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## RECENT WORK ON DIFFERENTIAL GALOIS THEORY

by Marius VAN DER PUT

### 1. INTRODUCTION

Linear differential equations have a rich variety of aspects: asymptotic theory, differential Galois theory, the construction of algorithms, arithmetic properties, Grothendieck's conjecture, rigid differential equations et cetera. We will restrict our attention to recent progress on the inverse problem for the differential Galois theory. For the history of this problem we refer to the excellent survey paper [S] and its references. Our purpose is to explain and partly prove the results obtained by J.-P. Ramis, C. Mitschi and M.F. Singer. Let us give two examples of the results that we hope to explain:

(1) Every connected reductive groups over  $\mathbf{C}$  can be realized as the differential Galois group of an equation  $y' = (A_0 + A_1 z)y$ , with constant matrices  $A_0, A_1$ .

(2) There is no linear differential equation over the field of convergent Laurent series  $\mathbf{C}(\{z\})$  with differential Galois group  $\mathbf{G}_a^2$ .

In this presentation we have taken the liberty of changing proofs, omitting material and adding new material with respect to the original papers.

### 2. PICARD-VESSIOT THEORY

A *differential field*  $K$  is a field equipped with a differentiation (or derivation)  $f \mapsto f'$  satisfying the usual rules:  $(f + g)' = f' + g'$  and  $(fg)' = f'g + fg'$ . The field of constants  $C$  is defined by  $C = \{a \in K \mid a' = 0\}$ . We will suppose in the sequel that  $C$  is an algebraically closed field of characteristic zero and that  $C \neq K$ . The derivatives of an element  $f$  will be denoted by  $f', f'', f''', \dots$  or  $f^{(1)}, f^{(2)}, f^{(3)}, \dots$ . A *scalar homogeneous linear differential equation of order  $n$*  is an equation of the form

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y^{(1)} + a_0y = 0 \text{ with } a_0, \dots, a_{n-1} \in K.$$

The solution space  $\{y \in K \mid L(y) = 0\}$  is a vector space over  $C$  of dimension  $\leq n$ . In general the dimension is strictly less than  $n$ . This leads to the definition of a *Picard-Vessiot ring*  $E \supset K$  for the equation  $L$ :

- (i) The  $K$ -algebra  $E$  is equipped with a differentiation (also denoted by  $'$ ) extending the differentiation of  $K$ .
- (ii)  $E$  has only trivial differential ideals.
- (iii) The space  $V := \{a \in E \mid L(a) = 0\}$  is a vector space over  $C$  of dimension  $n$ .
- (iv) The  $K$ -algebra  $E$  is generated by the elements of  $V$  and their derivatives.

It can be shown that a Picard-Vessiot ring exists and that it is unique up to differential isomorphism. Moreover  $E$  has no zero divisors and its field of fractions has  $C$  as set of constants. The field of fractions of a (or the) Picard-Vessiot ring is called a (or the) Picard-Vessiot field of the equation over  $K$ . The space  $V$  will be called the *solution space* of  $L$ . Let  $G$  denote the group of the differential automorphisms of  $E/K$ . This group acts as a group of  $C$ -linear automorphisms of  $V \subset E$ . The induced group homomorphism  $G \rightarrow \mathrm{GL}(V)$  turns out to be injective and its image is an *algebraic* subgroup of  $\mathrm{GL}(V)$ . The *differential Galois group of the equation  $L$  over  $K$*  is defined to be  $G$  with its structure of linear algebraic group given by the embedding  $G \subset \mathrm{GL}(V)$ . The group  $G$  also coincides with the group of all differential automorphisms of the Picard-Vessiot field over  $K$ .

The Tannakian approach to (linear homogeneous) differential equations over  $K$  is given in [De-M] and [De]. This approach can be described as follows. Let  $\mathrm{Diff}_K$  denote the category of the differential modules over  $K$ . Under the hypothesis that the field of constants  $C$  of  $K$  has characteristic 0 and is algebraically closed, one can show that this category is a neutral Tannakian category. In other words, there is an equivalence of Tannakian categories  $\mathrm{Diff}_K \rightarrow \mathrm{Repr}_H$ , where  $H$  is a certain affine group scheme over  $C$  and  $\mathrm{Repr}_H$  denotes the Tannakian category of the finite dimensional representations of  $H$ .

