Séminaire Delange-Pisot-Poitou.
Théorie des nombres

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Noncontinuable power series

Séminaire Delange-Pisot-Poitou. Théorie des nombres, tome 15, nº 1 (1973-1974), exp. nº 16, p. 1-7

<http://www.numdam.org/item?id=SDPP_1973-1974__15_1_A12_0>

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(Théorie des nombres)
15e année, 1973/74, n° 16, 7 p.

11 mars 1974

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by Rolf WALLISSER

1. Introduction.

In the last decade, there are many results on the classical problem to determine the global behaviour of a function, given by a power series, from certain properties of the coefficients of the series. It was especially one conjecture which occupied several authors :

Let $g \in \mathbb{C}[x]$ and $\Phi \in \mathbb{R}[x]$ be polynomials of degree greater than zero with complex respectively real coefficients. Let $[x]$ denote the largest integer which does not exceed x .

$$G(z) = \sum_{n=0}^{\infty} g([\Phi(n)])\ z^n$$

has $|z| = 1$ as a natural boundary if, and only if, at least one of the coefficients of $\Phi(x) - \Phi(0)$ is irrational.

The following authors solved the problem in special cases : First HECKE [3] showed, that

$$G(z) = \sum_{n=0}^{\infty} [\alpha n] z^n$$

is noncontinuable over the circle of convergence if, and only if, α is irrational. SALEM [5] generalized this, and proved the conjecture in the case

$$G(z) = \sum_{n=0}^{\infty} [\Phi(n)] z^n \ .$$

NEWMAN, MEIJER, POPKEN, CARROLL and KEMPERMAN, CANTOR and SCHWARZ were occupied with the irrationality of G . (For detailed references, see CARROLL and KEMPERMAN [2] or SCHWARZ [6].) In 1970, I could prove a general theorem on power series whose coefficients are uniformly distributed (WALLISSER [8]), and with the aid of this result, SCHWARZ and myself answered the problem mentioned above in the affirmative.

2. Noncontinuable power series and uniform distribution of coefficients.

The following result of WIENER is the main tool in the proof of our theorem :

LEMMA 1 (WIENER [9]). - Let $\Psi(z) = \sum_{n=1}^{\infty} a_n z^n$ have the unit circle as circle of convergence. Suppose that, for each integer $p \geqslant 0$, the limit

$$b_p = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_{n+p}\ \overline{a}_n$$

exists. If

$$A(\theta) = b_0\ \theta + \lim_{N\to\infty, \|\ \|_2} \sum_{|n|\leq N, n\neq 0} \frac{1}{in}\ b_n \exp in\theta \ , \quad (b_{-n} = \overline{b}_n) \ ,$$

is strictly monotone, then Ψ has $|z| = 1$ as a natural boundary.

We use this lemma to derive the following.

THEOREM 1. - Let f be a Riemann integrable function, not equivalent to a constant function. $(x_n)_N$ is a sequence of real numbers, such that, for each integer $p > 1$, $(\{x_n\}, \{x_{n+p}\})_N$ is uniformly distributed in the unit square. Then

$$f(z) = \sum_{n=0}^{\infty} (f(\{x_n\}) + o(1))z^n$$

has the unit circle as a natural boundary.

Proof. - We show, that

$$H(z) = F(z) - F(0) - \frac{a}{1-z} + a , \quad a = \int_0^1 f(x)\,dx ,$$

is noncontinuable beyond its circle of convergence. To apply lemma 1, we have to state the existence of

$$b_p = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (f(\{x_{k+p}\}) + o(1) - a)(\overline{f}(\{x_k\}) + o(1) - \overline{a}) .$$

With regard to the uniform distribution of the sequence $(\{x_{k+p}\}, \{x_k\})_N$ for each $p \geqslant 1$, we obtain, by a well known result of H. WEYL on uniform distribution of sequences,

$$b_p = (\int_0^1 f(x)\,dx - a)(\int_0^1 \overline{f}(x)\,dx - \overline{a}) = 0 , \quad p \geqslant 1 ,$$

$$b_0 = \int_0^1 |f(x) - a|^2\,dx .$$

b_0 is strictly positive because of the assumption that f is not equivalent to a constant function. The theorem is therefore an immediate consequence of lemma 1.

With the aid of this theorem, we can prove an analog onto our conjecture mentioned in the introduction.

