Astérisque

MARIUS VAN DER PUT

Recent work on differential Galois theory

Astérisque, tome 252 (1998), Séminaire Bourbaki,
exp. n° 849, p. 341-367

<http://www.numdam.org/item?id=SB_1997-1998__40__341_0>

© Société mathématique de France, 1998, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (http://smf4.emath.fr/Publications/Asterisque/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Séminaire BOURBAKI Juin 1998
50ème année, 1997-98, n° 849

RECENT WORK ON DIFFERENTIAL GALOIS THEORY

by **Marius VAN DER PUT**

1. INTRODUCTION

Linear differential equations have a rich variety of aspects: asymptotic theory, differential Galois theory, the construction of algorithms, arithmetic properties, Grothendieck's conjecture, rigid differential equations et cetera. We will restrict our attention to recent progress on the inverse problem for the differential Galois theory. For the history of this problem we refer to the excellent survey paper [S] and its references. Our purpose is to explain and partly prove the results obtained by J.-P. Ramis, C. Mitschi and M.F. Singer. Let us give two examples of the results that we hope to explain:

(1) Every connected reductive groups over $\mathbf{C}$ can be realized as the differential Galois group of an equation $y' = (A_0 + A_1 z)y$, with constant matrices A_0, A_1.

(2) There is no linear differential equation over the field of convergent Laurent series $\mathbf{C}(\{z\})$ with differential Galois group $\mathbf{G}_a^2$.

In this presentation we have taken the liberty of changing proofs, omitting material and adding new material with respect to the original papers.

2. PICARD-VESSIOT THEORY

A *differential field* K is a field equipped with a differentiation (or derivation) $f \mapsto f'$ satisfying the usual rules: $(f+g)' = f' + g'$ and $(fg)' = f'g + fg'$. The field of constants C is defined by $C = \{a \in K |\ a' = 0\}$. We will suppose in the sequel that C is an algebraically closed field of characteristic zero and that $C \neq K$. The derivatives of an element f will be denoted by $f', f'', f''', \ldots$ or $f^{(1)}, f^{(2)}, f^{(3)}, \ldots$. A *scalar homogeneous linear differential equation of order* n is an equation of the form

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = 0 \text{ with } a_0, \ldots, a_{n-1} \in K.$$

The solution space $\{y \in K |\ L(y) = 0\}$ is a vector space over C of dimension $\leq n$. In general the dimension is strictly less than n. This leads to the definition of a *Picard-Vessiot ring* $E \supset K$ for the equation L:

(i) The K-algebra E is equipped with a differentiation (also denoted by $'$) extending the differentiation of K.

(ii) E has only trivial differential ideals.

(iii) The space $V := \{a \in E |\ L(a) = 0\}$ is a vector space over C of dimension n.

(iv) The K-algebra E is generated by the elements of V and their derivatives.

It can be shown that a Picard-Vessiot ring exists and that it is unique up to differential isomorphism. Moreover E has no zero divisors and its field of fractions has C as set of constants. The field of fractions of a (or the) Picard-Vessiot ring is called a (or the) Picard-Vessiot field of the equation over K. The space V will be called the *solution space* of L. Let G denote the group of the differential automorphisms of E/K. This group acts as a group of C-linear automorphisms of $V \subset E$. The induced group homomorphism $G \to \mathrm{GL}(V)$ turns out to be injective and its image is an *algebraic* subgroup of $\mathrm{GL}(V)$. The *differential Galois group of the equation L over K* is defined to be G with its structure of linear algebraic group given by the embedding $G \subset \mathrm{GL}(V)$. The group G also coincides with the group of all differential automorphisms of the Picard-Vessiot field over K.

The Tannakian approach to (linear homogeneous) differential equations over K is given in [De-M] and [De]. This approach can be described as follows. Let Diff_K denote the category of the differential modules over K. Under the hypothesis that the field of constants C of K has characteristic 0 and is algebraically closed, one can show that this category is a neutral Tannakian category. In other words, there is an equivalence of Tannakian categories $\mathrm{Diff}_K \to \mathrm{Repr}_H$, where H is a certain affine group scheme over C and Repr_H denotes the Tannakian category of the finite dimensional representations of H.

For a fixed differential module M over K one can consider the full subcategory $\{\{M\}\}$ of Diff_K generated by all $M \otimes \cdots \otimes M \otimes M^* \otimes \cdots \otimes M^*$. This subcategory is also a neutral Tannakian category and thus there is an equivalence of Tannakian categories $\{\{M\}\} \to \mathrm{Repr}_G$ for some affine group scheme G over C. It turns out that G can be identified with the differential Galois group of M (and is thus a linear algebraic group). The equivalence can be made explicit by $N \mapsto ker(\partial, E \otimes_K N)$, where E is a Picard-Vessiot ring for M over K. The G action on E induces a G action on $E \otimes_K N$ which commutes with the action of ∂ on $E \otimes_K N$. In particular, $ker(\partial, E \otimes_K N)$ is a finite dimensional (algebraic) representation of the linear algebraic group G.

This more abstract Tannakian approach to differential modules has some advantages. An easy application is the following useful statement:

Suppose that G is a differential Galois group over K and let $N \subset G$ be a normal algebraic subgroup. Then the linear algebraic group G/N is also a differential Galois group over K.

Important examples of differential fields:

Let C denote an algebraically closed field of characteristic 0.

(1) $K = C((z))$, the field of the formal Laurent series in z over C with the derivation $\frac{d}{dz}$.

(2) $K = \mathbf{C}(\{z\})$, the field of the convergent Laurent series over the field of the complex numbers with the derivation $\frac{d}{dz}$.

(3) $K = C(z)$, the field of rational functions over C with the derivation $\frac{d}{dz}$.

(4) K a finite extension of $C(z)$, equipped with the unique derivation extending the derivation $\frac{d}{dz}$ of $C(z)$. In other words, K is the function field of a curve (irreducible, smooth, projective, connected) over C.

The *theme* of this exposition is to answer, for the differential fields K in the above list, the question:

Which linear algebraic groups G are differential Galois groups over K ?

3. THE THEOREMS

Let G be a linear algebraic group (over the field C). The subgroup $L(G)$ of G is defined as the group generated by all (maximal) tori lying in G. It is clear that $L(G)$ is a normal subgroup of G, contained in the component of the identity G^o of G. The group $L(G)$ is generated by algebraic subgroups of G and hence is itself algebraic. Thus the factor group $G/L(G)$ is again a linear algebraic group.

Theorem 3.1 (Ramis). **The local theorem.**— *A linear algebraic group G is a differential Galois group over the field $\mathbf{C}(\{z\})$ if and only if $G/L(G)$ is topologically (for the Zariski topology) generated by one element.*

Corollary 3.2 (Ramis).— *Suppose that G is a differential Galois group over the field $\mathbf{C}(\{z\})$, then $G/L(G)$ is topologically generated by the image of the topological monodromy.*

Explanation

The differential equation $L(y) = 0$ over $\mathbf{C}(\{z\})$ lives in fact as a meromorphic differential equation on a neighbourhood $\{z \in \mathbf{C} |\ |z| < \epsilon\}$ of $z = 0$, which has only $z = 0$ as singularity. The solution space V can be identified with the solutions of the equation at the point $\epsilon/2$. Analytical continuation of the solutions at $\epsilon/2$ along the circle around 0

and through $\epsilon/2$ (with positive orientation) induces a $\mathbf{C}$-linear automorphism of V. This automorphism can be shown to belong to $G \subset \mathrm{GL}(V)$. The element is called the *topological monodromy.* It is unique up to conjugation. (See section 7 for more information.)

Examples 3.3 : *Order two*

Again $K = \mathbf{C}(\{z\})$ and consider a differential equation over K of order two with differential Galois group in $\mathrm{SL}(2)$. The well known classification of the algebraic subgroups of $\mathrm{SL}(2)$ can be used in order to determine the possible differential Galois groups. The list (of conjugacy classes) that one finds is:

$$\mathrm{SL}(2), B, \mathbf{G}_a, \{\pm 1\} \times \mathbf{G}_a, \mathbf{G}_m, D_\infty, \text{ finite cyclic },$$

where B is the Borel subgroup, $\mathbf{G}_a$ the additive group, $\mathbf{G}_m$ the multiplicative group and D_∞ is the infinite dihedral group, i.e., the subgroup of $\mathrm{SL}(2)$ leaving the union of two lines $L_1 \cup L_2 \subset \mathbf{C}^2$ (through the origin) invariant.
Every group in the above list can be realized by a differential equation

$$y'' + mz^{-1}y' - a_0 y = 0 \text{ where } m \in \{0, 1\} \text{ and } a_0 \in \mathbf{C}[z, z^{-1}].$$

In fact the choices $(m, a_0) = (0, z)$, $(0, z^2+3z+5/4)$, $(1, 0)$, $(0, -\frac{z^{-2}}{4})$, $(0, z^{-2})$, $(0, -\frac{3}{16}z^{-2} + z^{-1})$, $(0, \frac{1}{4}(-1+(\frac{t}{n})^2)z^{-2})$ with $\frac{t}{n} \in \mathbf{Q}$ produce the above list of differential Galois groups.

Some definitions: regular, regular singular and irregular

Let $X/\mathbf{C}$ denote a curve (irreducible, smooth, projective) of genus g. At this point of the exposition it is convenient to define a (linear homogeneous) differential equation over X as a matrix differential equation $y' = Ay$, where A is a $n \times n$-matrix with coefficients in K, the function field of X. Consider a point $c \in X$ with local coordinate z_c. The field K is embedded in $K_c = \mathbf{C}(\{z_c\})$, the field of the convergent Laurent series in z_c. Let $B \in \mathrm{GL}(n, K_c)$. Put $v = By$. Then v satisfies the matrix differential equation $v' = (B^{-1}AB - B^{-1}B')v$. This new equation over K_c is called *locally equivalent* with $y' = Ay$. The equation $y' = Ay$ is called *regular, resp. regular singular* at c if there is a locally equivalent equation $v' = Dv$ such that $D \in M(n, \mathbf{C}\{z_c\})$ resp. $z_c D \in M(n, \mathbf{C}\{z_c\})$. Otherwise the point c will be called an *irregular singularity* of $y' = Ay$.

Theorem 3.4 (Ramis). **The global theorem.**— *Let $X/\mathbf{C}$ be a curve (irreducible, smooth, projective) of genus g and $S \subset X$ a finite set with cardinality $m \geq 1$. A linear algebraic group G can be realized as the differential Galois group of a differential equation over X with singularities in S if and only if $G/L(G)$ is topologically generated by $2g + m - 1$ elements.*

Remark.— The theorem remains valid for $S = \emptyset$ and $2g + m - 1$ replaced by $2g$.

