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# A GENERALIZED CHACON'S INEQUALITY AND ORDER CONVERGENCE OF PROCESSES

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Abstract. - We show that, if $(X_n)$ is a sequence of Bochner integrable random variables, almost everywhere order bounded and valued in a Banach lattice with an order continuous norm, then

$$\int (\limsup X_n - \liminf X_n) \leq \limsup_{\sigma,\tau\in T} \int (X_\sigma - X_\tau)$$

where $T$ is the set of bounded stopping times. We then obtain order convergence theorems for martingales, submartingales and subadditive processes.

## 1. Introduction.

Let $(\Omega , \mathfrak{F} , P)$ be a probability space, $(\mathfrak{F}_n)$ an increasing sequence of sub-$\sigma$-algebras of $\mathfrak{F}$ such that $\mathfrak{F} = \sigma(\bigcup_n \mathfrak{F}_n)$. Denote by $T$ the set of bounded stopping times. In [1], R. V. CHACON showed that, if $(X_n)$ is an $L^1$-bounded, $(\mathfrak{F}_n)$-adapted sequence of real random variables such that $(\int X_\sigma)_{\sigma\in T}$ is bounded, then the following inequality holds.

$$\int (\limsup X_n - \liminf X_n) dP \leq \limsup_{\sigma,\tau\in T} \int (X_\sigma - X_\tau) dP .$$

The aim of this paper is to show that this inequality still holds if one considers Bochner integrable random variables valued in certain Banach lattices, from which we can conclude order convergence of some vector valued processes as martingales and submartingales. The techniques that we use apply also to give the order convergence of the vector valued ergodic theorem and subadditive ergodic theorem.

The main idea of the proof is to reduce the case to an $L^1$-valued process and then, by a randomisation argument, to reduce it to the real case.

In the finite dimensional case, the $L^1$-boundedness implies, via the maximal inequality, that the process is bounded a.e. Since such a property is not satisfied in the infinite dimensional case, we must assume that, for almost all $w \in \Omega$, $\sup_n |X_n(w)|$ exists in the Banach lattice, at least to assure the existence of $\limsup X_n$ and $\liminf X_n$.

Clearly, the inequality is of interest in ordered spaces where the order convergence is stronger than the norm convergence. That is in spaces which have an order continuous norm. We will limit ourselves to this case. We shall start by recalling some well known facts.

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LEMMA 1. - *Let* $E$ *be an order continuous Banach lattice which has a weak unit, then there exists a probability space* $(X, \Sigma, \mu)$ *such that* $E$ *is order isometric to an ideal of* $L^1(X, \Sigma, \mu)$.

For a proof, see LINDENSTRAUSS, TZAFRIRI [4].

The following lemma is also standard.

LEMMA 2. - *For every Bochner integrable random variable*

$$X : (\Omega, \mathfrak{F}, P) \longrightarrow L^1[X, \Sigma, \mu],$$

*there exists* $Y : \Omega \times X \longrightarrow R$, $\mathfrak{F} \otimes \Sigma$ *measurable such that* $P$ *a. e.* $Y(w, .)$ *represents* $X(w)$. *Moreover,* $Y$ *is unique modulo a* $(P \otimes \mu)$*-negligible function. The following also holds*

(i) $X \in L^1_E[P] \Longleftrightarrow Y \in L^1_R[P \otimes \mu]$ *where* $E = L^1[X, \Sigma, \mu]$ ;

(ii) *if* $X \in L^1_E[P]$ *then* $\int X \, dP$ *is represented by* $t \longrightarrow \int Y(w, t) \, dP(w)$ ;

(iii) *if* $Y_n$ *is associated to a sequence* $X_n$ *such that for almost all* $w$ *the sequence* $|X_n|(w)$ *is order bounded in* $E$, *then* $\limsup Y_n$ *and* $\liminf Y_n$ *represents the functions* $w \longrightarrow \limsup X_n(w)$ *and* $w \longrightarrow \liminf_n X_n(w)$.

Consider now the filtration $(\mathfrak{F}_n \otimes \Sigma)$ on $\Omega \times X$. The next lemma describes an approximation process for the stopping times relative to this filtration. Since the arguments are standard, we give a sketch of the proof.

