# Séminaire Delange-Pisot-Poitou. Théorie des nombres



AUDREY TERRAS


## Applications of special functions for the general linear group to number theory


*Séminaire Delange-Pisot-Poitou. Théorie des nombres*, tome 18, nº 2 (1976-1977), exp. nº 23, p. 1-16

<http://www.numdam.org/item?id=SDPP_1976-1977__18_2_A4_0>



© Séminaire Delange-Pisot-Poitou. Théorie des nombres (Secrétariat mathématique, Paris), 1976-1977, tous droits réservés.

L'accès aux archives de la collection « Séminaire Delange-Pisot-Poitou. Théorie des nombres » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques*

http://www.numdam.org/

Séminaire DELANGE-PISOT-POITOU
(Théorie des Nombres)
18e année, 1976/77, n° 23, 16 p.



25 avril 1977

# APPLICATIONS OF SPECIAL FUNCTIONS FOR THE GENERAL LINEAR GROUP TO NUMBER THEORY

by Audrey TERRAS (1)

## 1. Automorphic forms for $Gl(n, \underset{\sim}{R})$-Eisenstein series.

We shall be considering special functions on the (weakly) symetric space $\mathcal{P}_n$ of positive definite (real) $n \times n$ matrices. Now $\mathcal{P}_n$ can be identified with the homogeneous space $Gl(n, \underset{\sim}{R})/O(n)$ via the map which sends $A$ in $Gl(n, \underset{\sim}{R})$ to ${}^tAA$, where ${}^tA$ denotes the transpose of the matrix $A$. For the basic results on symetric spaces see [8] and [22]. Note that $A$ in $Gl(n, \underset{\sim}{R})$ acts on $P$ in $\mathcal{P}_n$ via $P \longmapsto P[A] \equiv {}^tAPA$.

As in [3], an automorphic form $f$ in $\mathfrak{M}(Gl(n, \underset{\sim}{Z}), \lambda)$ for $\lambda = (\lambda_j) \in \underset{\sim}{C}^n$ is a $Gl(n, \underset{\sim}{Z})$-invariant function $f : \mathcal{P}_n \longmapsto C$ (satisfying certain growth conditions) which is an eigenfunction with eigenvalue $\lambda_j$ for the $Gl(n, \underset{\sim}{R})$-invariant differential operator $D_j$ defined by

$$D_j = \mathrm{Tr}((X \frac{\partial}{\partial X})^j) , \text{ where } \frac{\partial}{\partial X} = (\frac{1}{2}(1 + \delta_{ij}) \frac{\partial}{\partial x_{ij}})_{1 \leq i, j \leq n} \quad \text{for } X \in \mathcal{P}_n .$$

That is $f$ satisfies

$$(1) \quad \begin{cases} f(X[A]) = f(X) , \text{ for all } A \in Gl(n, \underset{\sim}{Z}) \text{ and } X \in \mathcal{P}_n . \\ D_j f = \lambda_j f , \text{ for all } j = 1, \ldots, n . \end{cases}$$

It is proved in [15](p. 64), that $D_1, \ldots, D_n$ generate all the $Gl(n, \underset{\sim}{R})$-invariant differential operators on $\mathcal{P}_n$. See [8] and [22] for the more general theory.

The analogy between the definition of $\mathfrak{M}(Gl(n, \underset{\sim}{Z}), \lambda)$ and the usual definition of automorphic forms is best understood by reading MAASS [16] where the theory is developed for $Sl(2)$. In this case one studies (instead of holomorphic functions on the upper half plane $\mathcal{H}$) eigenfunctions of the Laplacian on $\mathcal{H}$ which are invariant under $Sl(2, \underset{\sim}{Z})$. MAASS shows ([16], p. 197-218) that $\mathrm{Re}\, s > 1$ implies that $\dim \mathfrak{M}(Sl(2, \underset{\sim}{Z}), s(s-1)) = 1$ (2). That is, the only automorphic forms in this case are the Eisenstein series which we are about to write down. For $\mathrm{Re}\, s \leqslant 1$ the situation is very mysterious. One expects to find cusp forms but one is unable to write down the analogues of the explicit examples one had in the classical holomorphic case.

---

(1) Partially supported by an NSF Grant, and the "Sonderforschungsbereich für theoretische Mathematik an der Universität Bonn".

(2) Actually, one has to multiply Maass's functions $g$ in $[Sl(2, \underset{\sim}{Z}), s, s, 1]$ by $y^s$ in order to obtain our functions $f$ in $\mathfrak{M}(Sl(2, \underset{\sim}{Z}), s(s-1))$.

Connections with representation theory, classical PDE's, the Riemann hypothesis have made the subject interesting to many people. The possible generalization of the result quoted above to $Gl(n)$ appears to be open at the moment.

What are the Eisenstein series for $Gl(n, \mathbb{R})$ ? They are special functions on $\mathcal{P}_n$ generalizing Epstein's zeta function. Such functions have been investigated in [10], [12], [15](§ 17), [22], [29], [30], [32]. The simplest case involves only one complex variable $s$ and this is the only case that we shall consider here. Taking $k$ to be an integer between 1 and $n$, we define the Eisenstein series by

$$(2) \qquad Z_k(P^{(n)}, s) \equiv \sum_{A^{(n,k)} \text{rank } k, \bmod Gl(k,\mathbb{Z})} |P[A]|^{-s}, \text{ for } \operatorname{Re} s > \frac{n}{2}.$$

Here $|P|$ is the determinant of $P$ and the sum is over $n \times k$ matrices $A$ of rank $k$ inequivalent under right multiplication by elements of $Gl(n, \mathbb{Z})$. Using Minkowski's description of a fundamental domain for $\mathcal{P}_n$ modulo $Gl(n, \mathbb{Z})$, KOECHER shows, in [10], that one can bound $Z_k$ by a product of Epstein zeta functions $Z_1$, thus proving convergence in the stated region $\operatorname{Re} s > \frac{n}{2}$ (3).

If $P$ has first diagonal block $P_1^{(k)}$, i. e.

$$P^{(n)} = \begin{pmatrix} P_1^{(k)} & * \\ * & * \end{pmatrix},$$

then MAASS shows in [15](p. 69), that $|P_1|^{-s}$ is eigen for all the $Gl(n, \mathbb{R})$-invariant differential operators on $\mathcal{P}_n$. Thus the $Z_k$ satisfy (1).

The case $k = 1$ is Epstein's zeta function. When $n = 3$ for example, these functions have been used to express certain potential functions in crystal physics and they required the analytic continuation of the function beyond the region where (2) converges (Cf. [4]). There have been many number-theoretical applications of this case also. Many of these come from Hecke's integral formula ([7], p. 198 ff) which writes the Dedekind zeta function of an algebraic number field $K$ as a sum over the ideal class group of integrals of certain Epstein zeta functions $Z_1$. For imaginary quadratic field $K$ the integral disappears. See [23] and [26] for applications of this relation between $K$ and $Gl(n)$. We shall return to this subject in § 3.