THEOREM 2. - Let $g \in \mathbb{C}[x]$ be a polynomial with complex coefficients :

$$g(x) = g_0 + g_1 x + \dots + g_m x^m , \quad g_m \neq 0 , \quad m \geqslant 1 .$$

Let $\Phi \in \mathbb{R}[x]$ be a polynomial with real coefficients :

$$\Phi(x) = \Phi_0 + \Phi_1 x + \dots + \Phi_k x^k , \quad \Phi_k \neq 0 , \quad k > 1 .$$

Then the following statement is true :

$$G(z) = \sum_{n=0}^{\infty} g(\{\Phi(n)\})z^n$$

has the unit circle as a natural boundary if, and only if, $\Phi(x) - \Phi(0)$ has an irrational coefficient.

Proof.

(a) It is a mere calculation to show, that the condition at least one of the coefficients $\Phi_1, \dots, \Phi_k$ is irrational is necessary for G being noncontinuable. In fact, using the Taylor-development, we have

$$(1) \qquad \Phi(nQ + q) - \Phi(q) = \sum_{\nu=1}^{k} Q^{\nu-1} n^{\nu}\left(\frac{Q}{\nu!} \Phi^{(\nu)}(q)\right) .$$

If all coefficients $\Phi_1, \dots, \Phi_k$ are rational, and if Q is a least common multiple of the divisors, (1) is an integer, and we can write

$$(2) \qquad G(z) = \sum_{q=0}^{Q-1} g(\{\Phi(q)\}) \sum_{n=0, n\equiv q(Q)}^{\infty} z^n = \frac{1}{1 - z^Q} \sum_{q=0}^{Q-1} z^q \, g(\{\Phi(q)\}) .$$

G is therefore a rational function.

The main result of theorem 2 is the opposite direction.

(b_1) Φ_1 irrational, $\Phi_2, \dots, \Phi_k$ rational.

Here we can apply the method which HECKE [3] used in proving the noncontinuability of the series $\sum_{n=0}^{\infty} (\{\alpha n\}) z^n$. In this way, we can show (see SCHWARZ [6]) : There is a dense set M on the unit circle, such that for every $\varphi \in M$

$$\lim_{r\to 1} (1 - r)\, G(re^{i\varphi}) \neq 0 .$$

Therefore each point of the unit circle is a singular point for G, and the convergence circle becomes a natural boundary for the function.

(b_2) At least one of the numbers $\Phi_2, \dots, \Phi_k$ is irrational. To use theorem 1, we have to show the uniform distribution of $(\{\Phi(n)\}, \{\Phi(n + p)\})_{\underset{\sim}{N}}$ for each integer $p > 1$.

Following H. WEYL, we have to prove that, for each pair $(\ell_1, \ell_2) \neq (0, 0)$ of integers, the derivative of

$$(3) \qquad \Psi_{\ell_1 \ell_2}(x) = \ell_1 \, \Phi(x) + \ell_2 \, \Phi(x + p)$$

has an irrational coefficient. Let Φ_ν be the irrational coefficient of Φ with greatest index. Because of our assumption, that at least one of the reals $\Phi_2, \dots, \Phi_k$ is irrational, we have $\nu \geq 2$. The coefficients of x^ν or $x^{\nu-1}$ are of the form

$$(4) \qquad (\ell_1 + \ell_2)\Phi_\nu + \ell_2 \binom{\nu+1}{1} \Phi_{\nu+1} \, p + \dots + \ell_2 \binom{k}{\nu} \Phi_k \, p^{k-\nu} ,$$

or

$$(5) \qquad (\ell_1 + \ell_2)\Phi_{\nu-1} + \ell_2 \binom{\nu}{1} \Phi_\nu \, p + \dots + \ell_2 \binom{k}{\nu - 1} \Phi_k \, p^{k-\nu+1} .$$

Therefore, if $\ell_1 + \ell_2 \neq 0$, (4) is irrational, and if $\ell_1 + \ell_2 = 0$, (5) is irrational. Since we have $\nu \geq 2$, $\Psi'_{\ell_1 \ell_2}$ has in any case at least one irrational coefficient. We can now apply theorem 1 with $x_n = \Phi(n)$ and $f = g$, and our result follows.

3. On power series with coefficients of the form $f([x_n])$.

Using a special version of Hadamards multiplication theorem, we can derive from theorem 2 a result which answers our initial problem in the affirmative.

THEOREM 3. - Let $(x_n)_{\underset{\sim}{N}}$ be a sequence of real numbers with the following properties :

(I) $x_n \leq x_{n+1}$ for $n \geq n_0$,

(II) $\lim_{n\to\infty} x_n = \infty$,

(III) $(\{x_{n+p}\}, \{x_n\})$ is uniformly distributed for $p \geq 1$.

