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NUMBERS WITH GOOD FACTORIZATION PROPERTIES

by Władysław HARKIEWICZ

1. - F. JOGELS (1943, [2]) has shown that in $Q(\sqrt{-5})$, the simplest quadratic field with non-trivial class-group, almost no algebraic integer has unique factorization, that is to say, if $F(x)$ denotes the number of non-associated integers α with $|N(\alpha)| \leq x$, which have unique factorization, then $F(x)/x$ tends to zero.

He proved also, that the same holds for the number $H(x)$ of natural numbers $n \leq x$ with unique factorization in $Q(\sqrt{-5})$.

In fact, analogous results are true for all fields with non-trivial class-groups, as shown in [4],[6]. For $F(x)$, one gets evaluations of the form

$$F(x) \ll x/\log^\alpha x, \quad \alpha > 0,$$

whereas similar evaluations for $H(x)$ are obtained only for K/Q normal.

So we get the first question.

QUESTION I. - Is the evaluation $H(x) \ll x/\log^\alpha x$ ($\alpha = \alpha(K) > 0$), true for all fields K with non-trivial class-group?

2. - In 1960, L. CARLITZ [1] observed, that in an algebraic number field K with the class-number $h(K) \geq 3$, one can find integers α which have factorizations of different lengths, i. e. $\alpha = \pi_1 \dots \pi_r = \rho_1 \dots \rho_s$, with π_j, ρ_i irreducible, and $r \neq s$.

If $G(x)$ is the number of non-associated integers α with $|N(\alpha)| \leq x$, whose all factorizations are of the same length, then again (see [4]) one has

$$G(x) \ll x/\log^\beta x \quad (\beta = \beta(K) > 0),$$

and the analogue for natural numbers holds, provided K/Q is normal [5]. For non-normal K/Q , only $o(x)$ is proved at this moment [6].

So we have the second question.

QUESTION II. - Obtain the evaluation $O(x/\log^\beta x)$ ($\beta > 0$) for natural numbers having all factorizations of the same length, in a given field K with $h(K) \geq 3$.

3. - We shall now indicate the main points of the proof of the following theorem.

THEOREM. - If $h = h(K) \geq 2$, then

$$F(x) = (\underline{C} + o(1)) \frac{x(\log \log x)^M}{(\log x)^{1-(1/h)}},$$

where M is the maximal number of non-principal prime ideals, which can occur in a factorization of a number counted by $F(x)$ with the exponent one.

Let X_1, \dots, X_{h-1} be the non-principal ideal classes in K . If I is any ideal, without principal prime ideal factors, write it in the form

$$I = \prod_i (p_{i1}^{\alpha_{i1}} \dots p_{ik_i}^{\alpha_{ik_i}}) \cdot \mathcal{O}, \quad 1 \leq i \leq h-1,$$

with $p_{ij} \in X_i$, $\alpha_{ij} > 0$.

We say, that the system

$$\tau = \tau(I) = \langle \{\alpha_{11}, \dots, \alpha_{1k_1}\}, \dots, \{\alpha_{h-1,1}, \dots, \alpha_{h-1,k_{h-1}}\} \rangle$$

is the type of I . If I has no prime divisors from a class X_i say, then we write \emptyset in the place of $\{\alpha_{i1}, \dots\}$.

For a given type τ , let $d(\tau) = \mathcal{N}\{\alpha_{ij} = 1\}$ be its depth.

The proof of the theorem is based on the following result.

PROPOSITION. - Let \mathcal{A} be any set of principal ideals subject to the following conditions :

- (i) $I \in \mathcal{A}, \tau(I) = \tau(J) \implies J \in \mathcal{A}$;
- (ii) If all prime ideal factors of I are principal, then $I \in \mathcal{A}$;
- (iii) $\exists B, I \in \mathcal{A} \implies d(\tau(I)) \in B$, whenever $\tau(I)$ is defined.

Then

$$\mathcal{N}\{I = N(I) \leq x; I \in \mathcal{A}\} = (\underline{C} + o(1)) \frac{x(\log \log x)^M}{(\log x)^{1-(1/h)}}$$

with $\underline{C} = \underline{C}(\mathcal{A}) > 0$ and $M = \max\{d(\tau(I))\}; I \in \mathcal{A}\} \leq B$.

This implies immediately the theorem, as if α has a unique factorization, then α has at most $2h-1$ different prime ideal factors from a given class. Indeed, if it has $\geq 2h$, say $p_1, \dots, p_{2h}, \dots$ then

$$(p_1 \dots p_h)(p_{1+h} \dots p_{2h}) = (p_{1+h} p_2 \dots)(p_1 p_{2+h} \dots).$$

The proof of the proposition is based on the tauberian theorem of DELANGE.

One starts with the following lemma.

LEMMA. - If $X \in \mathfrak{X}(K)$, and $F_1(t), \dots, F_n(t)$ are real and

$$0 < F_i(t) \ll t^{-2}, \quad d \geq 1,$$

for $\text{Re } s > 1$ write

$$S(s) = \sum_{p_1, \dots, p_\alpha} 1/(N_{p_1}^s \dots N_{p_\alpha}^s) \sum_{q_1, \dots, q_n} F_1(N_{p_1}^{q_1}) \dots F_n(N_{p_n}^{q_n}),$$

with $p_1, \dots, p_\alpha \in X$, and distinct; $q_1, \dots, q_n \in \mathbb{N}$, distinct, and $q_i \neq p_j$.
 then $S(s) = P(\log(1/s - 1))$, $P \in \Omega[X]$, Ω ring of functions regular in $\text{Re } s \geq 1$, $\text{deg } P = d$, and the leading coefficient is positive at $s = 1$.

Proof. - Induction in d . This allows to show, that if τ is given, and

$$S_\tau = \{I : I = I_1 I_2, \tau(I_1) = \tau, I_2 \text{ has all prime factors principal}\},$$

then

$$\sum_{I \in S_\tau} N(I)^{-s} = (b(\log \frac{1}{s-1})) / (s-1)^{1/h},$$

$b \in \Omega[X]$, $\text{deg } b = d(\tau)$, leading coefficient of b positive at 1, and in fact the same result holds if we sum up this equality over any set of τ with $d(\tau)$ fixed.

There are also another applications of our proposition. Using it in the case $K = \mathbb{Q}$, one regains a theorem of L. MIRSKEY [3]:

$$\pi\{n \leq x : d(n) = k\} = (\underline{c} + o(1)) \frac{x^{1/(p-1)} (\log \log x)^m}{\log x},$$

whose $p = \min\{p : p|k\}$, $p^m \parallel k$.

As well as analogues of this for all multiplicative functions with $f(p^t) = a_k$.

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