Another application of the Eisenstein series (2) involves the case $n = k$ which can be shown to reduce to a product of Riemann zeta functions :

$$(3) \qquad Z_n(P^{(n)}, s) = |P|^{-s} \prod_{j=0}^{n-1} \zeta(2s - j).$$

See [10] for the proof. In [25](I, p. 459-468), SIEGEL used the analytic continuation of (3) to obtain a proof of Minkowski's formula for the volume of the funda-

(3) Using the Minkowski-Hlawka theorem (Cf. [41], or the case $k = 1$ of [42]), one can see that the integral of $Z_1(P, s)$ over $P$ in $\mathfrak{F}_n \equiv \mathcal{P}_n/Gl(n, \mathbb{Z})$ with $|P| = 1$ and $\ell_P \geq 1$ is finite for $\operatorname{Re} s > n/2$. Here $\ell_P$ is defined by (19).

mental domain of P in $\mathcal{P}_n$ of determinant 1 modulo $Gl(n, \underset{\sim}{Z})$ (4). Unfortunately, the proof had a gap for a long time (the same gap as in [10]), since the analytic continuation of $Z_k$ by Riemann's method of theta functions runs into trouble due to the singular matrices summed over in the theta series. Some methods of solving the problem were found by SIEGEL in [25](III, p. 328-333) and SELBERG [15](p. 209 ff) and [29](§ 1). The situation appears far more difficult in the case $k > 1$. We shall return to this subject in § 3.

## 2. Special functions arising in Fourier coefficients of automorphic forms - K-Bessel functions.

What do we mean by a Fourier expansion of f in $\mathfrak{M}(Gl(n, \underset{\sim}{Z}), \lambda)$ ? Given a decomposition $n = k + m$ of n as a sum of 2 positive integers, we can write a partial Iwasawa decomposition (Cf. [8])

(4) $$P^{(n)} = \begin{pmatrix} T^{(k)} & 0 \\ 0 & V^{(m)} \end{pmatrix} \begin{bmatrix} I & 0 \\ Q & I \end{bmatrix} .$$

Here I denotes the identity matrix, 0 the zero matrix. The invariance of f in $\mathfrak{M}(Gl(n, \underset{\sim}{Z}), \lambda)$ under $Gl(n, \underset{\sim}{Z})$ implies that f(P) is periodic of period 1 in Q. Thus, we have a Fourier expansion

(5) $$f(P) = \sum_{N^{(m,k)} \in \underset{\sim}{Z}^{m\times k}} c_N(T, V) \exp(2\pi i Tr(Q^t N)) .$$

The natural idea is to use the $c_N(T, V)$ to study f as MAASS does in [15] and [16], in analogous situations for other Lie groups. The differential equations f satisfies imply certain PDE's for the coefficients $c_N$. It is easy to see by separation of variables as in [16](p. 212) that an eigenfunction of $\Delta_{\mathcal{H}}$ invariant under $Sl(2, \underset{\sim}{Z})$ has K-Bessel functions as its Fourier coefficients. One wonders what happens for $\mathfrak{M}(Gl(n, \underset{\sim}{Z}), \lambda)$. Rather than considering arbitrary functions f in $\mathfrak{M}(Gl(n, \underset{\sim}{Z}), \lambda)$, we shall look at the special case of Eisenstein series. The Fourier coefficients of these Eisenstein series will be seen to be matrix argument generalizations of K-Bessel functions.

Our first example is Epstein's zeta function. Suppose $n = k + m$ and set

(6) $$\Lambda_1(P^{(n)}, s) \equiv \pi^{-s}\,\Gamma(s)\, Z_1(P^{(n)}, s) .$$

Then we have as in [31] (5)

(7) $$\Lambda_1(P, s) = \Lambda_1(V^{(m)}, s) + |V|^{-\frac{1}{2}}\, \Lambda_1(T^{(k)}, s - \tfrac{m}{2})$$
$$+ |V|^{-\frac{1}{2}} \sum_{a\in \underset{\sim}{Z}^k - 0,\, b\in \underset{\sim}{Z}^m - 0} K_{s-(m/2)}(\pi T[a], \pi V^{-1}[b]) \times \exp\{2\pi i Tr(Qa\, {}^t b)\}$$

---

(4) See BOREL [3](p. 20-25) and MAASS [15](p. 122 ff) for descriptions of fundamental domains for $\mathcal{P}_n$ modulo $Gl(n, \underset{\sim}{Z})$ and in greater generality.

(5) An easier proof can be had using the transformation formula of theta rather than applying Poisson summation directly to $Z_1(P, s)$, as H. RESNIKOFF has noted.

where for positive real $p$ and $q$

(8) $$K_s(p\ ,\ q) \equiv \int_0^\infty x^{s-1} \exp\{-\ (px + \frac{q}{x})\}\ dx = 2K_s(2\sqrt{pq})(\frac{p}{q})^{-s/2}\ .$$

Thus Epstein's zeta function is a singular form in the sense of [20] and [21]. That is, the Fourier series of $Z_1(P^{(n)}\ ,\ s)$ for $n = k + m$ involves only a sum over $N^{(m,k)}$ of rank 1. This means that 2 or higher dimensional Mellin transforms of Epstein zeta functions would diverge as we shall see later.

Formula (7) has seen many applications in number theory, e. g. [26] in the proof that there are exactly 9 imaginary quadratic fields of class number one. One can also use (7) to prove Kronecker's limit formula which expresses the constant term in the Laurent expansion of $Z_1(P^{(2)}\ ,\ s)$ in terms of the Dedekindzeta function (Cf. [31] p. 478, and [23]).

Now let us consider the Fourier expansion of $Z_k(P^{(n)}\ ,\ s)$ corresponding to the decomposition $n = k + m$ . We shall assume that $m \geqslant k$, which is no restriction since, as in [29](p. 174), the functions $Z_k(P^{(n)}\ ,\ s)$ and $Z_{n-k}(P^{(n)}\ ,\ s)$ are essentially the same. In this case, it has appeared easier to consider the Eisenstein series for $Sl(n)$ , rather than that for $Gl(n)$ . The relation between the two is the following (Cf. [29])

(9) $$Z_k(P^{(n)}\ ,\ s) \prod_{j=0}^{k-1} \zeta(2s-j)^{-1} = \sum_{(A^{(n,k)}*)\in Sl(n,\underset{\sim}{Z})/P(k,\underset{\sim}{Z})} |P[A]|^{-s} \equiv Z_k^*(P^{(n)}\ ,\ s)$$

where $P(k\ ,\ \underset{\sim}{Z})$ denotes the parabolic subgroup of matrices of the form

$$\begin{pmatrix} U & V \\ 0 & W \end{pmatrix}$$

with $U$ a $k \times k$ matrix and $W$ an $m \times m$ matrix. The Fourier expansion of (9) can be obtained by the method BAILY [2](p. 228-240) uses to obtain Siegel's result [25](II, p. 97-137) on Fourier coefficients of Eisenstein series for $Sp(n)$ . BAILY obtains Siegel's matrix decompositions from the Bruhat decomposition. The Fourier expansion is shown in [32] to be

(10) $$Z_k^*(P^{(n)}\ ,\ s)$$
$$= |V|^{-s} + |V|^{-s} \sum_{r=1}^{k} \frac{\pi^{(r^2+r)/4}}{\Gamma_r(s)} \sum_{\substack{A\in Sl(k,\underset{\sim}{Z})/P(r,\underset{\sim}{Z}) \\ B\in Sl(m,\underset{\sim}{Z})/P(r,\underset{\sim}{Z}) \\ U\in \underset{\sim}{Z}^{r\times r}}} |V^{-1}[A_r]|^{(r/2)-s}\ \sigma_s(U)$$
$$\times K^{(r)}_{s-(r/2)}\ (T[B_r]\ ,\ V^{-1}[A_r\ U]) \times \exp\{2\pi i Tr(QB_r\ {}^tU\ {}^tA_r)\}\ .$$

