Séminaire Delange-Pisot-Poitou.
Théorie des nombres

Jan Turk

Sets of integers composed of few prime numbers

Séminaire Delange-Pisot-Poitou. Théorie des nombres, tome 19, nº 2 (1977-1978), exp. nº 44, p. 1-6
<http://www.numdam.org/item?id=SDPP_1977-1978__19_2_A18_0>

Séminaire DELANGE-PISOT-POITOU
(Théorie des nombres)
19e année, 1977/78, n° 44, 6 p.

12 juin 1978

SETS OF INTEGERS COMPOSED OF FEW PRIME NUMBERS

by Jan TURK (*)

1. Introduction and statement of the results.

For any finite set X of positive integers, we denote the number of elements of X by $N(X)$, and the number of distinct primes of which the integers of X are composed by $\omega(X)$.

In [2], RAMACHANDRA, SHOREY and TIJDEMAN proved, in connection with a conjecture of C. A. GRIMM [1], the following theorem ($\log_2 n = \log\log n$, etc.).

THEOREM A. - Let n , k be positive integers, $n \geqslant 3$. If the interval $(n, n+k)$ contains a set X of integers, with $\omega(X) < N(X)$, then

$$k > c_0 (\log n)^3 (\log_2 n)^{-3} .$$

In theorem A, c_0 denotes an absolute positive constant. The proof of theorem A uses the theory of linear forms in logarithms of rational numbers with rational coefficients and two arithmetical lemmas of an elementary nature. These lemmas, essentially, suffice to obtain a lower bound for the length of an interval which contains a set X which satisfies a stronger condition on $\omega(X)$ than in theorem A.

THEOREM 1.

(a) For every $0 < c < 1$, there exists a number $c_1 > 0$, depending only on c , such that, if n , k are positive integers, with $n \geqslant 3$ with the property that $(n, n+k)$ contains a set X of integers with $\omega(X) < cN(X)$, then

$$k > c_1 (\log n)^3 (\log_2 n)^{-3} .$$

(b) For every $0 < \alpha < 1$, there exists a number $c_2 > 0$, depending only on α , such that, if n , k are positive integers, with $n \geqslant 3$ with the property that $(n, n+k)$ contains a set X of integers with $\omega(X) < (N(X))^\alpha$, then

$$k > c_2 (\log n)^c (\log_2 n)^{-c} ,$$

where $c = 2\alpha^{-1} + 1$.

Using a generalization of one of the above mentioned lemmas (see lemma 3), we can prove the following refinement.

THEOREM 2. - For every $0 < \alpha < 1$, there exists a number $c_3 > 0$, depending only on α , such that, if n , k are positive integers, with $n \geqslant 3$ with the property that $(n, n+k)$ contains a set X of integers with $\omega(X) < (N(X))^\alpha$, then

(*) Partially supported by the Netherlands Organization for the Advancement of Pure Research (Z. W. O.).

$k > c_3 (\log n)^c (\log_2 n)^{-c}$, where $c = \max\{2\alpha^{-1} + 1, 4\alpha^{-1} - 2\}$.

For $\alpha < 2/3$, this improves upon the lower bound for k of theorem 1 (b) ; for small values of α , the lower bound of theorem 2 is about the square of the lower bound of theorem 1 (b). Theorem 2 is not valid any longer if one replaces the lower bound for k by $\exp\{(\log n)^{(1/2)+\epsilon}\}$ in view of the following result.

For every $0 < \alpha \leq 1$, there exists a number $c_5 > 0$, depending only on α , such that there exist infinitely many integers n (≥ 3) with the property that $(n , n + k(n))$ contains a set X of integers with $\omega(X) < (N(X))^\alpha$, where $k(n) = \exp(c_5(\log n \log_2 n)^{1/2})$. The method of theorem 2 also works for small functions of $N(X)$ other than small powers. For larger functions of $N(X)$, the method of theorem A can be generalised, provided that also an appropriate upper bound for $P(X)$, the largest prime occuring in the prime decomposition of the integers of X , is given. These results will appear in the author's thesis.

2. Proofs.

Notation. - Let X be a finite subset of $\mathbb{N}$, the set of positive integers. We denote the number of elements of X by $N(X)$, and the set of primes which divide at leats one element of X by $\Omega(X)$. We write $\omega(X)$ for $N(\Omega(X))$. For integers x and primes p , we denote the exponent of p in the prime decomposition of x by $v_p(x)$. For real numbers y , we denote the largest integer not exceeding y by $[y]$.