For a fixed differential module  $M$  over  $K$  one can consider the full subcategory  $\{\{M\}\}$  of  $\mathrm{Diff}_K$  generated by all  $M \otimes \cdots \otimes M \otimes M^* \otimes \cdots \otimes M^*$ . This subcategory is also a neutral Tannakian category and thus there is an equivalence of Tannakian categories  $\{\{M\}\} \rightarrow \mathrm{Repr}_G$  for some affine group scheme  $G$  over  $C$ . It turns out that  $G$  can be identified with the differential Galois group of  $M$  (and is thus a linear algebraic group). The equivalence can be made explicit by  $N \mapsto \ker(\partial, E \otimes_K N)$ , where  $E$  is a Picard-Vessiot ring for  $M$  over  $K$ . The  $G$  action on  $E$  induces a  $G$  action on  $E \otimes_K N$  which commutes with the action of  $\partial$  on  $E \otimes_K N$ . In particular,  $\ker(\partial, E \otimes_K N)$  is a finite dimensional (algebraic) representation of the linear algebraic group  $G$ .

This more abstract Tannakian approach to differential modules has some advantages. An easy application is the following useful statement:

Suppose that  $G$  is a differential Galois group over  $K$  and let  $N \subset G$  be a normal algebraic subgroup. Then the linear algebraic group  $G/N$  is also a differential Galois group over  $K$ .

*Important examples of differential fields:*

Let  $C$  denote an algebraically closed field of characteristic 0.

- (1)  $K = C((z))$ , the field of the formal Laurent series in  $z$  over  $C$  with the derivation  $\frac{d}{dz}$ .
- (2)  $K = C(\{z\})$ , the field of the convergent Laurent series over the field of the complex numbers with the derivation  $\frac{d}{dz}$ .
- (3)  $K = C(z)$ , the field of rational functions over  $C$  with the derivation  $\frac{d}{dz}$ .
- (4)  $K$  a finite extension of  $C(z)$ , equipped with the unique derivation extending the derivation  $\frac{d}{dz}$  of  $C(z)$ . In other words,  $K$  is the function field of a curve (irreducible, smooth, projective, connected) over  $C$ .

The *theme* of this exposition is to answer, for the differential fields  $K$  in the above list, the question:

*Which linear algebraic groups  $G$  are differential Galois groups over  $K$  ?*

### 3. THE THEOREMS

Let  $G$  be a linear algebraic group (over the field  $C$ ). The subgroup  $L(G)$  of  $G$  is defined as the group generated by all (maximal) tori lying in  $G$ . It is clear that  $L(G)$  is a normal subgroup of  $G$ , contained in the component of the identity  $G^\circ$  of  $G$ . The group  $L(G)$  is generated by algebraic subgroups of  $G$  and hence is itself algebraic. Thus the factor group  $G/L(G)$  is again a linear algebraic group.

**Theorem 3.1 (Ramis). The local theorem.**— *A linear algebraic group  $G$  is a differential Galois group over the field  $C(\{z\})$  if and only if  $G/L(G)$  is topologically (for the Zariski topology) generated by one element.*

**Corollary 3.2 (Ramis).**— *Suppose that  $G$  is a differential Galois group over the field  $C(\{z\})$ , then  $G/L(G)$  is topologically generated by the image of the topological monodromy.*

*Explanation*

The differential equation  $L(y) = 0$  over  $C(\{z\})$  lives in fact as a meromorphic differential equation on a neighbourhood  $\{z \in C \mid |z| < \epsilon\}$  of  $z = 0$ , which has only  $z = 0$  as singularity. The solution space  $V$  can be identified with the solutions of the equation at the point  $\epsilon/2$ . Analytical continuation of the solutions at  $\epsilon/2$  along the circle around 0

and through  $\epsilon/2$  (with positive orientation) induces a  $\mathbf{C}$ -linear automorphism of  $V$ . This automorphism can be shown to belong to  $G \subset \mathrm{GL}(V)$ . The element is called the *topological monodromy*. It is unique up to conjugation. (See section 7 for more information.)