Here $A_r$ denotes the first $r$ columns of $A$ . And $\sigma_s(U)$ denotes an analogue of Siegel's singular series (Cf. [15] p. 30) :

(11) $$\sigma_s(U) = \sum_{R\in(\underset{\sim}{Q}/\underset{\sim}{Z})^{r\times r}} \nu(R)^{-2s} \exp\{2\pi i Tr({}^tRU)\}\ ,$$

where $\nu(R)$ is the product of the reduced denominators of the elementary divisors of $R$ . The other part of the Fourier coefficient is a special function for $\mathcal{P}_r$ , a K-Bessel function of symetric $r \times r$ matrix arguments $A$ and $B$ defined by

(12) $$K_s^{(r)}(A , B) \equiv \int_{\mathcal{P}_r} \exp\{- \mathrm{Tr}(AX + BX^{-1})\} |X|^{s-((k+1)/2)} dX .$$

Such functions were first considered by HERZ in [9]. They clearly generalize (8) and, moreover, Siegel's gamma function :

(13) $$\Gamma_r(s) \equiv \prod_{j=0}^{r-1} \Gamma(s - \tfrac{j}{2}) = K_s^{(r)}(I , 0) \pi^{r(1-r)/4} .$$

Note that $K_s^{(r)}(A , B)$ satisfies the differential equation

(14) $$\left|\frac{\partial}{\partial A}\right| \left|\frac{\partial}{\partial B}\right| K_s^{(r)}(A , B) = K_s^{(r)}(A , B) ,$$

where $|\partial/\partial A|$ denotes the partial differential operator obtained by taking the determinant of the matrix $\partial/\partial A$ of differential operators defined before (1).

One might expect to be able to use (10) in order to analytically continue $Z_k(P , s)$ as in the case $k = 1$. Unfortunately the singular series $\sigma_s(U)$ is not well enough understood to do this. The K-Bessel functions in (12) are better understood. See [32] for some of their properties. It would be useful to be able to imitate the proof of (7) in [31] to obtain the rest of the Fourier expansions of $Z_k(P^{(n)} , s)$ corresponding to the other decompositions of $n$ as a sum of two positive integers (neither equal to $k$ ) and in order to obtain a simpler result than (10), i. e. a result not explicitly involving the singular series.

Let us close this section with an application of Fourier expansions of Epstein zeta functions over number fields. One needs to define Eisenstein series for $\mathcal{P}_n$ over a number field $K$. This is done in [34], [35], [36], for example. Then the Fourier expansion of the Eisenstein series $Z_1^K(P^{(2)} , s)$ will have the Dedekind zeta function $\zeta_K(s)$ in its constant term and products of K-Bessel functions in its non-constant terms. This leads to various results for $\zeta_K(s)$ , for example, the functional equation, as Siegel noticed in [25](I, p. 177). It also implies a relation between $\zeta_K(s)$ and $\zeta_K(s - 1)$ which gives formulas reminiscent of those of Ramanujan for $\zeta(3)$ and Grosswald for algebraic number fields (Cf. [33]). One also obtains a formula for the product $h_K R_K$ of the class number and the regulator. For example, if $K$ is totally real, the result is

(15) $$h_K R_K = 4(2\pi)^{-n} d_K \zeta_K(2) - 2^{3-n} \sqrt{d_K} \pi \sum_{u \in \delta_K^{-1} - 0} |u^{(1)}| \sigma_{-1}(u\delta_K) \times \exp\{- 2\pi(|u^{(1)}| + \dots + |u^{(n)}|)\} .$$

Here $\delta_K$ is the different, $d_K$ the absolute value of the discriminant, $n$ the degree,

$$\sigma_s(\mathfrak{U}) = \sum_{\mathfrak{b}|\mathfrak{U}} N\mathfrak{b}^s \quad , \text{ for an ideal } \mathfrak{U} \text{ of } K .$$

See [19] for some analogous results for L-functions.

3. Complete and incomplete Mellin transforms and the analytic continuation of Eisenstein series via Riemann's method of theta functions - incomplete gamma functions.

Let us first consider the simplest example of Riemann's analytic continuation of Eisenstein series via the method of theta functions, as it has been called in crystal physics (Cf. [4] p. 388). LAVRIK and MONTGOMERY (Cf. [13] and [18]) have recently used the method to obtain estimates for zeta functions at $s = \sigma + it$, for large $t$, for example. Another application of the method can be found in [1], where class numbers of pure cubic fields are computed. The method is applied to Dirichlet L-functions, for example, in [5] and [38]. Finally, the method is used in [27] and [28] to evaluate Artin L-functions and thus find class fields of totally real quadratic and cubic fields. Let us use Riemann's idea first for the Epstein zeta function $Z_1$ and then look at the complications that ensue when the method is used on Koecher's zeta function $Z_k$ in (2).

First define for $P \in \mathcal{P}_n$, $t > 0$,

(16) $$\theta_1(P^{(n)}, t) \equiv \sum_{a \in \underset{\sim}{Z}^n} \exp\{-\pi P[a]t\} .$$

Then Poisson summation yields the transformation formula

$$\theta_1(P, t) = |P|^{-\frac{1}{2}} t^{-(n/2)} \theta(P^{-1}, t^{-1}) .$$

It follows that

$$\Lambda_1(P, s) \equiv \pi^{-s} \Gamma(s) Z_1(P, s) = \frac{1}{2} \int_0^\infty t^{s-1} \{\theta(P, t) - 1\} dt = \int_0^1 + \int_1^\infty .$$

Replacing $t$ by $t^{-1}$ in the first integral, and using the transformation formula of the theta function, one sees easily that

(17) $$2\Lambda_1(P, s) = \frac{|P|^{-\frac{1}{2}}}{s - (n/2)} - \frac{1}{s} + \sum_{a \in Z^n - 0} \{G(s, \pi P[a]) + |P|^{-\frac{1}{2}} G(\tfrac{n}{2} - s, \pi P^{-1}[a])\},$$

