : \underset{\sim}{Q}] \leqslant \delta$,

$$\underset{\sim}{\ell} \cap L^n \subseteq S[\delta] .$$

From this corollary, lemma 17 and the choice of $\underset{\sim}{w}_i$, $i = 1, \dots, n + 1$, it follows that, for $\delta \geqslant [K : \underset{\sim}{Q}]$, the set $S[\delta]$ is not contained in an algebraic hypersurface of any finite degree. This contradicts theorem 1 [11] (see supra). So, $\overrightarrow{w_1 w_2}, \dots, \overrightarrow{w_1 w_{n+1}}$ cannot be lineary independent, and $S_{\bar{\underset{\sim}{Q}}}(f)$ is contained in a hyperplane in $\underset{\sim}{C}^n$.

Using completely different methods, we can prove, for arbitrary meromorphic functions, a much stronger result.

THEOREM 20. – Let $f(\underset{\sim}{z})$ be a meromorphic transcendental function in $\underset{\sim}{C}^n$ of order $\leqslant (n + 1)/n$. Then, the set $S_{\bar{\underset{\sim}{Q}}}(f)$ of $\underset{\sim}{w} \in \bar{\underset{\sim}{Q}}^n$, such that

$$\partial^{\underset{\sim}{k}} f(\underset{\sim}{w}) \in \underset{\sim}{Z}, \text{ for all } \underset{\sim}{k} \in \underset{\sim}{N}^n,$$

is contained in an algebraic hypersurface in $\underset{\sim}{C}^n$ of degree 1 (i. e. in a hyperplane).

The proof of this theorem is exactly the same as for theorem 6. The only difference is contained in an application of SCHWARZ lemma in $\underset{\sim}{C}^n$. We apply a multidimensional SCHWARZ lemma in a very particular form.

LEMMA 21. – Let $S_0 \subset \underset{\sim}{C}^n$ be a set which is not contained in a hyperplane. Then, there exists $S \subset S_0$, $|S| = n + 1$, which is also not contained in a hyperplane in $\underset{\sim}{C}^n$ and such that

(S) For any $\epsilon > 0$, any entire function $F(\underset{\sim}{z})$ in $\underset{\sim}{C}^n$ having at any point $\underset{\sim}{x}_i \in S$ zero of order k_i, $i = 1, \dots, |S| = n + 1$, we have, for $i = 1, \dots, n + 1$, $\underset{\sim}{k} \in \underset{\sim}{N}^n$, $|\underset{\sim}{k}| = k_i$,

$$|\partial^{\underset{\sim}{k}} F(w_i)| \leqslant k_i!\, r^{c \sum_{i=1}^{n+1} k_i} |F|_R\, (R\epsilon/6nr)^{-((1-\epsilon)/n)\sum_{i=1}^{n+1} k_i},$$

for $r \geqslant r_0(S, n, \epsilon) > 0$, $c = c(S, n, \epsilon) > 0$ and $R > (6nr/\epsilon)$.

The proof of this lemma uses lemma 1 [11].

3. Arithmetical nature of numbers at which meromorphic functions have integral rational values.

Here, we collect several results, and open questions connected with the problem of determining the arithmetic nature of numbers $w \in \mathbb{C}$ such that $f^{(k)}(w) \in \mathbb{Z}$, for all $k \geqslant 0$, where f is a given meromorphic transcendental function.

We already know that there are examples of entire functions of order 2 (resp. any given order n, $n \geqslant 2$) possesing 2 integral points (resp. exactly n integral points). We suggest that, in such a situation, there are additional relations between integral points.

CONJECTURE 22. - If $f(z)$ has order $n \geqslant 1$, and $f(z)$ is a meromorphic function, satisfying algebraic differential equation $R(z, f(z), \ldots, f^{(q)}(z)) \equiv 0$, then, for n distinct integer points $w_1, \ldots, w_n$ of $f(z)$, $w_i \in \bar{\mathbb{Q}}$, $i = 1, \ldots, n$, we have

$$w_2 - w_1, \ldots, w_n - w_1 \in \mathbb{Q} .$$

On the other hand, not only functions of strict order < 2 admit not more than one integer point. For example, like in the proof of theorem 2, we obtain the following proposition.

PROPOSITION 23. - Let $f(z)$ be an entire transcendental function in $\mathbb{C}$ with order of growth

$$\log|f|_R \leqslant R^{2-(\log\log R)^{-1}}, \text{ for } R \longrightarrow \infty,$$

or a meromorphic transcendental function having the form $f_1(z)/f_2(z)$, where f_1, f_2 are such entire functions. Then, there is at most one integral point of $f(z)$.

Analogically, if we consider, in the scheme of the proof of theorem 6, $\rho = 2 - (\log\log u_i^{\tilde{a}})^{-1}$ and increasing N (cf. (W) supra), we obtain the same result for any order of growth.

THEOREM 24. - Let $f(z)$ be an entire transcendental function of growth

$$\log|f|_R \leqslant R^{n-(\log\log R)^{-1}}, \text{ for } R \longrightarrow \infty, \quad n \geqslant 1, \tag{3.1}$$

or a meromorphic transcendental function being the ratio of entire functions with such a growth. Then, there are no more than n integral points of $f(z)$.

In fact, these bounds for order of growth of $|f|_R$ can be easily improved. Instead of formulating such results (probably not very important), we propose the following problem for meromorphic functions of infinite order of growth.