LEMMA 3. - *Let* $\tau$ *be a stopping time for the filtration* $(\mathfrak{F}_n \times \Sigma)$ *such that* $p \leq \tau \leq q$. *Then, for all* $\varepsilon > 0$, *there exists a finite partition* $(A_\ell)$ *of* $X$ *by measurable sets and stopping times* $(\sigma_\ell)$ *on* $\Omega$ *such that* $p \leq \sigma_\ell \leq q$ *and that, if the stopping time* $\sigma$ *is defined by*

$$(*) \qquad \sigma(w, t) = \Sigma_\ell \, 1_{A_\ell}(t) \, \sigma_\ell(w),$$

*then*

$$P \otimes \mu\{\tau \neq \sigma\} \leq \varepsilon.$$

*Proof.* - Let $k \in N$, and $p \leq k \leq q$. The set $\{\tau = k\}$ belongs to $\mathfrak{F}_k \otimes \Sigma$, hence there exists a set $B_k$ which is a union of rectangles $C \times D$, with $c \in \mathfrak{F}_k$ and $D \in \Sigma$, and such that

$$P \otimes \mu(\{\tau = k\}) \Delta B_k) \leq \frac{\varepsilon}{q^2}.$$

Let $\sigma$ be such that $\{\sigma = k\} = B_k - \bigcup_{p \leq s < k} B_s$, for each $k$, $p \leq k \leq q$.

Clearly, $\sigma$ is a stopping time and $P \otimes \mu\{\tau \neq \sigma\} \leq \varepsilon$. If we denote by $A_\ell$ the atoms of the finite algebra of $X$ generated by the projections on $X$ of all the rectangles involved, then $\sigma$ can be written as in (*).

THEOREM 1. – If $E$ is a Banach lattice with an order continuous norm, and $(X_n)$ is a sequence of $E$-valued Bochner integrable random variables verifying

(i) $\sup_n |X_n(w)|$ exists for almost all $w \in \Omega$,

and

(ii) $(\int X_\sigma)_{\sigma \in T}$ is order bounded in $E$.

Then, the following inequality is satisfied.

$$\int (\limsup X_n - \liminf X_n)dP \leq \limsup_{\sigma,\tau \in T} \int (X_\sigma - X_\tau)dP .$$

Proof. – Since the $X_n$'s are almost separably valued, we can assume that $E$ is separable, hence with a weak order unit [4]. By lemma 1, it is enough to show the inequality in $L^1[X, \Sigma, \mu]$. Denote by $T'$ the set of bounded $(\mathcal{F}_n \times \Sigma)$ stopping times and by $T'_s$ the set of those which can be written as in formula (*).

For every $n \in \mathbb{N}$, let $Y_n$ be an $\mathcal{F}_n \otimes \Sigma$ representation of $X_n$. Let $\tau \in T'$, with $p \leq \tau \leq q$. It follows from lemma 3 that, for every $\epsilon > 0$, there exists $\tau' \in T'_s$, with $p \leq \tau' \leq q$, and

$$|\int Y_\tau - \int Y_{\tau'}| \leq \epsilon .$$

Hence, we have

$$\sup_{\tau \in T', \tau \geq p} \int Y_\tau = \sup_{\tau \in T'_s, \tau \geq p} \int Y_\tau .$$

Write now $\tau(w, t) = \sum_\ell 1_{A_\ell}(t)\, \sigma_\ell(w)$ $(\tau \in T'_s)$. For $\sigma_\ell$ such that $p = \inf \tau \leq \sigma_\ell \leq \sup \tau$,

$$\int Y_\tau \, d(P \otimes \mu) = \sum_\ell \int_{A_\ell} \int_\Omega Y_{\sigma_\ell(w)} (w, t)\, dP(w)\, d\mu(t)$$
$$= \sum_\ell 1_{A_\ell} \int_\Omega X_{\sigma_\ell} \, dP \leq \sup_{\sigma > p, \sigma \in T} \int_\Omega X_\sigma \, dP ,$$

since $t \longrightarrow \int_\Omega Y_{\sigma_\ell(w)} (w, t)\, dP(w)$ represents $\int_\Omega X_{\sigma_\ell} \, dP(w)$. This shows that $\iint Y_\tau \leq \int_X (\sup_{\sigma \geq p, \sigma \in T} \int_\Omega X_\sigma \, dP) d\mu(t)$. The same proof shows that

$$\iint Y_\tau \geq \int_X (\inf_{\sigma \geq p, \sigma \in T} \int_\Omega X_\sigma \, dP) d\mu(t) .$$