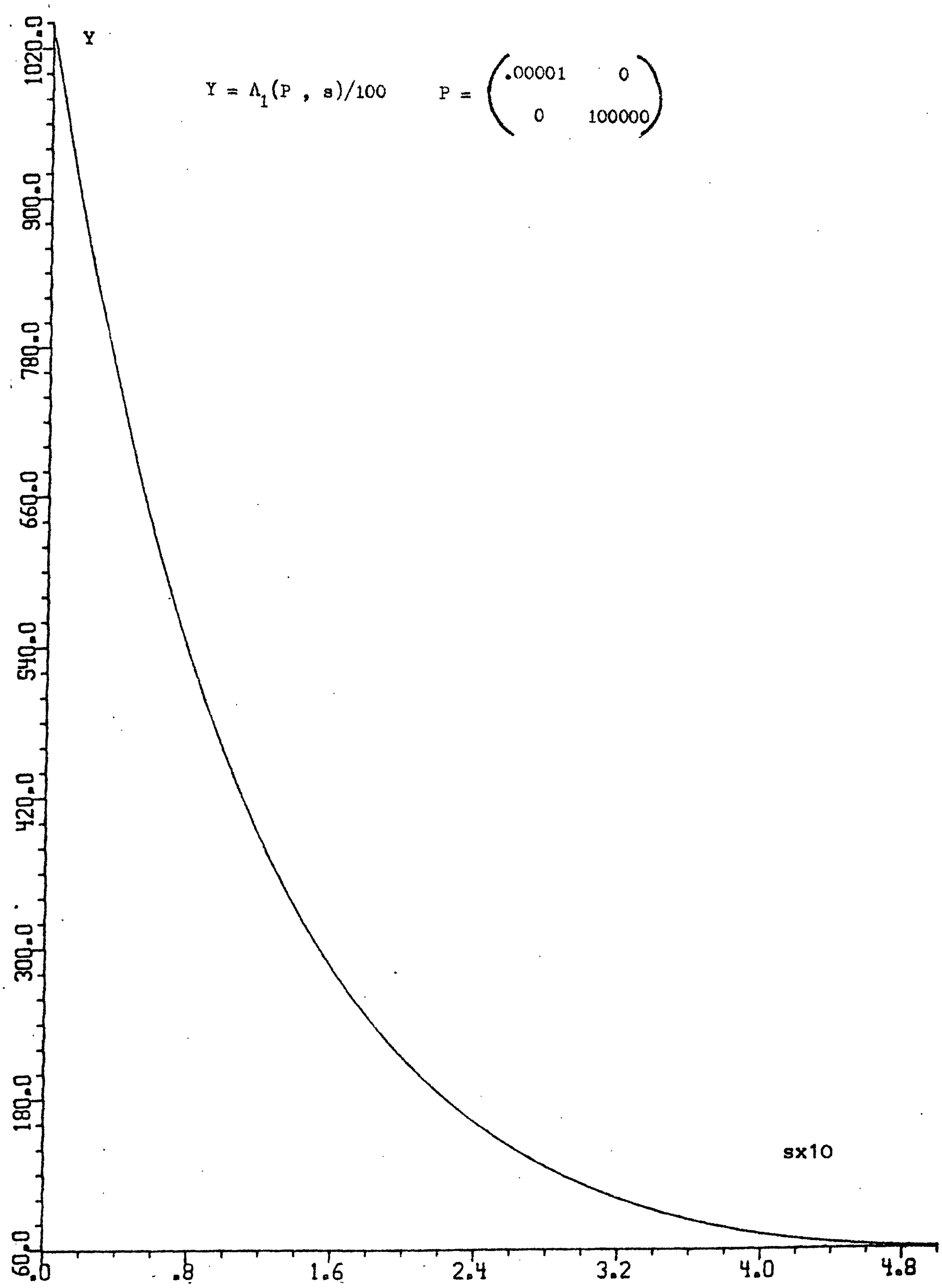
where the incomplete gamma function is defined by

(18) $$G(s, a) = \int_1^\infty t^{s-1} \exp(-at)\, dt , \text{ for } \operatorname{Re} a > 0 .$$

Since $G(s, a)$ dies exponentially as $|a|$ approaches infinity, convergence in (17) is exponentially faster than in the original Dirichlet series. Thus if one has a program to compute incomplete gamma functions, one can use (17) to compute Epstein's zeta function. Simple recursive procedures for the computation of $G(s,a)$ were developed by R. TERRAS in [39] and [40]. The programs are very short and would even fit into an HP-25. We used these programs to obtain graphs of $\Lambda_1(P, s)$ in [38] as well as graphs of L-functions of Kronecker symbols. A graph is included on the next page, showing $\Lambda_1(P, s)$ with

$$P = \begin{pmatrix} .00001 & 0 \\ 0 & 100000 \end{pmatrix}$$

and $s$ in $(0, 1/2)$. Note that $\Lambda_1(P, s)$ has a pole at $s = 0$ though it is not visible on the graph. Thus $Z_1$ has a zero very close to $s = 0$. Formula (17) will also allow us to produce contour maps of Epstein zeta functions for complex values of $s$, since complex argument incomplete gamma programs have already been developed by R. TERRAS. Such graphs would be interesting, since it is known that Epstein zeta functions of binary quadratic forms usually have other weird zeros besides the real ones we just found, i. e., zeros in $\operatorname{Re} s > 1$. Such results for higher degree forms appear to be open at the moment.




Another application of (17) is the deduction of the following relation between the size of the minimum of $P \in \mathcal{P}_n$ over the integer lattice and the sign of Epstein's zeta function. Suppose that $P$ is in $\mathcal{P}_n$ and set

(19) $$\ell_P = \min\{P[a] \mid a \in \mathbb{Z}^n - 0\} .$$

Then results of MINKOWSKI and BLICHFELDT (Cf. [6]) say that, for any $P \in \mathcal{P}_n$, $\ell_P < c_n |P|^{1/n}$, where $c_n$ is a positive constant asymptotic to $n/\pi e$ as $n \to \infty$. The theorem of Minkowski-Hlawka (Cf [41]) implies also that there exist $P \in \mathcal{P}_n$ such that

$$\ell_P > (n/2\pi e)|P|^{1/n} \text{ , if } n \text{ is sufficiently large.}$$

Using (17), it is easy to see that, if $P \in \mathcal{P}_n$ with $|P| = 1$ and $\ell_P < nu/2\pi e$ with $0 < u < 1$, then for $n$ sufficiently large (depending on $u$ and larger for $u$ nearer 1), $Z_1(P, nu/2) > 0$. It follows that $Z_1(P, s) = 0$ for some $s \in (nu/2, n/2)$. The details are in [37].

Thus if the Riemann hypothesis holds for the Dedekind zeta function of a number field then the quadratic forms appearing in Hecke's integral formula ([7], p. 198 ff) must (mostly) have large minima over the integer lattice.

Finally, note that (17) can be used to derive Hecke's integral formula for the analytic continuation of the Dedekind zeta function $\zeta_K(s)$ via higher dimensional incomplete gamma functions (Cf. [7], [11], [24]). First set

(20) $$\Lambda_K(s) \equiv (2^{-r_2} \sqrt{d_K}\, \pi^{-n/2})^s\, \Gamma(\tfrac{s}{2})^{r_1}\, \Gamma(s)^{r_2}\, \zeta_K(s) .$$

Then Hecke's integral formula ([7], p. 198 ff) says

(21) $$\frac{w_K}{2^{r_1}\, nR_K} \Lambda_K(s) \equiv \sum_{C \in I_K} \int_{[-\frac{1}{2},\frac{1}{2}]^r} \Lambda_1(P^{\circ}_{\alpha_C, x}\, , ns/2)\, dx .$$

Here $n$ is the degree, $I_K$ the ideal class group, $R_K$ the regulator, $w_K$ the number of roots of 1, $d_K$ the absolute value of the discriminant, $r_1$ the number of real conjugate fields, $r_2$ the number of pairs of complex conjugate fields, $r = r_1 + r_2 - 1$. The fundamental units will be $\varepsilon_1, \dots, \varepsilon_r$. For an ideal class $C$ take an ideal $\mathfrak{U}$ in $C$ with $\mathfrak{U} = \underset{\sim}{Z}\alpha_1 \oplus \dots \oplus \underset{\sim}{Z}\alpha_n$. Then the notation $P^{\circ}_{\alpha_C, x}$ is explained as follows :

(22) $$\begin{cases} \tau_j = \prod_{i=1}^{r} |\varepsilon_i^{(j)}|^{2x_i} \\ P\{A\} = {}^tAPA \ , \quad P^{\circ} = |P|^{-1/n}\, P \\ P_{\alpha_C, x} = \begin{pmatrix} \tau_1 & & 0 \\ & \ddots & \\ 0 & & \tau_n \end{pmatrix} \{\alpha_j^{(i)}\} . \end{cases}$$

It is then not hard to see that (17) implies

(23) $$\Lambda_K(s) = \frac{\lambda}{s(s-1)} + F_K(s) \text{ , where } \lambda = 2^{r_1} h_K R_K / w_K .$$

And $F_K$ is the following sum of higher dimensional incomplete gamma functions.