PROBLEM 25. - Let $f(z)$ be a meromorphic function of infinite order of growth. Let

$$S_{\mathbb{Z}} = \{w \in \mathbb{Z} : f^{(k)}(w) \in \mathbb{Z}, \text{ for all } k \geqslant 0\} ;$$

What is the density of S_Z : $|S_Z \cap (-R, R)|$ in terms of $\log|f|_R$, when $R \longrightarrow \infty$?

This problem is very interesting especially because of examples, constructed in [6] for functions of infinite order of growth having $S_Z = \mathbb{Z}$ etc.

The last part of this paper is devoted to problems of diophantine approximations of values of meromorphic functions. We consider the following question for functions of order less than two.

PROBLEM 26. - Let $f(z)$ be a meromorphic transcendental function of order $\rho < 2$, and suppose $f(z)$ has one integral point w (we can always take $w = 0$ by the change of variable $z \longrightarrow z_1 + w$). Then, for any $v \in C$ such that $v \neq w$, $f(z)$ is regular at v, and $f^{(k)}(v) \in \mathbb{Z}$, for all $k \geqslant 0$, we have by theorem 2

(3.2) v is transcendental.

What is the mesure of transcendence of v ?

Below we shall give an answer to problem 26, yielding a mesure of transcendence of v. The basic fact that leads to this estimate is the existence of a good upper bound for the numbers of zeroes of auxiliary functions $F(z) = P(z, f(z))$ in terms of $\deg(P)$. Such estimates can be proved only for functions satisfying algebraic differential equations.

It is real luck that functions $f(z)$ satisfying all assumptions of problem 26, also satisfy an algebraic differential equation.

PROPOSITION 27. - Let all the hypotheses of problem 26 and (3.2) be satisfied. Then, for some $d \leq 2/(2 - \rho)$, the function $f(z)$ satisfy an algebraic differential equation $R(f(z), f'(z), \dots, f^{(d)}(z)) \equiv 0$, where

$$R(z_0, \dots, z_d) \in \mathbb{Z}[z_0, \dots, z_d] .$$

Proposition 27 follows immediately from [12] applied to the system of functions $f(z), f'(z), \dots, f^{(d)}(z)$.

For $f(z)$ satisfying an algebraic differential equation, we use a method of D. W. BROWNAWELL-D. MASSER on estimates for the orders of zeroes (evaluated at $z = 0$, v) of the auxiliary function $F(z) = P(z, f(z))$ for $P(x, y) \in C[x, y]$. This yields the following lemma.

LEMMA 28. - Let $f(z)$ be a transcendental function satisfying an algebraic differential equation

$$R(z, f(z), f'(z), \dots, f^{(q)}(z)) \equiv 0 ,$$

for $R(x_1, x_2, \dots, x_{q+2}) \in C[x_1, \dots, x_{q+2}]$. If $P(x, y) \in C[x, y]$, $P(x, y) \neq 0$, and $w_1, \dots, w_m \in C$, then for the function $F(z) = P(z, f(z))$ and for the sum $\sum_{i=1}^{m} \mathrm{ord}_{w_i}(F)$ of orders of zeroes of $F(z)$ at $z = w_i$, $i = 1, \dots, m$, we have the bound

$$\sum_{i=1}^{m} \mathrm{ord}_{w_i}(F) \leqslant c_1(d_x(P)d_y(P) + d_x(P) + d_y(P)),$$

where $c_1 > 0$ depends only on $w_1, \dots, w_m$, $f(z)$, q and $R(x_1, \dots, x_{q+2})$.

For the proof of lemma 28, we consider $F'(z) = P_x + P_y f'$, $F''(z) = \dots$ etc., and consider resultants (on y) of $P(x, y)$ and some polynomials $R(x, y)$ obtained by differentiating $F(z)$ and taking into account the differential equation for $f(z)$.

For functions of order < 2, we obtain, using lemma 28, proposition 27 and the method of proof of theorem 6, described before (see also [3]), the following first result on the measure of transcendence of v in (3.2).

THEOREM 29. - Let $f(z)$ be a meromorphic transcendental function of order $\rho < 2$, having an integral point w ($w \in \mathfrak{Z}$). If $v \in \underset{\sim}{C}$, $v \neq w$ and $f^{(k)}(v) \in \underset{\sim}{Z}$, for all $k \geqslant 0$, then, for arbitrary algebraic ζ of degree $\leqslant d$ and of height $\leqslant H$, we have, for $H \geqslant c_0(d)$,

$$|v - \zeta| \geqslant \exp(-c_1(d)(\log H)^{2d/(2-\rho)}) . \tag{3.3}$$

Analogous results take place also for problem of simultaneous diophantine approximations. We use, in the same line, the analogue of proposition 27 and lemma 28. Instead of giving the final result, we formulate only a general but weak estimation.

THEOREM 30. - Let $f(z)$ be a meromorphic transcendental function of order $\rho < n$, where $n \geqslant 2$, and let $v_1, \dots, v_n$ be distinct complex numbers such that

$$f^{(k)}(v_i) \in \underset{\sim}{Z}, \text{ for all } k \geqslant 0 \text{ and } i = 1, \dots, n .$$

Then, for algebraic $\zeta_1, \dots, \zeta_n$ of degree $\leqslant d$ and of height $\leqslant H$, we have, for any $\epsilon > 0$,

$$\max_{i=1,\dots,n} |v_i - \zeta_i| \geqslant \exp(-\exp(\log H)^{\epsilon}),$$

provided $H \geqslant c_1(d, n, f(z), \epsilon) > 0$.

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