*Examples 3.3 : Order two*

Again  $K = \mathbf{C}(\{z\})$  and consider a differential equation over  $K$  of order two with differential Galois group in  $\mathrm{SL}(2)$ . The well known classification of the algebraic subgroups of  $\mathrm{SL}(2)$  can be used in order to determine the possible differential Galois groups. The list (of conjugacy classes) that one finds is:

$$\mathrm{SL}(2), B, \mathbf{G}_a, \{\pm 1\} \times \mathbf{G}_a, \mathbf{G}_m, D_\infty, \text{ finite cyclic},$$

where  $B$  is the Borel subgroup,  $\mathbf{G}_a$  the additive group,  $\mathbf{G}_m$  the multiplicative group and  $D_\infty$  is the infinite dihedral group, i.e., the subgroup of  $\mathrm{SL}(2)$  leaving the union of two lines  $L_1 \cup L_2 \subset \mathbf{C}^2$  (through the origin) invariant.

Every group in the above list can be realized by a differential equation

$$y'' + mz^{-1}y' - a_0y = 0 \text{ where } m \in \{0, 1\} \text{ and } a_0 \in \mathbf{C}[z, z^{-1}].$$

In fact the choices  $(m, a_0) = (0, z), (0, z^2 + 3z + 5/4), (1, 0), (0, -\frac{z^{-2}}{4}), (0, z^{-2}), (0, -\frac{3}{16}z^{-2} + z^{-1}), (0, \frac{1}{4}(-1 + (\frac{t}{n})^2)z^{-2})$  with  $\frac{t}{n} \in \mathbf{Q}$  produce the above list of differential Galois groups.

*Some definitions: regular, regular singular and irregular*

Let  $X/\mathbf{C}$  denote a curve (irreducible, smooth, projective) of genus  $g$ . At this point of the exposition it is convenient to define a (linear homogeneous) differential equation over  $X$  as a matrix differential equation  $y' = Ay$ , where  $A$  is a  $n \times n$ -matrix with coefficients in  $K$ , the function field of  $X$ . Consider a point  $c \in X$  with local coordinate  $z_c$ . The field  $K$  is embedded in  $K_c = \mathbf{C}(\{z_c\})$ , the field of the convergent Laurent series in  $z_c$ . Let  $B \in \mathrm{GL}(n, K_c)$ . Put  $v = By$ . Then  $v$  satisfies the matrix differential equation  $v' = (B^{-1}AB - B^{-1}B')v$ . This new equation over  $K_c$  is called *locally equivalent* with  $y' = Ay$ . The equation  $y' = Ay$  is called *regular*, *resp. regular singular* at  $c$  if there is a locally equivalent equation  $v' = Dv$  such that  $D \in M(n, \mathbf{C}\{z_c\})$  *resp.*  $z_c D \in M(n, \mathbf{C}\{z_c\})$ . Otherwise the point  $c$  will be called an *irregular singularity* of  $y' = Ay$ .

**Theorem 3.4 (Ramis). The global theorem.**— *Let  $X/\mathbf{C}$  be a curve (irreducible, smooth, projective) of genus  $g$  and  $S \subset X$  a finite set with cardinality  $m \geq 1$ . A linear algebraic group  $G$  can be realized as the differential Galois group of a differential equation over  $X$  with singularities in  $S$  if and only if  $G/L(G)$  is topologically generated by  $2g + m - 1$  elements.*

*Remark.*— The theorem remains valid for  $S = \emptyset$  and  $2g + m - 1$  replaced by  $2g$ .

**Corollary 3.5** (Ramis).— (1) Suppose that  $G$  is the differential Galois group for a differential equation on  $X$  with singularities in the finite set  $S$ . Then the natural homomorphism  $\pi_1(X \setminus S, *) \rightarrow G/L(G)$  has dense image for the Zariski topology.