(24) $$F_K(s) = \sum_{0 \neq \mathfrak{b} \text{ ideals}} \{ G_K(\tfrac{s}{2}, \pi(\tfrac{N\mathfrak{b}^2}{d_K})^{1/n}) + G_K(\tfrac{1-s}{2}, \pi(\tfrac{N\mathfrak{b}^2}{d_K})^{1/n}) \} ,$$

where

(25) $$G_K(s\ ,\ a) \equiv \int_{y\in(\underset{\sim}{R}^+)^{r_1+r_2}} Ny^s \exp\{-\ aTry\} \frac{dy}{y} .$$

Again the series for $F_K(s)$ converges exponentially faster than the original Dirichlet series for $\zeta_K(s)$ . For totally real fields $K$ , the incomplete gamma function defined by (25) is a special case of the following for $s$ , $a$ in $\underset{\sim}{C}^n$ with $\operatorname{Re} a > 0$ :

(26) $$G_n(s\ ,\ a) \equiv \int_{\prod y_j \geq 1, y_j > 0} \prod_{j=1}^{n} y_j^{s_j - 1} \exp(-\ a_j\ y_j)\ dy_j .$$

Note that LAVRIK (Cf. [13]) has obtained analogous expansions in general for Dirichlet series satisfying functional equations with multiple gamma factors, and used them to study the growth of the Dedekind zeta function with $|s|$ . The asymptotic behaviour of $G_K(s\ ,\ a)$ as $|a| \to 0$ is crucial for the Brauer-Siegel theorem on the growth of the product of the class number and the regulator with the absolute value of the discriminant. In particular,

(27) $$G_n(s\ ,\ a) \sim \prod_{j=1}^{n} a_j^{-s_j}\ \Gamma(s_j)\ ,\ \text{as}\ \ a \to 0\ ,\ \ a > 0\ ,$$

is easily proved, as well as a similar result for (25). Thus it appears useful to study the higher dimensional incomplete gamma functions. One can even compute some of them. For example, STARK computed some 3-dimensional incomplete gamma functions in [28] in order to find the Hilbert class field of a certain totally real cubic field (non abelian), obtained by adjoining a root of $x^3 - x^2 - 9x + 8 = 0$ to the rationals.

It is perhaps surprising to learn that the $G_n(s\ ,\ a)$ are actually special functions for $\mathcal{P}_n$ analogous to the K-Bessel functions defined by (12). Suppose that $A \in \mathcal{P}_n$ and $s \in \underset{\sim}{C}^n$ . Define the <u>incomplete gamma function of matrix argument</u> by

(28) $$I_n(s\ ,\ A) \equiv \int_{X\in\mathcal{P}_n, |X|\geq 1} \psi_s(X) \exp\{-\ Tr(AX)\} \frac{dX}{|X|^{(n+1)/2}} .$$

where $\psi_s(X)$ is the (right-) spherical function (Cf. [22]) (6)

(29) $$\psi_s(X) \equiv \prod_{j=1}^{n} |X_j|^{s_j} \ \text{ when }\ X = \begin{pmatrix} X_j^{(j)} & * \\ * & * \end{pmatrix} .$$

Making the change of variables $X = {}^tTT$ with $T$ upper triangular, one sees that

$$\left|\frac{\partial X}{\partial T}\right| = 2^n\ t_1^n\ t_2^{n-1}\ \ldots\ t_n$$

and thus

(30) $$I_n(s\ ,\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}) = \pi^{n(n-1)/4} \prod_{j=1}^{n} a_j^{(1-j)/2}\ G_n(f(s)\ ,\ a)\ ,$$

---

(6) Incidentally, $\psi_s$ is eigen for all the $Gl(n\ ,\ \underset{\sim}{R})$-invariant differential operators on $\mathcal{P}_n$ (Cf. [15], p. 69) and thus can be used to form Eisenstein series of more than one complex variable. These are studied in [22][12][15][29][30] for example.

if $f(s)_j = s_j + \dots + s_n + (1 - j)/2$ .

Before proceeding to the consideration of the Eisenstein series $Z_k$ , there is one more application of the preceeding to algebraic number theory. Let $x \in (\underset{\sim}{R}^+)^{r_1+r_2}$. Then from results of [35], it follows that

$$(31)\qquad \frac{2^{r_1+1}\, h_K\, R_K}{w_K}\left\{\frac{Nx + Nx^{-1}}{Nx - Nx^{-1}}\log(Nx) - 1\right\} + 2F_K(1)$$

$$= 2^{r_1+r_2}\sum_{0\neq u\in\delta_K^{-1}} \sigma_0(u\delta_K)\left\{\frac{Nx^{-1}\, \underset{\sim}{K}_0(2\pi x^{-2}\,|u|) - Nx\, \underset{\sim}{K}_0(2\pi x^{2}\,|u|)}{Nx - Nx^{-1}}\right\},$$

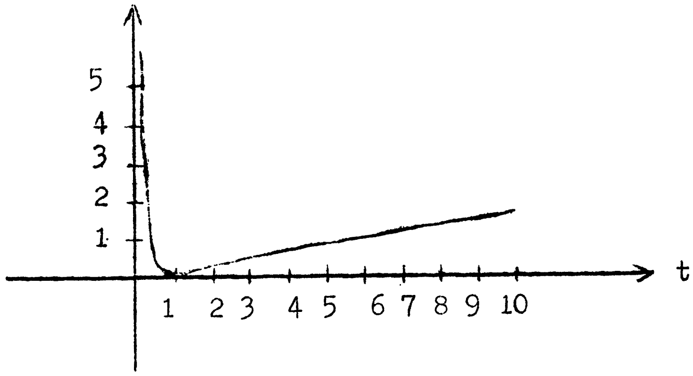
where $|u| = (|u^{(1)}| , \dots , |u^{(r_1+r_2)}|)$ , and

$$\underset{\sim}{K}_s(x) \equiv \prod_{j=1}^{r_1} K_{s_j}(x_j)\prod_{j=r_1+1}^{r_1+r_2} K_{2s_j}(2x_j) .$$

The graph of

$$f(t) = \left\{\frac{t + t^{-1}}{t - t^{-1}}\log t\right\} - 1$$

looks like




One might hope to be able to choose $x$ correctly in order to obtain arithmetical information about $K$ .