Let $G : (0, \infty) \to \mathbb{C}$ be a complex valued function which satisfies :

(IV) G is continuously differentiable for $x \geq x_0$,

(V) There exists $\delta > 0$, such that $|G'(x)| \geq \delta$ for $x \geq x_0$,

(VI) Uniformly, for all θ with $|\theta| \leq 1$, we have

$$\frac{G'(x+\theta)}{G'(x)} = 1 + o(1) \quad \text{for} \quad x \to \infty ,$$

(VII) $\sum_{n=1}^{\infty} G(x_n) z^n$ has the unit circle as circle of convergence and only a finite number of singularities with absolute value 1 .

(VIII) $\sum_{n>x_0}^{\infty} \frac{1}{G'(x_n)} z^n$ has 1 as radius of convergence, and $z = 1$ is the only singular point on the unit circle.

Then it follows :

$$F(z) = \sum_{n>n_0'} G([x_n]) + O(|G'(x_n)|) z^n ,$$

$n_0' \geq \max(n_0, x_0)$, has the unit circle as a natural boundary.

For a proof, we use the following result of Hadamard :

LEMMA 2 (HADAMARD, see [1]). - Let

$$f(z) = \sum_{n=0}^{\infty} f_n z^n , \quad g(z) = \sum_{n=0}^{\infty} \frac{1}{a_n} z^n$$

denote power series with $\overline{\lim}|f_n|^{1/n} = 1$, $\lim|a_n|^{1/n} = 1$. Let $g(z)$ be analytic in any point $z_0 \neq 1$ on the unit circle. Then each singularity z_0 with $|z_0| = 1$ of $f(z)$ is also a singularity of

$$f_1(z) = \sum_{n=0}^{\infty} a_n f_n z^n .$$

Proof of theorem 3. - We assume, that the conditions (I)-(VIII) are satisfied for $n_0 = x_0 = 1$. This is no limitation, because we can take all the series beginning with the same power of z .

If F would be continuable, then

$$\Delta(z) = \sum_{n=1}^{\infty} G(x_n) z^n - F(z) = \sum_{n=1}^{\infty} c_n z^n \tag{6}$$

would be continuable too, and the same holds in consequence of lemma 2 for the series

$$D(z) = \sum_{n=1}^{\infty} \frac{c_n}{G'(x_n)} z^n = \sum_{n=1} d_n z^n . \tag{7}$$

Regarding the assumptions on G' , we find

$$\begin{aligned} G(x_n) - G([x_n]) &= \int_{[x_n]}^{x_n} G'(x)\, dx = \int_{[x_n]}^{x_n} G'(x_n)(1 + o(1))\, dx \\ &= \{x_n\} G'(x_n) + o(|G'(x_n)|) , \end{aligned} \tag{8}$$

and so we get, for the coefficients d_n,

$$d_n = \frac{1}{G'(x_n)} (G(x_n) - G([x_n]) + O(|G'(x_n)|))$$

(9) $$d_n = \{x_n\} + O(1) .$$

According to theorem 2, $D(z)$ would be noncontinuable, a contradiction which shows, that F must have the unit circle as a natural boundary.

As an application of this theorem, we can get the following results (for others, see [7]) :

THEOREM 4. - <u>If $j : \underset{\sim}{N} \dashrightarrow \underset{\sim}{C}$ is a generalized polynomial of the form</u>

$$j(n) = j_0 + j_1 n^{\alpha_1} + \dots + j_m n^{\alpha_m} ,$$

$$j_m \neq 0 , \quad m \geq 1 , \quad 0 < \alpha_1 < \alpha_2 < \dots < \alpha_m , \quad \alpha_m \geq 1 ,$$

<u>and $\omega : \underset{\sim}{N} \dashrightarrow \underset{\sim}{R}$ is a generalized real valued polynomial with</u>

$$\omega(n) = \omega_0 + \omega_1 n^{\beta_1} + \dots + \omega_r n^{\beta_r} ,$$

$$\omega_r \neq 0 , \quad r \geq 1 , \quad 0 < \beta_1 < \beta_2 < \dots < \beta_r , \quad \beta_r \geq 1 ,$$

<u>and if at least one of the numbers $\omega_1 , \dots , \omega_r$ is irrational, the series</u>

$$j(z) = \sum_{n=0}^{\infty} j([\omega(n)]) z^n , \quad \hat{j}(z) = \sum_{n=0}^{\infty} j(\{\omega(n)\}) z^n$$