Corollary 3.5 (Ramis).— (1) *Suppose that G is the differential Galois group for a differential equation on X with singularities in the finite set S. Then the natural homomorphism $\pi_1(X \setminus S, *) \to G/L(G)$ has dense image for the Zariski topology.*

(2) *Suppose that the linear algebraic group G can be realized for the pair (X, S), then G can be realized by a differential equation with at most one irregular singular point (to be chosen freely in S) and the other singularities, regular singular (and in S).*

Examples 3.6 : *Order two, continued*

1) $(\mathbf{P}^1, \{0, \infty\})$ *and order two.*

The list of possible groups $G \subset \mathrm{SL}(2)$ coincides with the list that we have considered in 3.3. The above list is in fact the theoretical background for the simplification of the Kovacic algorithm for order two differential equations with at most two singular points, presented in [P2].

2) $(\mathbf{P}^1, \{0, 1, \infty\})$ *and order two.*

Every algebraic subgroup of $\mathrm{GL}(2)$ can be realized for this pair. More precisely, every algebraic subgroup $G \subset \mathrm{GL}(2)$ is the differential Galois group of an equation

$$y'' + \frac{a_1(z)}{z(z-1)}y' + \frac{a_2(z)}{z^2(z-1)^2}y = 0,$$

where $a_1(z), a_2(z) \in \mathbf{C}[z]$. (This equation is regular singular at $0, 1$ and has an arbitrary singularity at ∞).

Some definitions: The defect and the excess of a connected linear algebraic group.

C denotes an algebraically closed field of characteristic 0. Let G be a connected linear algebraic group over C. The unipotent radical of G is denoted by R_u. Let $P \subset G$ be a Levi factor, i.e., a closed subgroup such that G is the semi-direct product of R_u and P (see [H], p 184). The group $R_u/(R_u, R_u)$, where (R_u, R_u) is the (closed) commutator subgroup, is a commutative unipotent group and so isomorphic to $\mathbf{G}_{a,C}^n = C^n$. The group P acts on R_u by conjugation and this induces an action on $R_u/(R_u, R_u)$. Therefore we may write $R_u/(R_u, R_u) = U_1^{n_1} \oplus \cdots \oplus U_s^{n_s}$, where each U_i is an irreducible P-module. For notational convenience, we suppose that U_1 is the trivial 1-dimensional P-module and $n_1 \geq 0$. Since P is reductive, one can write $P = T \cdot H$, where T is a torus and H is a semi-simple group. Define $m_i := n_i$ if the action of H on U_i is trivial and $m_i := n_i + 1$ if the action of H on U_i is not trivial. Define $N = 0$ if H is trivial and $N = 1$ otherwise. The *defect* $d(G)$ is defined to be n_1 and the *excess* $e(G)$ is defined as $\max(N, m_2, \ldots, m_s)$. Since two Levi factors are conjugated, these numbers do not depend on the choice of P. Furthermore, one can show that $d(G)$ is the dimension of $R_u/(G, R_u)$.

Theorem 3.7 (Mitschi-Singer).— *Let G be a connected linear algebraic group over C. Then G is the differential Galois group of a matrix differential equation over $C(z)$ of the*

form

$$y' = (\frac{A_1}{z-a_1} + \cdots + \frac{A_{d(G)}}{z-a_{d(G)}} + A_\infty)y,$$

where A_i, $i = 1, \ldots, d(G)$ *are constant matrices and* A_∞ *is a matrix with polynomial entries of degree at most* $e(G)$. *In particular, the points* $a_1, \ldots, a_{d(G)}$ *are the singularities of the equation in* C. *These singularities are regular. The point* ∞ *is possibly an irregular singular point.*

Remarks

The proof of the result above is purely algebraic and moreover constructive. The statement is more precise than the one of Ramis' global theorem and more restricted in the sense that the group G is supposed to be connected and that the representation of G on the space of solutions is not prescribed. Using Tannakian arguments one may also prescribe the faithful representation of G on the space of solutions (at the cost of introducing possibly apparent singularities). In section 10 we will show that $d(G)$ is equal to the minimal number of topological generators of $G/L(G)$.

The special case of the theorem, where the group G is supposed to be connected and reductive, is rather striking. It states that G is the differential Galois group of a matrix differential equation $y' = (A + Bz)y$ with A and B constant matrices. In section 9 we will present an explicit proof of this statement in case G is connected and semi-simple.

4. ABHYANKAR'S CONJECTURE

The base field k is an algebraically closed field of characteristic $p > 0$, e.g. $\overline{\mathbf{F}}_p$. One considers a curve X/k (irreducible, smooth, projective) of genus g and a finite subset $S \subset X$ with cardinality $m \geq 1$. Abhyankar's conjecture is concerned with the Galois coverings of X which are unramified outside S. For a finite group G one denotes by $p(G)$ the subgroup generated by all the p-Sylow subgroups of G. In other words, $p(G)$ is the smallest normal subgroup such that $G/p(G)$ has order prime to p. We recall the well known theorems.

Theorem 4.1 (Raynaud).— *The finite group* G *is the Galois group of a covering of* $\mathbf{P}^1$, *unramified outside* ∞, *if and only if* $G = p(G)$.

Theorem 4.2 (Harbater).— (1) *Let the pair* (X, S) *be as above. The finite group* G *is a Galois group of a covering of* X, *unramified outside* S, *if and only if* $G/p(G)$ *is generated by* $2g + m - 1$ *elements.*

(2) *If* G *is a Galois group for the pair* (X, S), *then the natural homomorphism* $\pi^{(p)}(X \setminus S, *) \to G/p(G)$ *is surjective.*

(3) *Suppose that $G/p(G)$ is generated by $2g+m-1$ elements. Then there is a Galois covering of X with Galois group G, wildly ramified in at most one (prescribed) point of S, tamely ramified at the other points of S and unramified outside S.*

Remark.— The group $\pi^{(p)}(X \setminus S,^*)$ denotes the prime to p, algebraic fundamental group of $X \setminus S$. This group is known to be topologically generated by $2g+m-1$ elements and depends in fact only on the number $2g+m-1$.

The translation

During a dinner at Toulouse (nuit de la musique 1993), Raynaud and Ramis discussed the remarkable relation between the inverse problem for differential Galois theory and Abhyankar's conjecture. The following transformation rules seem to link to two subjects:

Differential	**C**haracteristic $p>0$
$X/\mathbf{C}$ curve, finite $S \subset X$, $S \neq \emptyset$	X/k curve, finite $S \subset X$, $S \neq \emptyset$
differential equation	equation with Galois covering of X
singular point (in S)	ramified point (in S)
local differential Galois group	inertia group
regular singular	tamely ramified
irregular singular	wildly ramified
linear algebraic group	finite group
$L(G)$	$p(G)$
$\pi_1(X \setminus S) \to G/L(G)$ Zariski dense	$\pi^{(p)}(X \setminus S) \to G/p(G)$ surjective

In the work of P. Deligne, N. Katz and G. Laumon on (rigid) differential equations there is also a link, this time more concrete, between differential equations and certain sheaves living in characteristic p. It would be interesting to find a mathematical theory explaining the philosophical link observed above.

5. FROM LOCAL TO GLOBAL

Proposition 5.1.— *Assume that the local theorem 3.1 is valid. Let the following data be given:*

1) *A compact Riemann surface X of genus g.*

2) *Points $p_1, \ldots, p_m$ on X with $m \geq 1$.*

3) *A vector space V of dimension n over $\mathbf{C}$.*

4) *An algebraic subgroup $G \subset \mathrm{GL}(V)$, such that $G/L(G)$ is topologically generated by $2g+m-1$ elements.*

Then there exists a differential equation over X, singular at most at the points $p_1, \ldots, p_m$, regular singular at $p_1, \ldots, p_{m-1}$, having a solution space, which is identified with V such that $G \subset \mathrm{GL}(V)$ is the differential Galois group.

Proof.— The function field of X will be denoted by K. For a point $p \in X$ we denote the field of the meromorphic functions at p by K_p.
First one chooses small disjoint disks $X_1, \ldots, X_m$ around the points $p_1, \ldots, p_m$. Put $X_i^* = X_i \setminus \{p_i\}$. In X_m^* one chooses a point c. The fundamental group $\pi_1(X_0, c)$, where $X_0 = X \setminus \{p_1, \ldots, p_m\}$, is generated by $a_1, b_1, \ldots, a_g, b_g, \lambda_1, \ldots, \lambda_m$ and has one relation $a_1 b_1 a_1^{-1} b_1^{-1} \cdots \lambda_1 \cdots \lambda_m = 1$. The element λ_m is a loop in X_m^* around p_m and the other λ_i are loops around $p_1, \ldots, p_{m-1}$. The differential module over X is constructed by glueing certain connections $M_0, \ldots, M_m$ (with possibly singularities), living above the spaces $X_0, \ldots, X_m$.

Let $pr : G \to G/L(G)$ denote the canonical homomorphism. One chooses a homomorphism $\rho : \pi_1(X_0, c) \to G \subset \mathrm{GL}(V)$, such that the homomorphism $pr\rho$ has Zariski dense image. Consider the algebraic group $G' = pr^{-1}(\ll pr\rho(\lambda_m) \gg)$, where $\ll a \gg$ denotes the algebraic subgroup generated by the element a. The group G' contains $L(G)$ and so $G'/L(G')$ is topologically generated by the image of $\rho(\lambda_m)$. According to the Ramis' local theorem, G' is the differential Galois group of a differential equation over the field K_{p_m}. One can extend this very local object to a differential module M_m, living above X_m, with only p_m as singular point. The solution space at the point $c \in X_m$ and the action of G' on this space can be identified with V and $G' \subset \mathrm{GL}(V)$. The topological monodromy corresponding to λ_m can be arranged to be $\rho(\lambda_m) \in G'$.

The usual solution of the Riemann-Hilbert problem (in weak form) provides a differential module M_0 above X_0 such that the monodromy action is equal to ρ.
The restrictions of M_0 to X_m^* and M_m to X_m^* are determined by their local monodromies. These are both equal to $\rho(\lambda_m) \in G \subset \mathrm{GL}(V)$. Thus we have a canonical way to glue M_0 and M_m over the open subset X_m^*. For each point p_i, $i = 1, \ldots, m-1$ one can consider the restriction of M_0 to X_i^*. This restriction is determined by its local monodromy around the point p_i. Clearly, the restriction of M_0 to X_i^* can be extended to a differential module M_i above X_i with a regular singular point at p_i.