(2) Suppose that the linear algebraic group  $G$  can be realized for the pair  $(X, S)$ , then  $G$  can be realized by a differential equation with at most one irregular singular point (to be chosen freely in  $S$ ) and the other singularities, regular singular (and in  $S$ ).

*Examples 3.6 : Order two, continued*

1)  $(\mathbf{P}^1, \{0, \infty\})$  and order two.

The list of possible groups  $G \subset \mathrm{SL}(2)$  coincides with the list that we have considered in 3.3. The above list is in fact the theoretical background for the simplification of the Kovacic algorithm for order two differential equations with at most two singular points, presented in [P2].

2)  $(\mathbf{P}^1, \{0, 1, \infty\})$  and order two.

Every algebraic subgroup of  $\mathrm{GL}(2)$  can be realized for this pair. More precisely, every algebraic subgroup  $G \subset \mathrm{GL}(2)$  is the differential Galois group of an equation

$$y'' + \frac{a_1(z)}{z(z-1)}y' + \frac{a_2(z)}{z^2(z-1)^2}y = 0,$$

where  $a_1(z), a_2(z) \in \mathbb{C}[z]$ . (This equation is regular singular at 0, 1 and has an arbitrary singularity at  $\infty$ ).

*Some definitions: The defect and the excess of a connected linear algebraic group.*

$C$  denotes an algebraically closed field of characteristic 0. Let  $G$  be a connected linear algebraic group over  $C$ . The unipotent radical of  $G$  is denoted by  $R_u$ . Let  $P \subset G$  be a Levi factor, i.e., a closed subgroup such that  $G$  is the semi-direct product of  $R_u$  and  $P$  (see [H], p 184). The group  $R_u/(R_u, R_u)$ , where  $(R_u, R_u)$  is the (closed) commutator subgroup, is a commutative unipotent group and so isomorphic to  $\mathbf{G}_{a,C}^n = C^n$ . The group  $P$  acts on  $R_u$  by conjugation and this induces an action on  $R_u/(R_u, R_u)$ . Therefore we may write  $R_u/(R_u, R_u) = U_1^{n_1} \oplus \cdots \oplus U_s^{n_s}$ , where each  $U_i$  is an irreducible  $P$ -module. For notational convenience, we suppose that  $U_1$  is the trivial 1-dimensional  $P$ -module and  $n_1 \geq 0$ . Since  $P$  is reductive, one can write  $P = T \cdot H$ , where  $T$  is a torus and  $H$  is a semi-simple group. Define  $m_i := n_i$  if the action of  $H$  on  $U_i$  is trivial and  $m_i := n_i + 1$  if the action of  $H$  on  $U_i$  is not trivial. Define  $N = 0$  if  $H$  is trivial and  $N = 1$  otherwise. The *defect*  $d(G)$  is defined to be  $n_1$  and the *excess*  $e(G)$  is defined as  $\max(N, m_2, \dots, m_s)$ . Since two Levi factors are conjugated, these numbers do not depend on the choice of  $P$ . Furthermore, one can show that  $d(G)$  is the dimension of  $R_u/(G, R_u)$ .

**Theorem 3.7** (Mitschi-Singer).— Let  $G$  be a connected linear algebraic group over  $C$ . Then  $G$  is the differential Galois group of a matrix differential equation over  $C(z)$  of the

form

$$y' = \left( \frac{A_1}{z - a_1} + \cdots + \frac{A_{d(G)}}{z - a_{d(G)}} + A_\infty \right) y,$$

where  $A_i$ ,  $i = 1, \dots, d(G)$  are constant matrices and  $A_\infty$  is a matrix with polynomial entries of degree at most  $e(G)$ . In particular, the points  $a_1, \dots, a_{d(G)}$  are the singularities of the equation in  $C$ . These singularities are regular. The point  $\infty$  is possibly an irregular singular point.

#### Remarks

The proof of the result above is purely algebraic and moreover constructive. The statement is more precise than the one of Ramis' global theorem and more restricted in the sense that the group  $G$  is supposed to be connected and that the representation of  $G$  on the space of solutions is not prescribed. Using Tannakian arguments one may also prescribe the faithful representation of  $G$  on the space of solutions (at the cost of introducing possibly apparent singularities). In section 10 we will show that  $d(G)$  is equal to the minimal number of topological generators of  $G/L(G)$ .