Using now Chacon's inequality on the real line, we get

$$\iint \limsup Y_n(w, t) - \liminf Y_n(w, t)$$
$$\leq \limsup_{\tau, \rho \in T'} \iint Y_\tau(w, t) - Y_\rho(w, t) d(P \otimes \mu)$$
$$\leq \int_X (\limsup_{\sigma \in T} \int_\Omega X_\sigma \, dP) d\mu - \int_X (\liminf_\sigma \int_\Omega X_\sigma \, dP) d\mu .$$

By lemma 2, we get :

$$\int_X [\int_\Omega (\limsup X_n(w) - \liminf X_n(w)) dP(w)] d\mu$$
$$\leq \int_X (\limsup_{\tau, \sigma \in T} \int_\Omega (X_\sigma - X_\tau) dP) d\mu .$$

Replacing $X$ by any measurable subset $A$, the inequality will still hold, hence

$$\int_\Omega (\limsup X_n - \liminf X_n) \leq \limsup_{\sigma, \tau} \int (X_\sigma - X_\tau) \quad \text{u. a. e.}$$

which concludes the proof.

Following [2], an $\mathfrak{F}_n$-adapted sequence in $L^1[E]$ is said to be an ordermart if the net $(\int X_\sigma)$ order converges in $E$. Vector valued martingales are clearly ordermarts : $(X_n)$ is said to be a supermartingales (resp. a submartingale) if the map $\sigma \longrightarrow (\int X_\sigma)$ is decreasing (resp. increasing). We have the immediate corollary.

COROLLARY 1.

(a) If $E$ has an order continuous norm and $(X_n)$ is an orderamart such that $\sup_n |X_n(w)|$ exists for almost all $w \in \Omega$, then $X_n$ order converges a. e.

(b) The same holds when we remplace the hypothesis $(X_n)$ orderamart by $X_n$ supermartingale.

Proof. - (a) follows immediately from the inequality. For (b) we notice that, for almost all $w$, $(X_n(w))$ is valued in a weakly compact order interval, hence $X_n$ converges weakly to $X \in L^1[E]$. Consider now $Z_n = X_n - E^{\mathfrak{F}_n}[X]$ ; we have $\inf_\sigma (\int Z_\sigma) = 0$, thus $(X_n)$ is an orderamart which is order convergent.

Recall also that if $\theta$ is a measure preserving point transformation on $(\Omega , \mathfrak{F} , P)$, we say that a sequence $(S_n)$ in $L^1[E]$ is a subadditive process if, for every $n$, $k$, we have $S_{n+k} \leq S_n + S_k \circ \theta^n$. See [3].

THEOREM 2. - If $E$ has an order continuous norm and $(S_n)$ is a subadditive process such that $\sup_n |S_n(w)|/n$ exists a. e., then $S_n/n$ is o-convergent a. e.

Sketch of proof. - Again, we can reduce the case to $E = L^1[X , \Sigma , \mu]$ and randomise the problem by considering $S'_n : \Omega \times \Sigma \longrightarrow R$ associated to $S_n$ by lemma 2. The sequence $(S'_n)$ is a real valued subadditive process with respect to the point transformation $\theta'(w , t) = (\theta(w) , t)$.

Using the convergence in the real case, it is enough now to notice that in $L^1(X , \Sigma , \mu)$ an order bounded and $\mu$-almost everywhere convergent sequence is order convergent.

## REFERENCES


[1] CHACON (R. V.). - A stopped proof of convergence, Advances in Math., t. 14, 1974, p. 365-368.

[2] GHOUSSOUB (N.). - Orderamarts : A class of asymptotic martingales, J. of multiv. Analysis (to appear).

[3] GHOUSSOUB (N.) and STEELE (J. M.). - Vector valued subadditive processes and applications in probability (to appear).

[4] LINDENSTRAUSS (J.) and TZAFRIRI (L.). - Classical Banach spaces, Vol. II : Function spaces (to appear).


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