The modules $M_0, M_1, \ldots, M_m$ (or rather the corresponding analytic vector bundles) are in this way glued to a vector bundle M above X. The connections can be written as $\nabla : M_0 \to \Omega_X \otimes M_0$, $\nabla : M_i \to \Omega_X(p_i) \otimes M_i$ for $i = 1, \ldots, m-1$ and $\nabla : M_m \to \Omega_X(dp_m) \otimes M_m$ for a suitable integer $d \geq 0$. The connections also glue to a connection $\nabla : M \to \Omega_X(p_1 + \cdots + p_{m-1} + dp_m) \otimes M$. Let M_* denote the set of all meromorphic sections of M. Then M_* is a vector space over K of dimension n with a connection $\nabla : M_* \to \Omega_{K/\mathbf{C}} \otimes M_*$. Thus we have found a differential module over K with the correct singularities. K has a natural embedding in the field K_c. The differential module is trivial over K_c and its Picard-Vessiot field PV over K can be seen as a subfield

of K_c. The solution space is $V \subset PV \subset K_c$.

Finally we have to show that the differential Galois group $H \subset \mathrm{GL}(V)$ is actually G. By construction $G' \subset H$ and also the image of ρ lies in H. This implies that $G \subset H$. In order to conclude that $G = H$ we can use the Galois correspondence. Thus we have finally to prove that an element $f \in PV \subset K_c$, which is invariant under G, belongs to K.

Since G' is by construction the differential Galois group of M_m above X_m, we conclude from the invariance of f under G' that f extends to a meromorphic solution of the differential equation above X_m. The invariance of f under the image of ρ implies that f extends to a meromorphic solution of the differential equation above $X_0 \cup X_m$. The points $p_1, \ldots, p_{m-1}$ are regular singular and any solution of the differential equation above X_i^*, $i = 1, \ldots, m-1$ extends meromorphically to X_i. Thus f is meromorphic on X and belongs to K.

Remark.— For the case of $X = \mathbf{P}^1$ and $p_m = \infty$, it is possible to refine the above reasoning to prove the following statement.

Corollary 5.2.— *Let $G \subset \mathrm{GL}(n, \mathbf{C})$ be an algebraic group such that $G/L(G)$ is topologically generated by $m-1$ elements. Then there are constant matrices $A_1, \ldots, A_{m-1}$ and there is a matrix A_∞ with polynomial coefficients (all matrices of order $n \times n$) such that the matrix differential equation*

$$y' = \Big(\frac{A_1}{z - p_1} + \cdots + \frac{A_{m-1}}{z - p_{m-1}} + A_\infty\Big) y$$

has differential Galois group $G \subset \mathrm{GL}(n, \mathbf{C})$.

This result is close to the one of Mitschi-Singer. We note however that there seems to be no bound on the degrees of the coefficients of the matrix A_∞.

6. THE FORMAL THEORY

In this section the differential field is $\hat{K} := \mathbf{C}((z))$. The classification of differential equations over $\hat{K}$ is well known (Birkhoff, Turrittin, Malgrange, Levelt, et al.). We will present this classification in a somewhat different form, which clarifies the connection with the differential Galois group. For this purpose we need a *universal differential extension $\hat{R}$ of $\hat{K}$* which can be defined as follows:

(i) The $\hat{K}$-algebra $\hat{R}$ has a differentiation extending the one of $\hat{K}$.

(ii) Every differential ideal of $\hat{R}$ is trivial, i.e., 0 or $\hat{R}$.

(iii) Every homogeneous linear differential equation of order n over $\hat{K}$ has a solution space in $\hat{R}$, which is a $\mathbf{C}$-vector space of dimension n.

(iv) $\hat{R}$ is minimal in the sense that $\hat{R}$ is generated over $\hat{K}$ by all the solutions and their

derivatives of all homogeneous linear equations over $\hat{K}$.
For any differential field one can show that there exists an universal differential extension (as above). Moreover one can show that this extension is unique up to differential isomorphism. The interesting thing is that one can write down this extension explicitly in the case of $\hat{K}$.

We introduce first some notations: δ is the derivation $z\frac{d}{dz}$. We will write f' for $z\frac{d}{dz}f$. Put $\mathcal{Q} = \cup_{m\geq 1} z^{-1/m}\mathbf{C}[z^{-1/m}]$. The $\hat{K}$-algebra $\hat{R}$ is represented by generators and relations as follows.
$\hat{R} = \hat{K}[\{z^a\}_{a\in\mathbf{C}}, \{e(q)\}_{q\in\mathcal{Q}}, l]$ and the *relations* are:
$z^{a+b} = z^a z^b$, $e(q_1+q_2) = e(q_1)e(q_2)$ and $z^a = z^a \in \hat{K}$ for $a \in \mathbf{Z}$.
The *differentiation* on $\hat{R}$ is given by:
$(z^a)' = az^a$, $e(q)' = qe(q)$, $l' = 1$.
The *intuitive interpretation of* $\hat{R}$ is:
z^a is the function $e^{a\log(z)}$, l is the function $\log(z)$ and $e(q)$ is the function $exp(\int q\frac{dz}{z})$.
In sectors at the point $z = 0$ this interpretation makes sense and can be combined with the lifting of certain formal Laurent series in $\hat{K}$ to actual meromorphic functions on a sector.

Some differential automorphisms of $\hat{R}/\hat{K}$:

The *formal monodromy* γ is the differential automorphism of $\hat{R}/\hat{K}$ defined by:
$\gamma(z^a) = e^{2\pi i a}z^a$ and $\gamma(l) = l + 2\pi i$,
in particular the action of γ on the algebraic closure of $\hat{K}$ and on $\mathcal{Q}$ is defined.
$\gamma(e(q)) = e(\gamma q)$.

The exponential torus $\mathrm{Hom}(\mathcal{Q}, \mathbf{C}^*)$ acts as a group of differential automorphisms of $\hat{R}/\hat{K}$. For an element c of this group, one defines the differential automorphism $\tilde{c}$ by
$\tilde{c}z^a = z^a$, $\tilde{c}l = l$, $\tilde{c}e(q) = c(q)e(q)$.
It is possible to write down all the differential automorphisms of $\hat{R}/\hat{K}$. This has the structure of an affine group scheme over $\mathbf{C}$. The subgroup generated by the formal monodromy and the exponential torus is a dense subgroup in a certain sense. For our purposes the formal monodromy and the exponential torus will suffice.

The triple $(V, \{V_q\}, \gamma_V)$ *associated to a differential equation over* $\hat{K}$.

Let $L(y) = 0$ be a homogeneous differential equation of order n over $\hat{K}$. Put $V := \{r \in \hat{R} |\ L(r) = 0\}$, the solution space of L in $\hat{R}$. Then V is a vector space over $\mathbf{C}$ of dimension n. The set V is invariant under the actions of the formal monodromy and the exponential torus. Let γ_V denote the restriction of γ to V. Put $V_q := V \cap \hat{K}[\{z^a\}_{a\in\mathbf{C}}, l]e(q)$. Then for almost all $q \in \mathcal{Q}$ the vector space $V_q = 0$. One has further $V = \oplus_{q\in\mathcal{Q}} V_q$ and $\gamma_V(V_q) = V_{\gamma q}$ for all $q \in \mathcal{Q}$. This defines the triple $(V, \{V_q\}, \gamma_V)$ associated to the differential equation over $\hat{K}$.

The important observation is:

The differential equation over $\hat{K}$ is equivalent to the associated triple $(V, \{V_q\}, \gamma_V)$.

We will make this statement more precise by considering the category $\mathbf{Gr}_1$, whose objects are the triples $(V, \{V_q\}, \gamma_V)$ satisfying:

(i) V is a finite dimensional vector space over $\mathbf{C}$,

(ii) $\{V_q\}_{q \in \mathcal{Q}}$ is a family of subspaces such that $V = \oplus V_q$.

(iii) γ_V is a $\mathbf{C}$-linear automorphism of V such that $\gamma_V(V_q) = V_{\gamma q}$ for all $q \in \mathcal{Q}$.

One can give $\mathbf{Gr}_1$ (in an obvious way) the structure of a (neutral) Tannakian category. Let $\mathrm{Diff}_{\hat{K}}$ denote the Tannakian category of the differential modules over $\hat{K}$.

Proposition 6.1.— *The above construction is an equivalence between the Tannakian categories* $\mathrm{Diff}_{\hat{K}}$ *and* $\mathbf{Gr}_1$.

Corollary 6.2.— *Let the triple $(V, \{V_q\}, \gamma_V)$ be associated to the differential equation $L(y) = 0$ over $\hat{K}$. Then the differential Galois group G of L can be identified with the* algebraic *subgroup of* $\mathrm{GL}(V)$ *generated by γ_V and the action of the exponential torus on V.*

Examples 6.3 : *The Airy equation*

The Airy equation $y'' = zy$ has a singular point at $z = \infty$. The triple associated to the equation is $V = V_{z^{3/2}} \oplus V_{-z^{3/2}}$. The two spaces $V_{z^{3/2}}, V_{-z^{3/2}}$ have both dimension 1. After a suitable choice of bases for the two spaces the formal monodromy has the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the exponential torus has the form $\left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbf{C}^* \right\}$. The differential Galois group is therefore the infinite Dihedral group $D_\infty \subset \mathrm{SL}(2)$.

The information in this section on the formal differential Galois group suffices to give a proof of the "easy halves" of the theorems.

Proposition 6.4. The easy implications for the local and the global theorem.— (1) *Let G be the differential Galois group of an equation over the field $\mathbf{C}(\{z\})$. Then $G/L(G)$ is topologically generated by one element.*

(2) *Let X be a compact Riemann surface of genus g, $S \subset X$ a finite subset with cardinality $m \geq 0$ and G the differential Galois group of a differential equation over X with singularities in S. Then the natural homomorphism $\pi_1(X \setminus S, *) \to G/L(G)$ has dense image.*

Proof.— (1) The group $G/L(G)$ is according to section 2 also the differential Galois group of some equation $y' = Ay$ over $\mathbf{C}(\{z\})$. The differential Galois group H of this equation over $\mathbf{C}((z))$ is a subgroup of $G/L(G)$. The latter group has a trivial maximal torus

and thus the exponential torus is not present in H. Therefore the differential equation is regular singular and is equivalent to $zy' = By$ where B is a constant matrix. It is well known that the differential Galois group (either over $\mathbf{C}(\{z\})$ or over $\mathbf{C}((z))$) is the algebraic subgroup of $\mathrm{GL}(n)$ generated by $e^{2\pi iB}$.