<u>are noncontinuable over the circle of convergence.</u>

THEOREM 5. - <u>The assumptions are the same as in theorem 2. Then it follows :</u>

$$\ddot{G}(z) = \sum_{n=0}^{\infty} g([\Phi(n)]) z^n$$

<u>has the unit circle as a natural boundary if, and only if, $\Phi(x) - \Phi(0)$ has an irrational coefficient.</u>

<u>Remark</u>. - The proofs of theorem 4 and theorem 5 are straightforward applications of theorem 3. In theorem 5, if only Φ_1 is irrational, we have again to use the method of Hecke (see SCHWARZ [6] for a detailed proof). If $\Phi_1 , \dots , \Phi_k$ are rational, we get from (1)

$$G(z) = \sum_{q=0}^{Q-1} \sum_{n'=0}^{\infty} g([\Phi(q)] + (\Phi(n' Q + q) - \Phi(q))) z^{n'Q+q}$$

$$= \sum_{q=0}^{Q-1} \sum_{\mu=0}^{m} \frac{1}{\mu!} g^{(\mu)}([\Phi(q)]) \sum_{n'=0}^{\infty} (\Phi(n' Q + q) - \Phi(q))^{\mu} z^{n'Q+q}$$

which is obviously a rational function.

4. "Almost all" results.

At last, we show that theorem 1 can also be used to derive results of the following type : The property of a power series to be noncontinuable over the unit circle is with regard to a special measure the normal case. So we get the following theorem.

THEOREM 6. - Let $f : [0, 1] \to \mathbb{C}$ be Riemann integrable and not equivalent to a constant function. Let $X = \{(x_1, x_2, \ldots) ; 0 \leq x_i \leq 1\}$ be the set of all sequences of real numbers with values in the unit interval. X becomes a measure space, if we take the product measure μ_∞ of the one dimensional Lebesgue measure μ. In $(X, \mathfrak{L}, \mu_\infty)$ the power series of the form

$$f(z) = \sum_{n=0}^{\infty} f(x_n) z^n$$

which are continuable over the unit circle form a set of measure zero.

Proof. - HLAWKA [4] has shown, that for μ_∞ nearly all sequences $(x_n)_{\underset{\sim}{N}}$ the sequence $(x_{n+p}, x_n)_{\underset{\sim}{N}}$ is, for every $p \geq 1$, uniformly distributed in the unit square. Therefore, we can use theorem 1 to prove the assertion.

Another application of theorem 2 is : For a convergent series $\sum a_n z^n$, we can choose signs $\varepsilon_n = \pm 1$ in such a way, that $\sum \varepsilon_n a_n z^n$ becomes noncontinuable.

THEOREM 7. - Let $(a_n)_{\underset{\sim}{N}}$ be a sequence of complex numbers with

$$|a_n| \leq 1 , \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} |a_n|^2 \neq 0 .$$

Let $r_n(t)$ be the Rademacher function : $r_n(t) = \operatorname{sgn} \sin 2^n \pi t$. Then for almost all $t \in [0, 1]$

$$F_t(z) = \sum_{n=0}^{\infty} a_n r_n(t) z^n$$

has the unit circle as an essential boundary.

Example : $a_n = 1$ for all n. Then almost all series with coefficients ± 1 are noncontinuable.

Proof. - We applicate lemma 1. For this, we have to calculate

$$b_p(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} a_{n+p} \overline{a_n} r_{n+p}(t) r_n(t) .$$

If we set

$$f_N(t) = \left|\frac{1}{N} \sum_{n<N} a_{n+p} \overline{a_n} r_{n+p}(t) r_n(t)\right|^4 ,$$

we get, using the relations of orthogonality for the Rademacher functions for $p \geq 1$:

$$\int_0^1 f_N(t)\, dt = \frac{1}{N^4} \sum_{n<N} |a_{n+p} \overline{a_n}|^4 + \frac{4}{2!\,2!} \sum_{i<k} |a_{i+p} \overline{a_i} a_{k+p} \overline{a_k}|^2$$

$$\leq N^{-4}\left(N + 1 + \binom{4}{2}\binom{N+1}{2}\right) \ll N^{-2} .$$

Therefore the series $\sum_{n=1}^{\infty} \int_0^1 f_N(t)\, dt$ is convergent, and from Lebesgues theorem of dominated convergence, we have

$$|b_p(t)|^4 = \lim_{N\to\infty} f_N(t) = 0 , \quad p \geq 1 ,$$

for almost all $t \in [0, 1]$. Because of the assumption $b_0 \neq 0$, the conditions of lemma 1 are satisfied and the theorem follows.

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(Texte reçu le 19 juin 1974)

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