Let us next seek to generalize the method of proof of (17) to obtain the analytic continuation of the Eisenstein series $Z_k(P^{(n)}, s)$ defined by (2). In general, we need to consider the question : How does one obtain the analytic continuation of a Dirichlet series corresponding to an automorphic form by (higher dimensional) Mellin transform ? We want to apply the theory not just to the forms $f$ satisfying (1) but also to theta functions which are restrictions of holomorphic forms for $Sp(n)$ . It would be interesting to consider the relation between automorphic forms for $Sp(n)$ and automorphic forms for $Gl(n)$ in some detail, but we shall not do this here. It would also be interesting to try to invert the Mellin transform. In the one variable case, that is essentially Fourier inversion. Thus in the higher dimensional case, one would expect to need Fourier inversion for symetric spaces as in [8].

Now let us consider the problem of obtaining the analytic continuation of the Eisenstein series $Z_k$ . The first observation is that $Z_k$ is essentially the

Mellin transform of the rank k terms in the theta function

(32) $$\theta(P^{(n)}, X^{(k)}) \equiv \sum_{A^{(n,k)}\in \underset{\sim}{Z}^{n\times k}} \exp\{-\pi \mathrm{Tr}(P[A]X)\} .$$

The transformation formula

(33) $$\theta(P^{(n)}, X^{(k)}) = |P|^{-k/2} |X|^{-n/2} \theta(P^{-1}, X^{-1})$$

is easily proved via the Poisson summation formula (Cf. [11]). Now let us denote by $\theta_r$ the rank r part of theta :

(34) $$\theta_r(P^{(n)}, X^{(k)}) \equiv \sum_{N\in \underset{\sim}{Z}^{n\times k}, \mathrm{rank}(N)=r} \exp\{-\pi \mathrm{Tr}(P[A]X)\} .$$

Then, for $1 \leqslant k \leqslant n$, set

(35) $$\Lambda_k(P^{(n)}, s) \equiv \pi^{k((k-1)/4-s)} \Gamma_k(s)\, Z_k(P^{(n)}, s) ,$$

with $\Gamma_k(s)$ as defined by (13). Then one has :

(36) $$2\,\Lambda_k(P^{(n)}, s) = \int_{X\in\mathfrak{F}_k} |X|^{s-(k+1)/2}\, \theta_k(P^{(n)}, X^{(k)})\, dX .$$

Here $\mathfrak{F}_k$ denotes a fundamental domain for $\mathcal{P}_k$ modulo $Gl(k, \underset{\sim}{Z})$, e. g. the convex cone of Minkowski-reduced matrices in [15](§ 9). Formula (36) says that $2\Lambda_k(P, s)$ is the k-dimensional Mellin transform of theta. Note that, when $k = 1$, you are just integrating over $\underset{\sim}{R}^+$. When $k > 1$, however, one must integrate over the complicated domain $\mathfrak{F}_k$ in order to compensate for the difference between the sum giving theta and the sum giving the Eisenstein series. And one has to take the Mellin transform of

$$\theta_k = \theta - \sum_{r=0}^{k-1} \theta_r \quad \text{and not just} \quad \theta - \theta_0 .$$

In fact, the Mellin transform of $\theta_r$ diverges for $0 \leqslant r < k$. And we shall soon see that even worse things happen. Replacing $|X|^s$ in (36) by $\psi_s(X)$ as in (29), and $\mathfrak{F}_k$ by the larger fundamental domain of $\mathcal{P}_k$ modulo the parabolic subgroup of $Gl(n, \underset{\sim}{Z})$ leaving $\psi_s$ invariant, one obtains Eisenstein series in several variables as Mellin transforms of theta.

Now let us attempt to generalize Riemann's method of analytic continuation which gave (17). In [10] and [25](I, p. 459-468) one attempted to imitate Riemann by pulling out the determinant of X and breaking the integral up into the region where $|X|$ is greater than one and the rest. Thus one obtained

$$\int_{X\in\mathfrak{F}_k} |X|^{s-(k+1)/2}\, \theta_k(P,X)\, dX = \int_{t=0}^{\infty} t^{s-1} \int_{\substack{W\in\mathfrak{F}_k\\ |W|=1}} \theta_k(P, t^{1/k} W)\, dW\, dt$$

$$= \int_{t=1}^{\infty} t^{s-1} \int_{\substack{W\in\mathfrak{F}_k\\ |W|=1}} \theta_k(P, t^{1/k} W)\, dW\, dt + \int_{t=1}^{\infty} t^{-s-1} \int_{\substack{W\in\mathfrak{F}_k\\ |W|=1}} \theta_k(P, t^{-1/k} W)\, dW\, dt .$$

Using the transformation formula of theta, it follows that

(37) $2\Lambda_k(P^{(n)}, s)$

$$= \int_{t=1}^{\infty} t^{-1} \int_{\substack{W\in\mathfrak{F}_k \\ |W|=1}} \{t^s\, \theta_k(P, t^{1/k} W) + |P|^{-k/2}\, t^{(n/2)-s}\, \theta_k(P^{-1}, t^{1/k} W^{-1})\}\, dW\, dt$$

$$+ \int_{t=1}^{\infty} t^{-1-s} \int_{\substack{W\in\mathfrak{F}_k \\ |W|=1}} \sum_{r=1}^{k-1} \{|P|^{-k/2}\, t^{n/2}\, \theta_r(P^{-1}, t^{1/k} W^{-1}) - \theta_r(P, t^{-1/k} W)\}\, dW\, dt$$

$$+ \int_{t=1}^{\infty} t^{-1-s} \{t^{n/2}\, |P|^{-k/2} - 1\}\, dt \int_{\substack{W\in\mathfrak{F}_k \\ |W|=1}} dW .$$

The first two integrals converge for all $s$ by the convergence of (2). And the last integral is obviously

$$v(k)\, \{\frac{|P|^{-k/2}}{s - (n/2)} - \frac{1}{s}\}$$

where

(38) $$v(k) \equiv \int_{W\in\mathfrak{F}_k, |W|=1} dW = c_k \operatorname{vol}(Sl(n, \mathbb{R})/Sl(n, \mathbb{Z})) .$$

Here $c_k$ denotes a positive constant.

Now one would hope to do something with the second integral. We know that it has to be finite for $\operatorname{Re} s > n/2$. If indeed it has no pole at $s = n/2$ then setting $k = n$ and using (3) yields the Minkowski formula for $v(k)$

(39) $$v(k) = \pi^{(2-k(k+1))/4} \prod_{j=2}^{k} \zeta(j)\, \Gamma(\tfrac{j}{2}) .$$

Unfortunately, however,

(40) $$S_r(P, s) \equiv \int_{t=1}^{\infty} t^{s-1} \int_{W\in\mathfrak{F}_k, |W|=1} \theta_r(P, t^{1/k} W)\, dW\, dt$$

is <u>infinite</u> if $1 \leqslant r < k$ for all values of $s$. This can be proved using two facts. First, one needs a decomposition of the rank $r$ integral matrices $N^{(n,k)}$, and second one needs a generalization of the theorem of Minkowski-Hlawka (Cf. [41]). We shall state these two results since they appear to be promising for future work on Eisenstein series for $Gl(n)$. One might also expect that there exist generalizations to other Lie groups.

The <u>decomposition theorem</u> for rank $r$ matrices $N \in \mathbb{Z}^{n\times k}$ says : $N$ can be uniquely written as

(41) $N = B\, {}^tA$,

where $(A^{(k,r)}*) \in Sl(k, \mathbb{Z})/H(r, \mathbb{Z})$, $B^{(n,r)} \in \mathbb{Z}^{n\times r}/Gl(r, \mathbb{Z})$, $rk(B) = r$.