(2) Let K denote the function field of X. The group $G/L(G)$ is the differential Galois group of a matrix equation $y' = Ay$ over X which has its singularities in the set S. As in (1), it follows that the equation has only regular singularities. Take a point $c \in X \setminus S$ and let V be the space of the local solutions of the equation. The Picard-Vessiot field L can be identified with the subfield of K_c generated by K and (the coordinates of the elements of) V. The monodromy of $y' = Ay$ on $X \setminus S$ induces a homomorphism $\pi_1(X \setminus S, c) \to \mathrm{GL}(V)$. The image of this homomorphism lies in $G/L(G)$. Let $f \in L$ be invariant under this image. Then f extends to a meromorphic function on $X \setminus S$. Since the singularities of $y' = Ay$ are regular singular, f extends to a meromorphic function on all of X. Therefore $f \in K$. From the Galois correspondence we draw the conclusion that the image of $\pi_1(X \setminus S, c)$ in $G/L(G)$ is Zariski dense.

7. THE ANALYTIC LOCAL THEORY

In this section the differential field K is $\mathbf{C}(\{z\})$. For notational convenience we take the derivation δ to be $f \mapsto \delta(f) = f' = z\frac{df}{dz}$. For the description of the differential modules over K we introduce a second category $\mathbf{Gr_2}$. First we will need some notation. Let $d \in \mathbf{R}$ and $q \in \mathcal{Q}$, $q \neq 0$. Then d (or rather e^{id}) is considered as a direction at the point $z = 0$. The *front of* q, in notation $Fr(q)$, consists of the d's such that the function $h := e^{\int q\frac{d}{dz}}$ has maximal decrease in the direction d. In other words $|h(re^{id})|$ has maximal decrease when $r > 0$ tends to zero. For the example $q = z^{-1}$ one finds $|h(re^{id})| = e^{r^{-1}\cos(d)}$ and the front $Fr(z^{-1})$ is thus $\pi + 2\pi\mathbf{Z}$. For any $q \in \mathcal{Q}$, one denotes the projection $V \to V_q$ (along the other $V_{q'}$'s) by pr_q. Using this terminology we can define an object $(V, \{V_q\}, \gamma_V, St_{V,d})$ of $\mathbf{Gr_2}$ as follows:

(1) $(V, \{V_q\}, \gamma_V)$ is an object of $\mathbf{Gr_1}$.

(2) For every $d \in \mathbf{R}$ there is given an automorphism $St_{V,d}$ of V of the form

$$id + \sum_{q_1,q_2;\ d \in Fr(q_2 - q_1)} N_{q_2,q_1},$$

where N_{q_2,q_1} is a $\mathbf{C}$-linear map $N_{q_2,q_1} : V \overset{pr_{q_2}}{\to} V_{q_2} \to V_{q_1} \subset V$.

(3) The $St_{V,d}$ should satisfy the relation $\gamma_V^{-1} St_{V,d} \gamma_V = St_{V,d+2\pi}$.

The $St_{V,d}$ are seen to be unipotent maps.

If $(V, \{V_q\}, \gamma_V)$ is fixed then there are modulo 2π only finitely many d's such that there exists q_2, q_1 with $d \in Fr(q_2 - q_1)$ and $V_{q_2} \neq 0$, $V_{q_1} \neq 0$. The possibilities for the $St_{V,d}$ form thus a finite dimensional vector space over $\mathbf{C}$.

Morphisms between two objects are $\mathbf{C}$-linear maps preserving all the data. The tensor product of two objects $V = (V, \{V_q\}, \gamma_V, St_{V,d})$ and $W = (W, \{W_q\}, \gamma_W, St_{W,d})$ is defined as the vector space $V \otimes W$ with the data:
$(V \otimes W)_q = \oplus_{q_1,q_2;q_1+q_2=q} V_{q_1} \otimes W_{q_2}$.
$\gamma_{V\otimes W} = \gamma_V \otimes \gamma_W$ and
$St_{V\otimes W,d} = St_{V,d} \otimes St_{W,d}$.

One can show that $\mathbf{Gr}_2$ is a neutral Tannakian category over $\mathbf{C}$. Let Diff_K denote the Tannakian category of the differential modules over $K = \mathbf{C}(\{z\})$. The main results of Martinet and Ramis on multisummation can now be phrased as:

Theorem 7.1.— *The multisummation operator provides an equivalence between the Tannakian categories* Diff_K *and* $\mathbf{Gr}_2$.

Corollary 7.2.— *Let L be a linear homogeneous differential equation over K and let $(V, \{V_q\}, \gamma_V, St_{V,d})$ denote the object of* $\mathbf{Gr}_2$ *associated to L. Then the differential Galois group of L can be identified with the Zariski closure of the subgroup of* $\mathrm{GL}(V)$ *generated by:*

(1) *The formal monodromy* γ_V.
(2) *The exponential torus.*
(3) *The collection* $\{St_{V,d}\}$, *called the* Stokes matrices *of L.*

Remark.— It must be stressed that the theorem and the corollary are at the heart of the theory of multisummation and form moreover the essential part of the proof of Ramis' local theorem.

The category $\mathbf{Gr}_3$

The categories $\mathbf{Gr}_1$ and $\mathbf{Gr}_2$ were introduced in [P1] in order to obtain a nice and compact formulation of the work of Martinet and Ramis. A further reformulation, due to Ramis, will simplify the proof of the local theorem even further. The category $\mathbf{Gr}_3$ is defined by:
the objects are tuples $(V, \{V_q\}, \gamma_V, st_{V,d})$ with

(1) $(V, \{V_q\}, \gamma_V)$ is an object of $\mathbf{Gr}_1$.
(2) For every $d \in \mathbf{R}$ there is given a $st_{V,d} \in \oplus_{q_1,q_2;d\in Fr(q_2-q_1)} \mathrm{Hom}(V_{q_1}, V_{q_2})$.
(3) The $st_{V,d}$ should satisfy $\gamma_V^{-1} st_{V,d} \gamma_V = st_{V,d+2\pi}$.

We identify, as before, $\mathrm{Hom}(V_{q_1}, V_{q_2})$ with a linear subspace of $\mathrm{End}(V)$. A morphism $f : (V, \{V_q\}, \gamma_V, st_{V,d}) \to (W, \{W_q\}, \gamma_W, st_{W,d})$ is a linear map $V \to W$ satisfying $f(V_q) \subset W_q$, $f \circ \gamma_V = \gamma_W \circ f$, $f \circ st_{V,d} = st_{W,d} \circ f$. The tensor product of two objects $(V, \{V_q\}, \gamma_V, st_{V,d}), (W, \{W_q\}, \gamma_W, st_{W,d})$ is the vector space $V \otimes W$ with the data $(V\otimes W)_q = \oplus_{q_1,q_2;q_1+q_2=q} V_{q_1}\otimes W_{q_2}$, $\gamma_{V\otimes W} = \gamma_V \otimes \gamma_W$ and $st_{V\otimes W,d} = st_{V,d}\otimes id_W + id_V \otimes st_{W,d}$.

It is easily seen that $\mathbf{Gr}_3$ is again a neutral Tannakian category. In fact, the Tannakian categories $\mathbf{Gr}_2$ and $\mathbf{Gr}_3$ are isomorphic.

Lemma 7.3.— (1) *The exponential map induces an equivalence of Tannakian categories* $\mathbf{Exp} : \mathbf{Gr}_3 \to \mathbf{Gr}_2$.

(2) *Let the object* $(V, \{V_q\}, \gamma_V, st_{V,d})$ *be associated to a differential equation over* $\mathbf{C}(\{z\})$. *Then the differential Galois group* $G \subset \mathrm{GL}(V)$ *is the smallest algebraic subgroup such that:*

(a) *The exponential torus and* γ_V *belong to* G.

(b) *All* $st_{V,d}$ *belong to the Lie algebra of* G.

Proof.— The exponential map associates to $(V, \{V_q\}, \gamma_V, st_{V,d})$ the object $(V, \{V_q\}, \gamma_V, St_{V,d})$ with $St_{V,d} = exp(st_{V,d})$. It is easily seen that this results into an equivalence of Tannakian categories. The second part of the lemma is a reformulation of 7.2.

Multisummation and the Stokes matrices $St_{V,d}$

Consider a differential equation $L(y) = g$ over K and with $g \in K$. Let $\hat{f} \in \hat{K}$ be a formal solution of the equation. The *main theorem of the asymptotic theory of differential equations* states that for a sector S at 0 with small enough opening, there exists a meromorphic function f on S with asymptotic expansion $\hat{f}$ and still satisfying $L(f) = g$. In general this f is not unique. Multisummation produces a unique choice for the f above. For $d \in \mathbf{R}$ one considers the direction e^{id} at 0. There are, modulo $2\pi\mathbf{Z}$, finitely many *singular directions* d for the differential equation $L(y) = g$. For a direction d, which is not singular, there is a multisummation operator S_d which maps a formal solution $\hat{f}$ to a solution $S_d(\hat{f})$ of the equation, which lives on a certain sector with bisector d and has $\hat{f}$ as asymptotic expansion. In general $S_d(\hat{f})$ depends on the choice of d. This is usually called the *Stokes phenomenon.* However $S_d(\hat{f})$ does not depend on $d \in (a, b)$ if this interval does not contain any singular direction.

Let d be a singular direction. Then we define $S_{d+}(\hat{f})$ and $S_{d-}(\hat{f})$ as $S_{d+\epsilon}(\hat{f})$ and $S_{d-\epsilon}(\hat{f})$ for $\epsilon > 0$ and small enough. The difference between $S_{d+}(\hat{f})$ and $S_{d-}(\hat{f})$ is measured by the Stokes matrix $St_{V,d}$. We will make this more precise.