LEMMA 1. - Let $n > 1$. Let X be a finite set of integers which are not smaller than n . For every prime p and every positive integer, j , we denote $\max\{0 , N\{x \in X | p^j \text{ divides } x\} - 1\}$ by $N(p^j)$. Then

(1) $$N(X) \leq \omega(X) + \sum_p \sum_j N(p^j) (\log p)(\log n)^{-1} .$$

The sum over p is over the prime numbers, the sum over j over the positive integers ; of course, there are only finitely many pairs (p , j) with $N(p^j) \neq 0$.

Proof. - For every p in $\Omega(X)$, let $n(p)$ be some element of X with $v_p(n(p)) \geq v_p(x)$, for every x in X . Let X' be the set of those elements x in X , with $x \neq n(p)$, for every p in $\Omega(X)$. We have $N(X') \geq N(X) - \omega(X)$. We denote the number of elements of X' which are divisible by p^j with $M(p^j)$, for every prime p and every positive integer j . We have

$$n^{N(X)-\omega(X)} \leq n^{N(X')} \leq \prod_{x\in X'} x = \prod_{p\in\Omega(X')} p^{(\sum_{x\in X'} v_p(x))} = \prod_{p\in\Omega(X')} p^{(\sum_{j=1}^{\infty} M(p^j))} .$$

From the definition of X' follows immediately that $M(p^j) \leq N(p^j)$, for every prime p and every positive integer j . Thus

$$(N(X) - \omega(X)) \log n \leq \sum_p \log p \sum_j M(p^j) \leq \sum_p \sum_j N(p^j) \log p .$$

COROLLARY. - <u>Let</u> n, k <u>be positive integers with</u> $n \geq 2$, <u>and let</u> X <u>be a set of integers contained in the interval</u> $(n, n+k)$. <u>Then</u>

$$N(X) \leq \omega(X) + k(\log k)(\log n)^{-1} . \tag{2}$$

<u>Proof</u>. - The number of integers in $(n, n+k)$ divisible by p^j is at most $[kp^{-j}] + 1$. It follows that $N(p^j) \leq [kp^{-j}]$, for every prime p and every positive integer j. We infer from (1) that

$$N(X) \leq \omega(X) + \sum_p \sum_j N(p^j)(\log p)(\log n)^{-1}$$
$$\leq \omega(X) + \sum_p \sum_j [kp^{-j}](\log p)(\log n)^{-1} = \omega(X) + \log (k!)(\log n)^{-1}$$
$$\leq \omega(X) + k(\log k)(\log n)^{-1} .$$

LEMMA 2. - <u>Let</u> n, k <u>be integers greater than</u> 1, <u>and let</u> X <u>be a set of integers contained in the interval</u> $(n, n+k)$. <u>If</u> $\omega(X) < N(X)$, <u>then</u> $\omega(X) \geq (\log n)(\log k)^{-1}$. <u>If</u> $\omega(X) + [(2\omega(X))^{1/2}] < N(X)$, <u>then</u>

$$\omega(X) > (1/2)(\log n)^2 (\log k)^{-2} .$$

<u>Proof</u>. - For every finite set Y of positive integers, we have

$$\prod_{y\in Y} y \leq \mathrm{LCM}(y) \prod_{y_1<y_2} \mathrm{GCD}(y_1, y_2) ,$$

where $\mathrm{LCM}(y)$ is the least common multiple of the elements of Y, and $\mathrm{GCD}(y_1, y_2)$ is the greatest common divisor of y_1 and y_2. The product is over all pairs (y_1, y_2), with y_1, y_2 in Y and $y_1 < y_2$. Define the integers $n(p)$, p from $\Omega(X)$, and the set X' as in the proof of lemma 1. We say already that the number of elements of X' divisible by p^j is at most $[kp^{-j}]$ for every prime p and every positive integer j. It follows that, for every $x \in X'$ and every prime p, we have $p^{v_p(x)} \leq k$, hence $\mathrm{LCM}(Y) \leq k^{\omega(Y)}$, for every subset Y of X'. Every common divisor of two distinct integers in $(n, n+k)$ divides the absolute value of their difference, which is one of the integers $1, 2, \ldots, k$. Therefore $\mathrm{GCD}(y_1, y_2) \leq k$, for every $y_1 < y_2$, y_1, y_2 in Y for every subset Y of X'. We infer that $N(Y) \log n \leq (\omega(Y) + (1/2)N(Y)(N(Y)-1))\log k$, for every $Y \subset X'$, hence $(\omega(X)/N(Y)) + (1/2)(N(Y)-1) \geq (\log n)(\log k)^{-1}$, for every $Y \subset X'$. If $\omega(X) < N(X)$, then X' has at least one element, and we choose for Y a subset of X' with $N(Y) = 1$ element. This gives $\omega(X) \geq (\log n)(\log k)^{-1}$. If $\omega(W) + [(2\omega(X))^{1/2}] < N(X)$, then we take for Y a subset of X', with $N(Y) = 1 + [(2\omega(X))^{1/2}]$ elements. This gives $\omega(X) > (1/2)(\log n)^2 \log k)^{-2}$.