Here $H(r, \mathbb{Z})$ denotes the subgroup of $Sl(k, \mathbb{Z})$ of matrices of the form

$$\begin{pmatrix} I^{(r)} & * \\ 0 & * \end{pmatrix} .$$

The decomposition (41) is proved easily using elementary divisor theory.

The generalization of the Minkowski-Hlawka theorem which we need says that for suficiently nice functions $f : \mathbb{R}^{k\times r} \longrightarrow \mathbb{C}$ $(k > r)$ there is a constant $c_{k,r} > 0$ (independent of $f$) such that

$$(42) \quad c_{k,r} \int_{Sl(k,\mathbb{R})/Sl(k,\mathbb{Z})} \sum_{(A^{(k,r)*})\in Sl(k,\mathbb{Z})/H(r,\mathbb{Z})} f(gA)\, d\bar{g} = \int_{\mathbb{R}^{k\times r}} f(X)\, dX .$$

The proof of (42) goes exactly as in [41]. For $f(X) = g({}^tXX)$ one would expect to say something nicer. Formula (42) can be found in [17].

Now, let us examine $S_r(P^{(n)}, s)$ with $1 \leqslant r \leqslant k-1$. Using (41) and (42), we find that

$$S_r(P,s) = \int_{t=1}^{\infty} t^{s-1} \int_{\substack{W\in\mathcal{P}_k\\ |W|=1}} \Big( \sum_{\substack{(A^{(k,r)*})\in Sl(k,\mathbb{Z})/H(r,\mathbb{Z})\\ B^{(n,r)}\in\mathbb{Z}^{n\times r}/Gl(r,\mathbb{Z}),\, \mathrm{rank} B=r}} \exp\{-\pi t^{1/k}\, \mathrm{Tr}(P[B]\, W[A])\}\Big)\, dW\, dt$$

$$= c_{k,r} \int_{t=1}^{\infty} t^{s-1} \int_{\mathbb{R}^{k\times r}} \Big( \sum_{\substack{B^{(n,r)} \mathrm{mod} Gl(r,\mathbb{Z})\\ \mathrm{rank}\ B=r}} \exp\{-\pi t^{1/k}\, \mathrm{Tr}(P[B]\, {}^tUU)\}\Big)\, dU\, dt .$$

Thus letting $R$ be a positive symmetric matrix such that $R^2 = t^{1/k}\, P[B]$ and then changing variables in the inner integral via $V = UR$, one obtains

$$\left|\frac{\partial U}{\partial V}\right| = |R|^{-k} = t^{-r/2}\, |P[B]|^{-k/2} .$$

Since the Dirichlet series defining the Eisenstein series in (2) diverges at $s = k/2$ by comparison with a product of Epstein zeta function (Cf. [10], p. 7), one sees that $S_r(P^{(n)}, s)$ is infinite.

Thus we conclude that cancellation must be occuring in the second integral in (37). There are several possible directions in which one can proceed in order to deal with this problem, as we mentioned at the end of § 1. However, none of these methods appear to lead to a simple result for $Z_k$ unless $k = 1$ (See [12], p. 260-266, for example). Thus it appears useful to attempt to use (41) and (42) to obtain a direct evaluation of the 3rd integral in (37). Perhaps one should attempt to break up the integral in (36) differently. For example, one can show that

$$\int_{X\in\mathcal{P}_k,\, \ell_X \geq |X|^{1/k}} |X|^{s-(k+1)/2}\, \theta_1(P, X)\, dX = c_k \frac{\Lambda_1(P, ks)}{s-(1/2)} ,$$

where $c_k > 0$ depends only on $k$. Here $\ell_X$ is defined by (19). Note that if $n = k = 2$ then the above integral has a double pole at $s = 1/2$, which is the correct behaviour of $\Lambda_2(P^{(2)}, s)$ at $s = 1/2$ by (3). We must unfortunately leave the reader dangling at this point and put off the complete solution to the problem of finding a direct analytic continuation of these Eisenstein series at least until the spring snows in Europe cease.

Our final remarks concern one-dimensional Mellin transforms of Epstein zeta functions. For example, write

(43) $$c_s(T\ ,\ V) = \Lambda_1(V\ ,\ s) + |V|^{-\frac{1}{2}}\ \Lambda_1(T\ ,\ s - \tfrac{m}{2})$$

for the constant term in the Fourier expansion (7). Then

$$\int_{t=0} t^{s-1}\ \{\Lambda_r(\begin{pmatrix} tT & 0 \\ 0 & t^{-1}V \end{pmatrix}\ ,\ \alpha) - c_\alpha(tT\ ,\ t^{-1}\ V)\}\ dt$$

$$= |V|^{-\frac{1}{2}} \sum_{\substack{a\in Z^k-0 \\ b\in \tilde{Z}^m-0}} \int_{t=0} t^{s-1+(m/2)} \int_{u=0} u^{\alpha-(m/2)-1}\ \exp\{-\ \pi T[a]\ ut\ -\ \pi V^{-1}[b]\ \tfrac{t}{u}\}\ du\ dt$$

$$= \tfrac{1}{2}\ \Lambda_1(T\ ,\ \tfrac{\alpha + s}{2})\ \Lambda_1(V\ ,\ \tfrac{\alpha - s}{2})\ ,$$

upon changing variables in the double integral via $y = t/u$ , $z = ut$ .

Now apply Riemann's trick to this Mellin transform and obtain

$$\tfrac{1}{2}\ \Lambda_1(T\ ,\ \tfrac{\alpha + s}{2})\ \Lambda_1(V\ ,\ \tfrac{\alpha - s}{2})$$

$$= |V|^{-\frac{1}{2}} \sum_{\substack{a\neq 0 \\ b\neq 0}} G_2(\tfrac{\alpha + s}{2}, \tfrac{m - \alpha + s}{2}, \pi T[a], \pi V^{-1}[b]) + \frac{\Lambda_1(T,\alpha)}{s - \alpha} + \frac{\Lambda_1(T,\alpha - \frac{1}{2})|V|^{-\frac{1}{2}}}{m - \alpha + s}$$

+ the same with $V$ and $T$ exchanged and $-s$ for $s$ .

Here $G_2$ denotes the higher dimensional incomplete gamma function defined by (26). Such a result would also follow from Lavrik's general theory [13]. In the special case that $k = m = 1$ and $V = T = 1$ , $\alpha = 0$ one obtains the following formula for the Riemann zeta function (involving only the 1-dimensional incomplete gamma function $G_1 = G$ ) $(\Lambda_1(1,s/2) = \pi^{-s/2}\ \Gamma(s/2)\ \zeta(s))$

(44) $$\tfrac{1}{2}\ \Lambda_1(1\ ,\ \tfrac{s}{2})\ \Lambda_1(1\ ,\ -\tfrac{s}{2})$$

$$= 4 \sum_{a\geq 1, b\geq 1} b^{-1}\ \{G(s\ ,\ 2\pi ab) + G(-s\ ,\ 2\pi ab)\} - \frac{1}{s^2} + \frac{\pi}{3(1 - s)(1 + s)}\ .$$

Such results should be useful for the study of such Eisenstein series and admit many possible generalizations, e. g. to L-functions and to the other Eisenstein series, taking higher dimensional Mellin transforms.