Let M be a differential module over $K = \mathbf{C}(\{z\})$. The solution space V is defined as $V = \ker(\partial, \hat{R} \otimes_K M)$. The ring $\hat{R}$ is given a basis over $\hat{K} = \mathbf{C}((z))$, e.g. $z^a l^n e(q)$ with $a \in \mathbf{C}$ and $0 \leq Re(a) < 1$; $n \in \mathbf{Z}, n \geq 0$ and $q \in \mathcal{Q}$. Further M can be given a basis over K. Combining this one obtains a basis $\{B_i\}_{i \in I}$ of $\hat{R} \otimes_K M$ over $\hat{K}$. The elements $v \in V$ can be written as finite sums $\sum \hat{f}_i B_i$ with $\hat{f}_i \in \hat{K}$. The $\hat{f}_i$ are formal solutions of a set of (inhomogeneous) linear differential equations over K. For a direction d, which is not singular, one defines $S_d(v)$ as $\sum S_d(\hat{f}_i)\mathrm{int}_d(B_i)$, where $\mathrm{int}_d(B_i)$ is the interpretation of B_i (this involves z^a and the logarithm) in the direction d. There results a $\mathbf{C}$-linear bijection $S_d : V \to \mathrm{Sol}_d$, where Sol_d is the vector space of the solutions of

M on a suitable sector containing the direction d. For *any* direction d, there are two isomorphisms $S_{d^+}, S_{d^-} : V \to \mathrm{Sol}_d$. One defines $St_{V,d} = S_{d^-}^{-1} S_{d^+} \in \mathrm{GL}(V)$. Clearly $St_{V,d}$ is the identity if d is not a singular direction. It is not difficult to prove that the object $(V, \{V_q\}, \gamma_V, St_{V,d})$ belongs to the category $\mathbf{Gr}_2$, and that multisummation produces in fact a functor $F_2 : \mathrm{Diff}_K \to \mathbf{Gr}_2$ of Tannakian categories. The proof of 7.1, i.e., proving that F_2 is an equivalence of categories, is rather involved (see [P3] and [R2]).

The relation between the formal and the topological monodromy

Let the differential module M over K be given with $F_2(M) = (V, \{V_q\}, \gamma_V, St_{V,d})$. Let $0 \leq d_1 < d_2 < \cdots < d_m < 2\pi$ denote the singular directions (in fact for the differential module $\mathrm{Hom}(M, M)$) lying in $[0, 2\pi)$. The topological monodromy γ_M of M is easily seen to satisfy (up to conjugation) the formula

$$\gamma_M = \gamma_V \cdot St_{V,d_1} St_{V,d_2} \cdots St_{V,d_m}.$$

Examples 7.4 : *The Airy equation, continued*

We consider again the Airy equation $y'' = zy$. There are two singular directions $d = 0, \pi$ modulo $2\pi\mathbf{Z}$. The topological monodromy is trivial and the formal monodromy is not trivial. The formula for the topological monodromy implies that both Stokes matrices $St_{V,0}$ and $St_{V,\pi}$ are different from 1. The two matrices have the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ with respect to the decomposition $V = V_{z^{3/2}} \oplus V_{-z^{3/2}}$. It follows from 7.2 that the differential Galois group of the Airy equation over $\mathbf{C}(z)$ is SL(2).

The same argument shows that the differential Galois group of any equation $y'' = ry$ with $r \in \mathbf{C}[z]$ a polynomial of odd degree is SL(2).

8. THE PROOF OF THE LOCAL THEOREM

The following lemma will be a guide for the construction of a suitable object in the category $\mathbf{Gr}_3$.

The data

V is a finite dimensional vector space over $\mathbf{C}$ and $G \subset \mathrm{GL}(V)$ is an algebraic subgroup. T denotes a maximal torus and $\underline{g}, \underline{t} \subset \mathrm{End}(V)$ are the Lie algebras of G and T.

The action of T on V yields a decomposition $V = \oplus_{i=1}^{s} V_{\chi_i}$, where the χ_i are distinct characters of T and the non-trivial spaces V_{χ_i} are defined as $\{v \in V |\ tv = \chi_i(t)v$ for all $t \in T\}$.

For each i, j one identifies $\mathrm{Hom}(V_{\chi_i}, V_{\chi_j})$ with a linear subset of $\mathrm{End}(V)$ by identifying

$L \in \mathrm{Hom}(V_{\chi_i}, V_{\chi_j})$ with $V \stackrel{pr_i}{\to} V_{\chi_i} \stackrel{L}{\to} V_{\chi_j} \subset V$, where pr_i denotes the projection onto V_{χ_i}, along $\oplus_{k \neq i} V_{\chi_k}$.

The adjoint action of T on $\underline{g}$ yields a decomposition $\underline{g} = \underline{g}_0 \oplus \sum_{\alpha \neq 0} \underline{g}_\alpha$. By definition, the adjoint action of T on $\underline{g}_0$ is the identity and is multiplication by the character $\alpha \neq 0$ on the spaces $\underline{g}_\alpha$. (We note that here the additive notation for characters is used. In particular, $\alpha \neq 0$ means that α is not the trivial character.)

Any $B \in \underline{g}$ can be written as $\sum B_{i,j}$ with $B_{i,j} \in \mathrm{Hom}(V_{\chi_i}, V_{\chi_j})$. The adjoint action of $t \in T$ on B has the form $Ad(t)B = \sum \chi_i^{-1}\chi_j(t)B_{i,j}$. It follows that the $\alpha \neq 0$ with $\underline{g}_\alpha \neq 0$ have the form $\chi_i^{-1}\chi_j$. In particular $\sum_{i,j;\chi_i^{-1}\chi_j=\alpha} B_{i,j} \in \underline{g}_\alpha \subset \underline{g}$. Let $L(G)$ denote, as before, the subgroup of G generated by all the conjugates of T.

Lemma 8.1 (Ramis).— (1) *$L(G)$ is a normal algebraic subgroup of G (and of G^o).*

(2) *The Lie algebra of $L(G)$ is generated by the subspaces $\underline{t}$ and $\{\underline{g}_\alpha\}_{\alpha \neq 0}$.*

Proof.— The first statement follows from [Bo], Ch 2, Prop.(7.5) p.190 and Thm.(7.6) p.192. Consider some $\alpha \neq 0$ and a non-zero element $\xi \in \underline{g}_\alpha$. From the definition of $\underline{g}_\alpha$ and $\alpha \neq 0$ it follows that there is an ordering, denoted by $V_1, \ldots, V_s$, of the spaces $\{V_{\chi_i}\}$, such that ξ maps each V_i into some V_j with $j > i$. In particular, ξ is nilpotent and $\mathbf{C}\xi$ is an algebraic Lie algebra corresponding to the algebraic subgroup $\{exp(c\xi) | c \in \mathbf{C}\}$ of G. Let $\underline{h}$ denote the Lie algebra generated by the algebraic Lie algebras $\underline{t}$ and $\mathbf{C}\xi$ for all $\xi \in \underline{g}_\alpha$ with $\alpha \neq 0$. Then $\underline{h}$ is an algebraic Lie algebra. ([Bo], loc. cit.).

Take an element $t \in T$ such that all $\chi_i(t)$ are distinct. Then clearly $t \cdot exp(\xi)$ is semi-simple and lies therefore in a conjugate of the maximal torus T. Thus $exp(\xi) = t^{-1} \cdot (t \cdot exp(\xi)) \in L(G)$ and ξ lies in the Lie algebra of $L(G)$. This proves that the Lie algebra $\underline{h}$ is a subset of the Lie algebra of $L(G)$.

On the other hand, $\underline{h}$ is easily seen to be an ideal in $\underline{g}$. The connected normal algebraic subgroup $H \subset G^o$ corresponding to $\underline{h}$ contains T and therefore $L(G)$. This proves the other inclusion.

We start the proof of Ramis' local theorem by considering the data:
V is a finite dimensional vector space over $\mathbf{C}$ and $G \subset \mathrm{GL}(V)$ is an algebraic group such that $G/L(G)$ is topologically generated by one element a. It suffices to construct an object $(V, \{V_q\}, \gamma_V, st_{V,d})$ in the category $\mathbf{Gr}_3$ with group G, i.e., G is the smallest algebraic subgroup of $\mathrm{GL}(V)$ such that:

(1) G contains γ_V and the exponential torus.

(2) The Lie algebra $\underline{g}$ of G contains all $st_{V,d}$.

$*$ Choose a maximal torus $T \subset G$ and a representative $A \in G$ of $a \in G/L(G)$. Since T is also a maximal torus of $L(G)$, there exists $B \in L(G)$ such that $ATA^{-1} = BTB^{-1}$. After replacing A by $B^{-1}A$ we may suppose that $ATA^{-1} = T$.

* Let $V = V_{\chi_1} \oplus \cdots \oplus V_{\chi_s}$ denote the decomposition of V with respect to the action of T.
* The element $A \in G$ permutes the spaces V_{χ_i}. More precisely, $A(V_{\chi_i}) = V_{\chi_j}$ where the character χ_j is given by $\chi_j(t) = \chi_i(A^{-1}tA)$. We will write $\tilde{A}\chi_i = \chi_j$. Indeed, for $t \in T, v \in V_{\chi_i}$ one has

$$tAv = A(A^{-1}tA)v = A\chi_i(A^{-1}tA)v = \chi_i(A^{-1}tA)Av.$$

* We recall that $\mathcal{Q} = \cup_{m \geq 1} z^{-1/m}\mathbf{C}[z^{-1/m}]$ carries an action of γ given by $\gamma z^{\lambda} = e^{2\pi i\lambda} z^{\lambda}$.

Lemma 8.2.— *There are elements $q_1, \ldots, q_s \in \mathcal{Q}$ such that*

(1) *If $\tilde{A}\chi_i = \chi_j$ then $\gamma(q_i) = q_j$.*

(2) *If $n_1\chi_1 + \cdots + n_s\chi_s = 0$ (here with additive notation for characters) for some $n_1, \ldots, n_s \in \mathbf{Z}$, then $n_1q_1 + \cdots + n_sq_s = 0$.*

Let $N \geq 1$ be any integer, then there exists $q_1, \ldots, q_s$ satisfying (1) *and* (2) *and such that for any $i \neq j$ the degrees of $q_i - q_j$ in z^{-1} are $\geq N$.*

Proof.— Conditions (1) and (2) can be translated into: the $\mathbf{Z}$-module $\mathbf{Z}\chi_1 + \cdots + \mathbf{Z}\chi_s$ can be embedded into $\mathcal{Q}$ such that the action of $\tilde{A}$ is compatible with the action of γ on $\mathcal{Q}$.

Let $\overline{\mathbf{Q}}$ denote the algebraic closure of $\mathbf{Q}$. Consider $M := \overline{\mathbf{Q}} \otimes (\mathbf{Z}\chi_1 + \cdots + \mathbf{Z}\chi_s) \supset \mathbf{Z}\chi_1 + \cdots + \mathbf{Z}\chi_s$ with the induced $\tilde{A}$ action. Since a power of $\tilde{A}$ acts trivially on M we may present this action by $M = \oplus_{\zeta_j} M_{\zeta_j}$, where ζ_j runs in a finite set of all roots of unity and $\tilde{A}$ acts on M_{ζ_j} as multiplication by ζ_j.