<u>Proof of theorem 1</u>.

(a) Let $\gamma > 1$ be given. Suppose n, k are positive intergers, $n \geq 3$, with the property that $(n, n+k)$ contains a set X of integers with $\gamma\omega(X) < N(X)$. Then $k \geq 2$. From (2) we deduce that $k \geq (\gamma - 1)\omega(X)(\log n)(\log k)^{-1}$. If $\omega(X) \geq \delta$, where δ is an appropriate constant depending only on γ (for example, $\delta = 2(\gamma-1)^{-2}$), then $N(X) > \gamma\omega(X) \geq \omega(X) + [(2\omega(X))^{1/2}]$, hence, by the second part of lemma 2, $\omega(X) > (1/2)(\log n)^2 (\log k)^{-2}$ and therefore

$$k > (1/2)(\gamma - 1)(\log n)^3 (\log k)^{-3} .$$

If $\omega(X) < \delta$, then, by the first part of lemma 2, $k \geqslant n^{\delta-1}$. Both inequalities imply $k > c_1(\log n)^3 (\log_2 n)^{-3}$ for a suitable constant c_1 which depends only on γ.

(b) Let $\beta > 1$ be given. Suppose n, k are positive integers, $n \geqslant 3$, with the property that $(n, n+k)$ contains a set X of integers with $(\omega(X))^\beta < N(X)$. Then $k \geqslant 2$. From (2) we deduce that $k \geqslant (\omega(X))^\beta (1 - (\omega(X))^{1-\beta})(\log n)(\log k)^{-1}$.

If $\omega(X) \geqslant \delta$, where δ is an appropriate constant ($\geqslant 2$) depending only on β, then $N(X) > (\omega(X))^\beta \geqslant \omega(X) + [(2\omega(X))^{1/2}]$, hence, by the second part of lemma 2, $\omega(X) > (1/2)(\log n)^2 (\log k)^{-2}$, and therefore

$$k > 2^{-\beta}(1 - \delta^{1-\beta})(\log n)^{2\beta+1} (\log k)^{-(2\beta+1)} .$$

If $\omega(X) < \delta$, then, by the first part of lemma 2, $k > n^{\delta^{-1}}$. Both inequalities imply $k > c_2(\log n)^{2\beta+1} (\log_2 n)^{-(2\beta+1)}$ for a suitable positive number c_2, which depends only on β.

LEMMA 3. - <u>For every non-negative integer</u> λ, <u>we have</u>

(3) $\quad N(X) \leqslant \omega(X) \sum_{j=0}^{\lambda} (\omega(X)(\log k)(\log n)^{-1})^j + k(\log k)^{\lambda+1} (\log n)^{-(\lambda+1)}$,

<u>for any</u> $n, k \in \underset{\sim}{N}$, <u>with</u> $n \geqslant 2$ <u>and any subset</u> X <u>of</u> $\{n, n+1, \dots, n+k\}$.

<u>Proof</u>. - By induction on λ. For $\lambda = 0$, the assertion follows from the corollary of lemma 1. Suppose λ_0 is a non-negative integer for which the assertion holds. We prove that the assertion also holds for the integer $\lambda_0 + 1$. Let $n, k \in \underset{\sim}{N}$, $n \geqslant 2$ and $X \subset \{n, n+1, \dots, n+k\}$. To prove assertion (3) with λ replaced by $\lambda_0 + 1$, we may assume without loss of generality that $k < n$. Let $p \in \Omega(X)$, and $j \in \underset{\sim}{N}$ be such that $N(p^j) \geqslant 1$. Then $p^j \leqslant k$ since $N(p^j) \leqslant [kp^{-j}]$, and consequently $j \leqslant [(\log k)(\log p)^{-1}]$. Let $p^j m_1 < \dots < p^j m_N$, with $N = N(p^j)$, be integers in X which are divisible by p^j. Then $\{m_1, \dots, m_N\} =: Y$ is contained in $\{m_1, m_1 + 1, \dots, m_1 + [kp^{-j}]\}$, and $m_1 \geqslant np^{-j} \geqslant nk^{-1} > 1$.