## REFERENCES


[1] BARRUCAND (P.), WILLIAMS (H.), and BANIUK (L.). - A computational technique for determining the class number of a pure cubic field, Math. of Comp., t. 30, 1976, p. 312-323.

[2] BAILY (W.). - Introductory lectures on automorphic forms. - Princeton, Princeton University Press, 1973 (Publications of the Mathematical Society of Japan, 12 ; Kano Memorial Lectures, 2).

[3] BOREL (A.). - Reduction theory for arithmetic groups ; and Introduction to automorphic forms, "Algebraic groups and discontinuous subgroups", p. 20-25, and p. 199-210. - Providence, American mathematical Society, 1966 (Pro-

ceedings of Symposia in pure Mathematics, 9).

[4] BORN (M.) and HUANG (K.). - Dynamical theory of crystal lattices. - London, Oxford University Press, 1956.

[5] DAVENPORT (H.). - Multiplicative number theory. - Chicago, Markham, 1967 (Lectures in advanced Mathematics, 1).

[6] DAVENPORT (H.). - Stanford lectures on selected topics in the geometry of numbers, Stanford University, 1950.

[7] HECKE (E.). - Mathematische Werke. - Göttingen, Vandenhoeck und Ruprecht, 1959.

[8] HELGASON (S.). - Lie groups and symmetric spaces, "Battelle Rencontres, 1967, Lectures in Mathematics and physics", p. 1-71. - New York, W. A. Benjamin, 1968.

[9] HERZ (C. S.). - Bessel functions of matrix argument, Annals of Math., Series 2, t. 61, 1955, p. 474-523.

[10] KOECHER (M.). - Über Dirichlet-Reihen mit Funktionalgleichung, J. reine und angew. Math., t. 192, 1953, p. 1-23.

[11] LANG (S.). - Algebraic number theory. - Reading, Addison-Wesley, 1970.

[12] LANGLANDS (R.). - On the functional equations satisfied by Eisenstein series. - Berlin, Springer-Verlag, 1976 (Lecture Notes in Mathematics, 544).

[13] LAVRIK (A. F.). - Approximate functional equations for Dirichlet functions, Math. U. S. S. R.-Isvestija, t. 2, 1968, p. 129-179.

[14] LAVRIK (A. F.). - An approximate functional equation for the Dirichlet L-function, Trans. Moscow Math. Soc., t. 18, 1968, p. 101-115.

[15] MAASS (H.). - Siegel's modular forms and Dirichlet series. - Berlin, Springer-Verlag, 1971 (Lecture Notes in Mathematics, 216).

[16] MAASS (H.). - On modular functions of one complex variable. - Bombay, Tata Institute, 1964 (Tata Institute of fundamental Research. Lectures on Mathematics, 29).

[17] MACBEATH (A. M.) and ROGERS (C. A.). - Siegel's mean value theorem in the geometry of numbers, Proc. Camb. phil. Soc., t. 54, 1958, p. 139-151.

[18] MONTGOMERY (H.). - Adam's Prize Essay.

[19] MORDELL (L. J.). - On Hecke's modular functions, zeta functions, and some other analytic functions in the theory of numbers, Proc. London math. Soc., Series 2, t. 32, 1931, p. 501-556.

[20] RESNIKOFF (H.). - Automorphic forms of singular weight are singular forms, Math. Annalen, t. 215, 1975, p. 173-193.

[21] RESNIKOFF (H.). - Stable spaces of modular forms (to appear).

[22] SELBERG (A.). - Harmonic analysis and discontinuous groups in weakly symmetric riemannian spaces with applications to Dirichlet series, J. Indian math. Soc., t. 20, 1956, p. 47-87.

[23] SIEGEL (C. L.). - Lectures on advanced analytic number theory. - Bombay, Tata Institute, 1957 (Tata Institute of fundamental Research. Lectures in Mathematics, 23).

[24] SIEGEL (C. L.). - Analytische Zahlentheorie. - Göttingen, Mathematisches Institut, 1963/64 (Universität Göttingen).

[25] SIEGEL (C. L.). - Gesammelte Abhandlungen. - Berlin, Springer-Verlag, 1966.

[26] STARK (H.). - On the problem of unique factorization in complex quadratic fields, "Number theory", p. 41-56. - Providence, American mathematical Society, 1969 (Proceedings of Symposia in pure Mathematics, 12).

[27] STARK (H.). - Values of L-functions at $s = 1$, III : Totally real fields and Hilbert's 12th problem, Advances in Math., t. 22, 1976, p. 64-84.

[28] STARK (H.). - Hilbert's 12th problem and L-series (to appear).

[29] TERRAS (A.). - A generalization of Epstein's zeta function, Nagoya math. J., t. 42, 1971, p. 173-188.

[30] TERRAS (A.). - Functional equations of generalized Epstein zeta functions in several variables, Nagoya math. J., t. 44, 1971, p. 89-95.

[31] TERRAS (A.). - Bessel series expansions of the Epstein zeta function and the functional equation, Trans. Amer. math. Soc., t. 183, 1973, p. 477-486.

[32] TERRAS (A.). - Fourier coefficients of Eisenstein series of one complex variable for the special linear group, Trans. Amer. math. Soc., t. 205, 1975, p. 97-114.

[33] TERRAS (A.). - Some formulas for the Riemann zeta function at odd integer argument resulting from Fourier expansions of the Epstein zeta function, Acta Arithm., Warszawa, t. 29, 1975, p. 181-189.

[34] TERRAS (A.). - The Fourier expansion of Epstein's zeta function for totally real algebraic number fields and some consequences for Dedekind's zeta function, Acta Arithm., Warszawa, t. 30, 1976, p. 187-197.

[35] TERRAS (A.). - The Fourier expansion of Epstein's zeta function over an algebraic number field and its consequences for algebraic number theory, Acta Arithm., Warszawa, t. 32, 1976, p. 37-53.

[36] TERRAS (A.). - A relation between $\zeta_K(s)$ and $\zeta_K(s-1)$ for any algebraic number field $K$, "Algebraic number fields ( L-functions and Galois properties) [1975. Durham], p. 475-483. - London, Academic Press, 1977.

[37] TERRAS (A.). - On a relation between the size of the minima of quadratic forms and the behaviour of Epstein and Dedekind zeta functions in the real part of the critical strip (to appear).

[38] TERRAS (A.), PURDY (G.), TERRAS (R.), WILLIAMS (H.). - Graphing L-functions of Kronecker symbols in the real part of the critical strip (to appear).

[39] TERRAS (R.). - On the convergence of the continued fraction for the incomplete gamma function and an algorithm for the Riemann zeta function (to appear).

[40] TERRAS (R.). - Determination of incomplete gamma functions by recursion in the argument (to appear).

[41] WEIL (A.). - Sur quelques résultats de Siegel, Summa Bras. Math., t. 1, 1946, p. 21-39.

A reference on application of special functions for $Gl(n)$ in statistics is

[42] JAMES (A. T.). - Special functions of matrix and single argument in statistics, "Theory and applications of special functions", p. 497-520. - New York, Academic Press, 1975.

(Texte reçu le 13 juin 1977)

Audrey TERRAS

Mathematics Department
University of California at San Diego
LA JOLLA, Calif. 92093 (Etats-Unis)

and

Mathematisches Institut der Universität Bonn
Wegelerstrasse 10,
D-5300 BONN (Allemagne fédérale)