Further $\mathcal{Q} = \oplus_{\lambda \in \mathbf{Q},\, \lambda < 0} \mathbf{C}z^{\lambda}$ and γ acts on $\mathbf{C}z^{\lambda}$ as multiplication by $e^{2\pi i\lambda}$. Define $\lambda_j \in \mathbf{Q}, \lambda_j < 0$ by $e^{2\pi i\lambda_j} = \zeta_j$ and λ_j is maximal. Choose for every j an embedding of $\overline{\mathbf{Q}}$-vector spaces $M_{\zeta_j} \subset z^{\lambda_j} z^{-N}\mathbf{C}[z^{-1}]$. Then the resulting embedding $M \subset \mathcal{Q}$ has the required properties. Moreover, any non-zero element of M is mapped to an element of degree $\geq N$ in the variable z^{-1}.

We continue now the proof of the Ramis' local theorem. Define the formal structure $(V, \{V_q\}, \gamma_V)$ (i.e., an object of $\mathbf{Gr}_1$) by $V_{\chi_i} = V_{q_i}$ and $\gamma_V = A$. As before $\underline{g}$ denotes the Lie algebra of G (or G^o). The decomposition of $\underline{g}$ with respect to the adjoint action of the torus T has already been made explicit, namely

$$\underline{g}_{\alpha} = \oplus_{i,j;\chi_i^{-1}\chi_j = \alpha} \underline{g} \cap \mathrm{Hom}(V_{q_i}, V_{q_j}).$$

For the definition of the structure of an object from the category $\mathbf{Gr}_3$ on V we may choose arbitrary elements $st_{V,d} \in \underline{g}_{\alpha}$, where $\alpha = \chi_i^{-1}\chi_j$, $d \in Fr(q_i - q_j)$, $0 \leq d < 2\pi$. The number of directions d modulo 2π in $Fr(q_i - q_j)$ is by construction sufficiently large to ensure a choice of the set $\{st_{V,d}\}$ such that these elements generate the vector space $\oplus_{\alpha \neq 0} \underline{g}_{\alpha}$.

Finally, we verify that the algebraic group $\tilde{G}$, associated to the object $(V, \{V_q\}, \gamma_V, st_{V,d})$, is equal to G. By construction $\tilde{G} \subset G$ and $\tilde{G}$ is the smallest algebraic group with:

(a) The exponential torus and γ_V lie in $\tilde{G}$.

(b) The Lie algebra of $\tilde{G}$ contains all $st_{V,d}$.

By construction, the exponential torus is equal to T and its Lie algebra $\underline{t}$ lies in $\tilde{G}$. Again by construction, the $\underline{g}_\alpha$ (with $\alpha \neq 0$) belong to the Lie algebra of $\tilde{G}$. Thus $\tilde{G}$ contains $L(G)$. The choice of γ_V implies that $\tilde{G} = G$.

9. THE CONSTRUCTIVE INVERSE PROBLEM

In this section we will present a constructive proof of the following "gem" which is at the heart of the work of Mitschi and Singer.

Theorem 9.1 (Mitschi-Singer).— *The field C is supposed to be algebraically closed and of characteristic 0. Every connected semi-simple linear algebraic group is the differential Galois group of an equation $y' = (A_0 + A_1 z)y$ over $C(z)$, where A_0, A_1 are constant matrices.*

The proof given here uses the language of Tannakian categories (see [De-M]). For a differential module M (over $C(z)$) one denotes by $\{\{M\}\}$ the full Tannakian subcategory of $\mathrm{Diff}_{C(z)}$ generated by M. By definition the objects of $\{\{M\}\}$ are the differential modules isomorphic to a finite direct sum of subquotients M_1/M_2 of some $M \otimes \cdots \otimes M \otimes M^* \otimes \cdots \otimes M^*$ (i.e., $M_2 \subset M_1$ are (differential) submodules of such a tensor product). For any linear algebraic group L (over C) one denotes by Repr_L the Tannakian category of the finite dimensional representations of L (over C). The differential Galois group of M is L if there is an equivalence between the Tannakian categories $\{\{M\}\}$ and Repr_L.

Let a finite dimensional vector space V over C and an algebraic subgroup $G \subset \mathrm{GL}(V)$ be given. Our first aim is to produce a differential module $M = (C(z) \otimes V, \partial)$ and a functor of Tannakian categories $\mathrm{Repr}_G \to \{\{M\}\}$.

9.2. The functor $\mathrm{Repr}_G \to \{\{M\}\}$

The Lie algebra of G will be written as $\underline{g} \subset \mathrm{End}(V)$. One chooses a matrix $A(z) \in C(z) \otimes \underline{g} \subset C(z) \otimes \mathrm{End}(V)$. To this choice there corresponds a differential equation $y' = A(z)y$ and a differential module $M := (C(z) \otimes V, \partial)$ with ∂ defined by $\partial(v) = -A(z)v$ for all $v \in V$. Let a representation $\rho : G \to \mathrm{GL}(T)$ be given. The induced maps $\underline{g} \to \mathrm{End}(T)$ and $C(z) \otimes \underline{g} \to C(z) \otimes \mathrm{End}(T)$ are also denoted by ρ. One associates to (T, ρ) the differential module $(C(z) \otimes T, \partial)$ with $\partial(t) = -\rho(A(z))t$ for all $t \in T$. The corresponding differential equation is $y' = \rho(A(z))y$. In this way one obtains a functor of Tannakian categories $\mathrm{Repr}_G \to \mathrm{Diff}_{C(z)}$. We claim that every $(C(z) \otimes T, \partial)$, as above, lies in fact in $\{\{M\}\}$.

Indeed, consider $\{\{V\}\}$, the full subcategory of Repr_G generated by V (the definition is similar to the definition of $\{\{M\}\}$). It is known that $\{\{V\}\} = \mathrm{Repr}_D$ (see

[De-M]). The representation $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$ is mapped to the differential module $M \otimes \cdots \otimes M \otimes M^* \otimes \cdots \otimes M^*$. Let $V_2 \subset V_1$ be G-invariant subspaces of $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$. Then V_2 and V_1 are invariant under the action of $\underline{g}$. Since $A(z) \in C(z) \otimes \underline{g}$, one has that $C(z) \otimes V_2$ and $C(z) \otimes V_1$ are (differential) submodules of $M \otimes \cdots \otimes M \otimes M^* \otimes \cdots \otimes M^*$. It follows that the differential module $(C(z) \otimes V_1/V_2, \partial)$ lies in $\{\{M\}\}$. This proves the claim.

The next step is to make the differential Galois group L of M and the equivalence $\{\{M\}\} \to \mathrm{Repr}_L$ as concrete as we can.

9.3. Differential modules over O

Fix some $c \in C$ and let O denote the localization of $C[z]$ at $(z-c)$. A differential module (N, ∂) over O is a finitely generated O-module equipped with a C-linear map ∂ satisfying $\partial(fn) = f'n + f\partial(n)$ for all $f \in O$ and $n \in N$. It is an exercise to show that N has no torsion elements. It follows that N is a free, finitely generated O-module. Let Diff_O denote the category of the differential modules over O. The functor $N \mapsto C(z) \otimes_O N$ induces an equivalence of Diff_O with a full subcategory of $\mathrm{Diff}_{C(z)}$. It is easily seen that this full subcategory has as objects the differential modules over $C(z)$ which are regular at $z = c$.

Let $\hat{O} = C[[z-c]]$ denote the completion of O. For any differential module N over O of rank n, one writes $\hat{N}$ for $\hat{O} \otimes N$. We note that $\hat{N}$ is a differential module over $\hat{O}$ and that $\hat{N}$ is in fact a trivial differential module. The space $ker(\partial, \hat{N})$ is a vector space over C of dimension n, which will be called $\mathrm{Sol}_c(N)$, the solution space of N over $C[[z-c]]$ (and also over $C((z-c))$). The canonical map $\mathrm{Sol}_c(N) \to \hat{N}/(z-c)\hat{N} = N/(z-c)N$ is an isomorphism. Let Vect_C denote the Tannakian category of the finite dimensional vector spaces over C. The above construction $N \mapsto N/(z-c)N$ is a fibre functor of $\mathrm{Diff}_O \to \mathrm{Vect}_C$. For a fixed object M of Diff_O one can consider the restriction $\omega : \{\{M\}\} \to \mathrm{Vect}_C$, which is again a fibre functor. The differential Galois group L of M is defined in [De-M] as $\underline{\mathrm{Aut}}^{\otimes}(\omega)$. By definition L acts on $\omega(N)$ for every object N of $\{\{M\}\}$. Thus we find a functor $\{\{M\}\} \to \mathrm{Repr}_L$, which is an equivalence of Tannakian categories. The composition $\{\{M\}\} \to \mathrm{Repr}_L \to \mathrm{Vect}_C$ (where the last arrow is the forgetful functor) is the same as ω. (We note that the above remains valid if we replace O by any localization of $C[z]$.)

9.4. Some observations

(9.4.1) Let G again be an algebraic subgroup of $\mathrm{GL}(V)$, let $A(z) \in C(z) \otimes \underline{g}$ be chosen. Suppose that the matrix $A(z)$ has no poles at $z = c$. Then we have functors of Tannakian categories $\mathrm{Repr}_G \to \{\{M\}\}$ and $\{\{M\}\} \to \mathrm{Repr}_L$. The last functor is made by considering M as differential module over O. The composition of the two functors

maps a representation (W, ρ) of G to a representation of L on $O/(z-c)O \otimes W$ which is canonically isomorphic to W. It follows that L is an algebraic subgroup of G.

(9.4.2) Suppose that L is a proper subgroup of G, then there exists a representation (W, ρ) of G and a line $\tilde{W} \subset W$ such that L stabilizes $\tilde{W}$ and G does not. The differential module $(C(z) \otimes W, \partial)$ has as image in Repr_L the space W with its L-action. The L-invariant subspace $\tilde{W}$ corresponds with a one-dimensional (differential) submodule $C(z)w \subset C(z) \otimes W$. After multiplication of w by an element in $C(z)$, we may suppose that $w \in C[z] \otimes W$ and that the coordinates of w with respect to a basis of W have g.c.d. 1. Let us write $\frac{d}{dz}$ for the differentiation on $C(z) \otimes W$, given by $\frac{d}{dz} fa = f'a$ for $f \in C(z)$ and $a \in W$. Then one finds the equation

$$\Big[\frac{d}{dz} - \rho(A(z))\Big] w = cw \text{ for some } c \in C(z).$$

(9.4.3) The idea for the rest of the proof is to make a choice for $A(z)$ which contradicts the equation for w above. For a given proper algebraic subgroup H of G one can produce a suitable $A(z)$ which contradicts the statement that L lies in a conjugate of H. In general however one has to consider infinitely many (conjugacy classes) of proper algebraic subgroups of G. This will probably not lead to a *construction* of the matrix $A(z)$. In the sequel we will make two restrictions, namely $A(z)$ is a polynomial matrix (i.e., $A(z) \in C[z] \otimes \underline{g}$) and that G is connected and semi-simple. As we will see the first restriction implies that the differential Galois group is a *connected* algebraic subgroup of G. The second restriction implies that G has finitely many conjugacy classes of maximal proper connected subgroups.