From the induction hypothesis, we infer

$$\begin{aligned} N(p^j) &= N(Y) \\ &\leqslant \omega(Y) \sum_{\sigma=0}^{\lambda_0} (\omega(Y)\log[kp^{-j}](\log m_1)^{-1})^\sigma + [kp^{-j}](\log[kp^{-j}](\log m_1)^{-1})^{\lambda_0+1} \\ &\leqslant \omega(X) \sum_{\sigma=0}^{\lambda_0} (\omega(X)(\log kp^{-j})(\log np^{-j})^{-1})^\sigma + [kp^{-j}]((\log kp^{-j})(\log np^{-j})^{-1})^{\lambda_0+1} \\ &\leqslant \omega(X) \sum_{\sigma=0}^{\lambda_0} (\omega(X)(\log k)(\log n)^{-1})^\sigma + [kp^{-j}]((\log k)(\log n)^{-1})^{\lambda_0+1} . \end{aligned}$$

From lemma 1 and these inequalities we deduce

$$N(X) \leq \omega(X) + \sum_{p\in\Omega(X)} \sum_{j=1}^{\infty} N(p^j)(\log p)(\log n)^{-1}$$

$$\leq \omega(X) + \sum_p \sum_{j=1}^{j=[(\log k)(\log p)^{-1}]} (\log p)(\log n)^{-1}\ \omega(X) \sum_{\sigma=0}^{\lambda_0} (\omega(X)(\log k)(\log n)^{-1})^{\sigma}$$

$$+ \sum_p \sum_j [kp^{-j}](\log p)(\log k)^{\lambda_0+1} (\log n)^{-(\lambda_0+2)} .$$

Using $\sum_{p\in\Omega(X)} \sum_{j=1}^{[(\log k)(\log p^{-1}]} \log p \leq \omega(X) \log k$, and

$$\sum_p \sum_j [kp^{-j}] \log p \leq \log k! \leq k \log k ,$$

we derive

$$N(X) \leq \omega(X) + \omega(X)(\log k)(\log n)^{-1}\ \omega(X) \sum_{\sigma=0}^{\lambda_0} (\omega(X)(\log k)(\log n)^{-1})^{\sigma}$$

$$+ (k \log k)(\log k)^{\lambda_0+1} (\log n)^{-(\lambda_0+2)}$$

$$= \omega(X) \sum_{\sigma=0}^{\lambda_0+1} (\omega(X)(\log k)(\log n)^{-1})^{\sigma} + k(\log k)^{\lambda_0+2} (\log n)^{-(\lambda_0+2)} ,$$

which proves (3) with λ replaced by $\lambda_0 + 1$.

<u>Proof of theorem</u> 2. – Let $\beta > 1$. Let n , k be positive integers , $n \geq 3$, with the property that $(n , n + k]$ contains a set X of intergers with $(\omega(X))^{\beta} < N(X)$. We will prove that $k > c_3(\log n)^c (\log_2 n)^{-c}$, where $c = \max\{2\beta + 1 , 4\beta - 2\}$, and where c_3 is a certain positive number which depends only on β . Theorem 2 follows by taking $\beta = \alpha^{-1}$. For $1 < \beta \leq 3/2$ the assertion follows from theorem 1 (b) with $\alpha = \beta^{-1}$. Suppose $\beta > 3/2$. Let $c_1 > 1$, $0 < \delta < 1$ be real numbers which satisfy

$$c_1(c_1 - 1)^{-1} (1 - \delta)^{-1} \leq 2^{1+[\beta-1]-(\beta-1)} . \tag{4}$$

We assume first that

$$k < n^{(2c_1)^{-1}} . \tag{5}$$

Clearly $k \geq 2$. From the first part of lemma 2, we obtain, by (5), that $\omega(X) \geq 3$. Hence, using $\beta > 3/2$, we have $N(X) > (\omega(X))^{\beta} \geq \omega(X) + [(2\omega(X))^{1/2}]$. From the second part of lemma 2, we infer

$$\omega(X) > (1/2)(\log n)^2 (\log k)^{-2} . \tag{6}$$

From (5) and (6) we deduce $\omega(X)(\log k)(\log n)^{-1} \geq c_1$, hence

$$\sum_{j=0}^{\lambda} (\omega(X)(\log k)(\log n)^{-1})^j \leq c_1(c_1 - 1)^{-1} (\omega(X)(\log k)(\log n)^{-1})^{\lambda} .$$

So we obtain from lemma 3 that

$$k \geq \{(\omega(X))^{\beta} - c_1(c_1 - 1)^{-1}\ \omega(X)(\omega(X)(\log k)(\log n)^{-1})^{\lambda}\}(\log n)^{\lambda+1} (\log k)^{-(\lambda+1)} ,$$

for every non-negative integer λ . Assume that λ satisfies

$$(\omega(X))^{\beta} - c_1(c_1 - 1)^{-1}\ \omega(X)(\omega(X)(\log k)(\log n)^{-1})^{\lambda} \geq \delta(\omega(X))^{\beta} . \tag{7}$$

Then we obtain

$$k \geq \delta(\omega(X))^{\beta} (\log n)^{\lambda+1} (\log k)^{-(\lambda+1)} . \tag{8}$$

For convenience, we define the real number a by

(9) $$\omega(X) = (1/2)(\log n)^{a} (\log k)^{-a} .$$

It follows from (6) that $a > 2$. We rewrite condition (7) as

(10) $$2^{\lambda-(\beta-1)} ((\log n)(\log k)^{-1})^{a(\beta-1)-\lambda(a-1)} \geq c_1(c_1 - 1)^{-1} (1 - \delta)^{-1} .$$

Put $\lambda = [a(a-1)^{-1} (\beta - 1)]$. We show that (10) is satisfied. Observe that $\lambda \geq [\beta - 1]$. From (5) it follows that $(\log n)(\log k)^{-1} > 2$. The exponent of $(\log n)(\log k)^{-1}$ in the left hand side of (10) is non-negative by the choice of λ. Therefore if $\lambda \geq [\beta - 1] + 1$, then the lefthandside of (10) is greater than

$$2^{[\beta-1]+1-(\beta-1)} \geq c_1(c_1 - 1)^{-1} (1 - \delta)^{-1}$$

by (4) and (10) is satisfied. If $\lambda = [\beta - 1]$, then the lefthandside of (10) equals

$$2^{[\beta-1]-(\beta-1)} ((\log n)(\log k)^{-1})^{a((\beta-1)-[\beta-1])+[\beta-1]} .$$

In view of $a > 2$, $\beta > 3/2$ the exponent of $(\log n)(\log k)^{-1}$ is at least 1 and therefore the lefthandside of (10) is greater than $2^{[\beta-1]-(\beta-1)+1}$ and, as before, (10) is satisfied. We conclude from (8), (9) and the choice of λ that

$$k \geq \delta 2^{-\beta} ((\log n)(\log k)^{-1})^{\beta a+[a(a-1)^{-1}(\beta-1)]+1} \geq \delta 2^{-\beta} ((\log n)(\log k)^{-1})^{c(\beta)},$$

where $c(\beta) := \inf_{a>2} \{\beta a + [a(a-1)^{-1} (\beta - 1)] + 1\}$.

If the assumption (5) is not satisfied, then $k \geq n^{(2c_1)^{-1}}$. Both inequalities imply that $k \geq c_3((\log n)(\log_2 n)^{-1})^{c(\beta)}$ for some suitable positive number c_3 which depends only on β. Finally, we observe that

$$c(\beta) \geq \inf_{a>2} \{\beta a + a(a-1)^{-1} (\beta - 1)\} = 2\beta + 2(\beta - 1) = 4\beta - 2 .$$

This proves theorem 2.

Remark. - In fact, one has

$$c(\beta) = 2\beta + [2(\beta - 1)] + \min\{1, \beta.(2(\beta - 1) - [2(\beta - 1)])([2(\beta - 1)] - (\beta - 1))^{-1}\} ,$$

for every $\beta > 3/2$. Hence $c(\beta) = 4\beta - 2$ if, and only if, $\beta = m/2$, for some $m \in \mathbb{N}$, $m \geq 4$. For values of α between 0 and 2/3 which are not of the form $2/m$, for some $m \in \mathbb{N}$, $m \geq 4$, we have therefore a somewhat better exponent than $4\alpha^{-1} - 2$ in the lower bound for k.

REFERENCES

[1] GRIMM (C. A.). - A conjecture on composite numbers, Amer. math. Monthly, t. 76, 1969, p. 1126-1128.

[2] RAMACHANDRA (K.), SHOREY (T. N.) and TIJDEMAN (R.). - On Grimm's problem relating to factorisation of a block of consecutive integers, J. für die reine und angew. Math., Part I : t. 273, 1975, p. 109-124 ; Part II : t. 288, 1976, p. 192-201.

(Texte reçu le 9 octobre 1978)

Jan TURK, Mathematisch Instituut der Rijksuniversiteit Leiden, Wassenaarseweg 80, LEIDEN (Nederland).