(9.4.4) *The differential Galois group M of the differential equation $y' = (A_0 + A_1 z + \cdots + A_m z^m)y$, where $A_0, \ldots, A_m \in \mathrm{End}(W)$ and W is a finite dimensional vector space over C, is connected.*

Indeed, let E denote the Picard-Vessiot ring. Put $M^o =$ the component of the identity of M and consider $F = E^{M^o}$. This is a finite Galois extension F of $C(z)$ with Galois group M/M^o. The extension $C(z) \subset F$ can be ramified only above the singular points of the differential equation. The only singular point of the differential equation is ∞. It follows that $C(z) = F$ and by Galois correspondence $M = M^o$.

(9.4.5) A faithful representation $\rho : G \to \mathrm{GL}(W)$, in other words a faithful G-module W, will be called a *Chevalley module* if:

(a) G leaves no line in W invariant.

(b) Any proper connected closed subgroup of G has an invariant line.

We will postpone the proof that a connected semi-simple G has a Chevalley module.

9.5. The choices for A_0 and A_1 in $\underline{g} \subset \mathrm{End}(V)$

The connected semi-simple group G is given as an algebraic subgroup $G \subset \mathrm{GL}(V)$, where V is a finite dimensional vector space over C. We recall that G is semi-simple if and only if its Lie algebra $\underline{g}$ is semi-simple.

For the construction of the equation we will need *the root space decomposition of* $\underline{g}$. This decomposition reads (see [J] and [F-H]): $\underline{g} = \underline{h} \oplus (\oplus_\alpha \underline{g}_\alpha)$, where $\underline{h}$ is a Cartan subalgebra and the one dimensional spaces $\underline{g}_\alpha = CX_\alpha$ are the eigenspaces for the adjoint action of $\underline{h}$ on $\underline{g}$ corresponding to the non-zero roots $\alpha : \underline{h} \to C$. More precisely, the adjoint action of $\underline{h}$ on $\underline{h}$ is zero and for any $\alpha \neq 0$ one has $[h, X_\alpha] = \alpha(h)X_\alpha$ for all $h \in \underline{h}$.

We fix a Chevalley module $\rho : G \to \mathrm{GL}(W)$. The induced (injective) morphism of Lie algebras $\underline{g} \to \mathrm{End}(W)$ is also denoted by ρ. The action of $\underline{h}$ on W gives a decomposition of $W = \oplus W_\beta$ into eigenspaces for a collection of linear maps $\beta : \underline{h} \to C$. (The β's are called the weights of the representation.)

For A_0 one chooses $\sum_{\alpha \neq 0} X_\alpha$. For A_1 one chooses an element in $\underline{h}$ satisfying conditions (a), (b) and (c) below.

(a) The $\alpha(A_1)$ are distinct and different from 0 (for the non-zero roots α of $\underline{g}$).

(b) The $\beta(A_1)$ are distinct and different from 0 (for the non-zero weights β of the representation)

It is clear that A_1 with these properties exists. Choose such an A_1. We want A_1 to satisfy the more technical condition:

(c) If the integer m is an eigenvalue of the operator $\sum_{\alpha \neq 0} \frac{1}{-\alpha(A_1)} \rho(X_{-\alpha})\rho(X_\alpha)$ on W, then $m = 0$.

If A_1 does not yet satisfy this last condition then a suitable multiple cA_1, with $c \in C^*$, satisfies all three conditions.

Claim: *Let $A_0, A_1 \in \underline{g} \subset \mathrm{End}(V)$ be chosen as above, then the action of the differential Galois group of $y' = (A_0 + A_1 z)y$ on the solution space can be identified with $G \subset \mathrm{GL}(V)$.*

Three equations

Then equation (9.4.2) reads now $[\frac{d}{dz} - (\rho(A_0) + \rho(A_1)z)]w = cw$ with $c \in C(z)$. Clearly $c \in C[z]$ and by comparing the degrees one finds that the degree of c is at most 1. More explicitly, one has

$$\Big[\frac{d}{dz} - (\rho(A_0) + \rho(A_1)z)\Big]w = (c_0 + c_1 z)w \text{ with } w = w_m z^m + \cdots + w_1 z + w_0,$$

with all $w_i \in W$, $w_m \neq 0$ and $c_0, c_1 \in C$. Comparing the coefficients of z^{m+1}, z^m, z^{m-1} one obtains the relations

$$\rho(A_1)(w_m) = -c_1 w_m$$

$$\rho(A_0)(w_m) + \rho(A_1)(w_{m-1}) = -c_0 w_m - c_1 w_{m-1}$$

$$-mw_m + \rho(A_0)(w_{m-1}) + \rho(A_1)(w_{m-2}) = -c_0 w_{m-1} - c_1 w_{m-2}.$$

The eigenspaces for the action of $\rho(A_1)$ on W will be denoted by W_b with $b = \beta(A_1)$ the corresponding eigenvalue of $\rho(A_1)$. Any element $\tilde{w} \in W$ is written as $\tilde{w} = \sum_b \tilde{w}_b$, with $\tilde{w}_b \in W_b$. The relation $[A_1, X_\alpha] = \alpha(A_1)X_\alpha$ implies that $\rho(X_\alpha)(W_d) \subset W_{d+\alpha(A_1)}$. This implies that A_0 has the property $\rho(A_0)(W_d) \subset \oplus_{b \neq d} W_b$.

We analyze now the three equations. The first equation can only be solved with $w_m \in W_d, w_m \neq 0$ and $d = -c_1$. The second equation, which can be read as $c_0 w_m = -\rho(A_0)(w_m) + (-\rho(A_1) - c_1)w_{m-1}$, imposes $c_0 = 0$. Indeed, the two right hand side terms $-\rho(A_0)(w_m)$ and $(-\rho(A_1) - c_1)w_{m-1}$ have no component in the eigenspace W_d for $\rho(A_1)$ to which w_m belongs. Further

$$w_{m-1} = \sum_{b \neq d} \frac{1}{-b+d} \rho(A_0)(w_m)_b + v_d = \sum_{\alpha \neq 0} \frac{1}{-\alpha(A_1)} \rho(X_\alpha)(w_m) + v_d,$$

for some $v_d \in W_d$.
The third equation can be read as

$$-mw_m + \rho(A_0)(w_{m-1}) = (-\rho(A_1) - c_1)w_{m-2}.$$

A necessary condition for this equation to have a solution w_{m-2} is that the left hand side has 0 as component in W_d. The component in W_d of the left hand side is easily calculated to be

$$-mw_m + (\rho(A_0)(w_{m-1}))_d = (-m + \sum_{\alpha \neq 0} \frac{1}{-\alpha(A_1)} \rho(X_{-\alpha})\rho(X_\alpha))(w_m).$$

Since this is zero, m is an eigenvalue of the operator $\sum_{\alpha \neq 0} \frac{1}{-\alpha(A_1)} \rho(X_{-\alpha})\rho(X_\alpha)$. It follows from our assumption on A_1 that $m = 0$.
This leaves us with the equation $[\frac{d}{dz} - (\rho(A_0) + \rho(A_1)z)]w = c_1 zw$ and $w \in W$. Since $\frac{d}{dz}w = 0$, one finds that Cw is invariant under $\rho(A_0)$ and $\rho(A_1)$. The Lie algebra $\underline{g}$ is generated by A_0 and A_1. Thus Cw is invariant under $\underline{g}$ and under G. Our assumptions on the G-module W imply that $w = 0$. The proof of theorem 9.1 is completed by a proof of the existence of a Chevalley module.

Lemma 9.6 (Mitschi and Singer [M-S3]).— *Every connected semi-simple linear algebraic group has a Chevalley module.*

Proof.— Let the connected semi-simple closed subgroup $G \subset \mathrm{GL}(V)$ be given. Chevalley's theorem (see [H]) states that for any proper algebraic subgroup M there is a G-module E and a line $L \subset E$ such that M is the stabilizer of that line. Since G is semi-simple, E is a direct sum of irreducible modules. The projection of L to one of these irreducible

components is again a line. Thus we find that M stabilizes a line in some irreducible G-module E of dimension greater than one. Any subgroup of G, conjugated to M, stabilizes also a line in E. Dynkin's theorem [D] implies that there are only finitely many conjugacy classes of maximal connected proper algebraic subgroups of G. One chooses an irreducible G-module W_i, $i = 1, \ldots, m$ for each class and one chooses an irreducible faithful module W_0. Then $W = W_0 \oplus \cdots \oplus W_m$ has the required properties.

Examples 9.7. Chevalley modules for SL(2) and SL(3).— (1) The standard action of $\mathrm{SL}(2, C)$ on C^2 is a Chevalley module.

(2) The standard action of $\mathrm{SL}(3, C)$ on C^3 will be called V. The induced representation on

$$W = V \oplus (\Lambda^2 V) \oplus (V \otimes_s V) = V \oplus (V \otimes V)$$

is a Chevalley module. Here $\Lambda^2 V$ is the second exterior power and $V \otimes_s V$ is the second symmetric power. Indeed, let M be a maximal proper connected subgroup of SL(3). Then M leaves a line in V invariant or leaves a plane in V invariant or M is conjugated with $\mathrm{PSL}(2) \subset \mathrm{SL}(3)$. In the second case M leaves a line in $\Lambda^2 V$ invariant and in the third case M leaves a line in $V \otimes_s V$ invariant. Further SL(3) leaves no line in W invariant.

10. OTHER FORMULATIONS

Ramis' first formulation of the inverse problem for the differential field $\mathbf{C}(\{z\})$ was the following theorem.

Theorem 10.1.— *A linear algebraic group G over $\mathbf{C}$ is a differential Galois group over $\mathbf{C}(\{z\})$ if and only if G has a local Galois structure.*

For the rather complicated definition of a local Galois structure $(T, a, \mathcal{N})$ we refer to [R1]. The next proposition simplifies matters.

Proposition 10.2 (Mitschi and Singer [M-S3]).— *Let G be a linear algebraic group over an algebraically closed field of characteristic 0. Let $R_u = R_u(G)$ denote the unipotent radical of G and let G^o be the component of the identity of G. The following statements are equivalent:*

(1) *G has a local Galois structure.*

(2) (a) *G/G^o is cyclic,* (b) *the dimension of $R_u/(R_u, G^o)$ is ≤ 1, and* (c) *the action of G/G^o on $R_u/(R_u, G^o)$ by conjugation, is trivial.*

Let again G be a linear algebraic group over an algebraically closed field of characteristic 0. A further investigation of Ramis leads to the definition of the linear algebraic group

$S(G) = R_u/(R_u, G^o) \cdot G/G^o$, which is the semi-direct product of $R_u/(R_u, G^o)$ and G/G^o, with respect to the action (by conjugation) of G/G^o on $R_u/(R_u, G^o)$.
One easily sees that condition (2) in the last proposition is equivalent to $S(G)$ is topologically (for the Zariski topology) generated by one element.
The connection with the linear algebraic group $V(G) := G/L(G)$ ($L(G)$ is defined in section 3) is the following.

Proposition 10.3 (Ramis [R2]).— *There is an isomorphism* $S(G) \to V(G)/(V(G)^o, V(G)^o)$.

Proof.— (1) We start by proving that a reductive group M has the property $L(M) = M^o$. We recall that in characteristic 0, reductive for a linear algebraic group is equivalent to the complete reducibility of any finite dimensional representation. It follows at once that $N := M/L(M)$ is also reductive. Thus the unipotent radical of N is trivial. By construction N^o has a trivial maximal torus and is therefore unipotent and equal to the unipotent radical of N. Thus $N^o = 1$. Since $L(M)$ is connected, the statement follows.

(2) Let G be a connected linear algebraic group and R_u its unipotent radical. Then G is the semi-direct $R_u \cdot M$, where M is some reductive subgroup of G, called a Levi subgroup or Levi-factor (see [H] p. 184). We note that a maximal torus of M is also a maximal torus of G. Let $\tau_1 : G = R_u \cdot M \to V(G) := G/L(G)$ denote the canonical map. Since $L(M) = M$, the kernel of τ_1 is the smallest normal subgroup of G containing M. The kernel of the natural map $\tau_2 : G = R_u \cdot M \to V(G)/(V(G), V(G))$ is the smallest normal subgroup containing M and (G, G). This is the same as the smallest normal subgroup containing M and (R_u, G), since $G = R_u \cdot M$. Thus the map τ_2 induces an isomorphism $R_u/(R_u, G) \to V(G)/(V(G), V(G))$.

(3) Let G be any linear algebraic group. We consider $V(G) := G/L(G)$ and $S'(G) := V(G)/(V(G)^o, V(G)^o)$. The component of the identity $S'(G)^o$ is easily identified with $S'(G^o)$ and according to (2) isomorphic to the unipotent group $R_u/(R_u, G^o)$. Further $S'(G)/S'(G)^o$ is canonically isomorphic to G/G^o. Using that $S'(G)^o$ is unipotent, one can construct a left inverse for the surjective homomorphism $S'(G) \to G/G^o$ (see lemma 10.4). Thus $S'(G)$ is isomorphic to the semi-direct product of $R_u/(G^o, R_u)$ and G/G^o, given by the action of G/G^o on $R_u/(R_u, G^o)$, defined by conjugation. Therefore $S(G)$ and $S'(G)$ are isomorphic.

Finally we will need a lemma, which is probably well known.

Lemma 10.4.— *Let H be a linear algebraic group such that H^o is unipotent. Then*
(a) *H is a semi-direct product of H^o and H/H^o.*
(b) *H and $H/(H^o, H^o)$ have the same minimal number of topological (for the Zariski topology) generators.*

Proof.— (a) Define the closed normal subgroups H_k^o of H by $H_0^o = H^o$ and $H_{k+1}^o = (H^o, H_k^o)$ for $k \geq 0$. Since H^o is a unipotent group $H_k^o = \{1\}$ for large k. Let $l(H)$ denote the smallest integer $m \geq 0$ such that $H_k^o = \{1\}$ for all $m > k$. By induction on $l(H)$ we will prove the existence of a homomorphism $s : H/H^o \to H$ such that $H/H^o \to H \to H/H^o$ is the identity. For $l(H) = 0$ the statement is trivial. Suppose that $l(H) = n \geq 1$. There is a map $s : H/H^o \to H$ such that s is a left inverse of $H \to H/H^o$ and $H/H^o \to H \to H/H_n^o$ is a homomorphism. Consider the map $f : H/H^o \times H/H^o \to H_n^o$, given by $f(a,b) = s(a)s(b)s(ab)^{-1}$. The group H_n^o is isomorphic to a finite dimensional vector space over the base field. The group H/H^o acts on this vector space by conjugation and f is a 2-cocycle for this H/H^o-module. Since the base field is supposed to have characteristic 0, this 2-cocycle is trivial. Thus for a suitable map $r : H/H^o \to H_n^o$, the map $a \mapsto s(a)r(a)$ is a homomorphism. This proves the induction step.

(b) It suffices to show the following:
"Let $a_1, \ldots, a_n \in H$ be such that their images in $H/(H^o, H^o)$ are topological generators, then $a_1, \ldots, a_n$ are topological generators of H itself".
We will prove the above statement by induction on $l(H)$. The cases $l(H) = 0, 1$ are trivial. Suppose $l(H) = n > 1$ and let M denote the closed subgroup of H generated by $a_1, \ldots, a_n$. The induction hypothesis implies that the natural homomorphism $M \to H/H_n^o$ is surjective. It suffices to show that $M \supset H_n^o$. Take $a \in H^o$, $b \in H_{n-1}^o$ and consider the element $aba^{-1}b^{-1} \in H_n^o$. One can write $a = m_1 A$, $b = m_2 B$ with $m_1, m_2 \in M$ and $A, B \in H_n^o$. Since H_n^o lies in the center of H^o, one has $aba^{-1}b^{-1} = m_1 A m_2 B A^{-1} m_1^{-1} B^{-1} m_2^{-1} = m_1 m_2 m_1^{-1} m_2^{-1} \in M$.

Conclusions : The groups $S(G)$, $V(G) := G/L(G)$ and $V(G)/(V(G)^o, V(G)^o)$ have the same minimal number of topological generators. The theorems 3.1 and 3.4 can therefore also be formulated with $S(G)$ or $V(G)/(V(G)^o, V(G)^o)$ instead of $G/L(G)$. Moreover the invariant $d(G)$ in theorem 3.7 coincides with the minimal number of topological generators of $G/L(G)$.

REFERENCES

[A1] Y. André - *Séries de Gevrey de type arithmétique*, Preprint May 1997.

[A2] Y. André - *Sur la conjecture des p-courbures de Grothendieck-Katz*, Preprint October 1997.

[Bo] A. Borel - *Linear Algebraic Groups*, W.A. Benjamin, Inc. New York, Amsterdam, 1969.

[D] E. Dynkin - *Maximal subgroups of the classical groups*, Trudy Moskov. Mat. Obschetsva, **1**, 39-160, Amer. Math. Soc. Transl., Ser. 2, **6** (1957), 245-378.

[De] P. Deligne - *Catégories Tannakiennes*, The Grothendieck Festschrift 2, p 111-195, Progress in Math. **87**, 1990, Birkhäuser.

[De-M] P. Deligne and M. Milne - *Tannakian categories*, Springer Lecture Notes in Math. vol 900, 1982, 101-228.

[F-H] W. Fulton and J. Harris - *Representation Theory*, Graduate Texts in Mathematics **129**, Springer Verlag, 1991.

[H] J.E. Humphreys - *Linear Algebraic Groups*, Second edition, Springer Verlag, New York, 1981.

[H-P] P.A. Hendriks and M. van der Put - *Galois action on solutions of a differential equation*, Journal of Symbolic Computation **19** (1995), no 6, 559-576.

[J] N. Jacobson - *Lie Algebras*, Dover Publications, Inc. New York, 1962.

[K] I. Kaplansky - *An Introduction to Differential Algebra*, Second edition, Hermann, Paris, 1976.

[Ka] N. Katz - *Rigid Local Systems*, Annals of Mathematical Studies, 1996

[Ko] E.R. Kolchin - *Differential Algebra and Algebraic Groups*, Academic Press, New York, 1973.

[Kov] J.J. Kovacic - *The inverse problem in the Galois theory of differential fields*, Ann. of Math **89** (1969), 583-608.

[Ma] A. Magid - *Lectures on Differential Galois Theory*, AMS University Lecture series **7**, 1994.

[M-R1] J. Martinet and J.-P. Ramis - *Théorie de Galois différentielle et resommation*, Computer Algebra and Differential Equations (E. Tournier ed), Academic Press, 1989, 115-214.

[M-R2] J. Martinet and J.-P. Ramis - *Elementary acceleration and multisummability* - Annales de l'Institut Henri Poincaré, Physique Théorique **54** (1991), no. 4, 331-401.

[M-S1] C. Mitschi and M.F. Singer - *Connected Linear Groups as Differential Galois Groups*, Journal of Algebra **184** (1996), 333 -361.

[M-S2] C. Mitschi and M.F. Singer - *The inverse problem in differential Galois theory*, The Stokes Phenomenon and Hilbert's 16th Problem (B.L.J. Braaksma et al ed.), World Scientific, River Edge, NJ, 1996, 185-197.

[M-S3] C. Mitschi and M.F. Singer - *On Ramis' solution of the local inverse problem of differential Galois theory*, Journal of Pure and Applied Algebra **110** (1996), 185-184.

[P1] M. van der Put - *Singular complex differential equations; An Introduction*, Nieuw Archief voor Wiskunde, vierde serie **13** (1995), no. 3, 451-470.

[P2] M. van der Put - *Symbolic analysis of differential equations*, to appear in "Some Tapas of Computer Algebra" A.M. Cohen, H. Cuypers, H. Sterk eds., 1998.

[P3] M. van der Put - *Galois theory of differential equations, Algebraic groups and Lie algebras*, preprint Groningen, Sept 1997.

[R1] J.-P. Ramis - *About the Inverse Problem in Differential Galois Theory: The Differential Abhyankar Conjecture*, In "The Stokes Phenomenon and Hilbert's 16th Problem", World Scientific Publ. (Editors B.L.J. Braaksma, G.K. Immink, M. van der Put), Singapore 1996.

[R2] J.-P. Ramis - *About the Inverse Problem in Differential Galois Theory: The Differential Abhyankar Conjecture*, Manuscript for Astérisque, version 16-1-1995.

[S] M.F. Singer - *Direct and Inverse Problems in Differential Galois Theory*, to be published in the introduction to Kolchin's collected works 1998.

Marius VAN DER PUT

Department of Mathematics
University of Groningen
P.O. Box 800
NL-9700 AV GRONINGEN
The Netherlands
E-mail : M.van.der.Put@math.rug.nl