The special case of the theorem, where the group  $G$  is supposed to be connected and reductive, is rather striking. It states that  $G$  is the differential Galois group of a matrix differential equation  $y' = (A + Bz)y$  with  $A$  and  $B$  constant matrices. In section 9 we will present an explicit proof of this statement in case  $G$  is connected and semi-simple.

## 4. ABHYANKAR'S CONJECTURE

The base field  $k$  is an algebraically closed field of characteristic  $p > 0$ , e.g.  $\overline{\mathbf{F}}_p$ . One considers a curve  $X/k$  (irreducible, smooth, projective) of genus  $g$  and a finite subset  $S \subset X$  with cardinality  $m \geq 1$ . Abhyankar's conjecture is concerned with the Galois coverings of  $X$  which are unramified outside  $S$ . For a finite group  $G$  one denotes by  $p(G)$  the subgroup generated by all the  $p$ -Sylow subgroups of  $G$ . In other words,  $p(G)$  is the smallest normal subgroup such that  $G/p(G)$  has order prime to  $p$ . We recall the well known theorems.

**Theorem 4.1** (Raynaud).— *The finite group  $G$  is the Galois group of a covering of  $\mathbf{P}^1$ , unramified outside  $\infty$ , if and only if  $G = p(G)$ .*

**Theorem 4.2** (Harbater).— (1) *Let the pair  $(X, S)$  be as above. The finite group  $G$  is a Galois group of a covering of  $X$ , unramified outside  $S$ , if and only if  $G/p(G)$  is generated by  $2g + m - 1$  elements.*

(2) *If  $G$  is a Galois group for the pair  $(X, S)$ , then the natural homomorphism  $\pi^{(p)}(X \setminus S, *) \rightarrow G/p(G)$  is surjective.*

(3) Suppose that  $G/p(G)$  is generated by  $2g + m - 1$  elements. Then there is a Galois covering of  $X$  with Galois group  $G$ , wildly ramified in at most one (prescribed) point of  $S$ , tamely ramified at the other points of  $S$  and unramified outside  $S$ .

*Remark.*— The group  $\pi^{(p)}(X \setminus S, *)$  denotes the prime to  $p$ , algebraic fundamental group of  $X \setminus S$ . This group is known to be topologically generated by  $2g + m - 1$  elements and depends in fact only on the number  $2g + m - 1$ .

#### *The translation*

During a dinner at Toulouse (nuit de la musique 1993), Raynaud and Ramis discussed the remarkable relation between the inverse problem for differential Galois theory and Abhyankar's conjecture. The following transformation rules seem to link to two subjects:

| Differential                                                    | Characteristic $p > 0$                                   |
|-----------------------------------------------------------------|----------------------------------------------------------|
| $X/\mathbb{C}$ curve, finite $S \subset X$ , $S \neq \emptyset$ | $X/k$ curve, finite $S \subset X$ , $S \neq \emptyset$   |
| differential equation                                           | equation with Galois covering of $X$                     |
| singular point (in $S$ )                                        | ramified point (in $S$ )                                 |
| local differential Galois group                                 | inertia group                                            |
| regular singular                                                | tamely ramified                                          |
| irregular singular                                              | wildly ramified                                          |
| linear algebraic group                                          | finite group                                             |
| $L(G)$                                                          | $p(G)$                                                   |
| $\pi_1(X \setminus S) \rightarrow G/L(G)$ Zariski dense         | $\pi^{(p)}(X \setminus S) \rightarrow G/p(G)$ surjective |

In the work of P. Deligne, N. Katz and G. Laumon on (rigid) differential equations there is also a link, this time more concrete, between differential equations and certain sheaves living in characteristic  $p$ . It would be interesting to find a mathematical theory explaining the philosophical link observed above.

## 5. FROM LOCAL TO GLOBAL

**Proposition 5.1.**— Assume that the local theorem 3.1 is valid. Let the following data be given:

- 1) A compact Riemann surface  $X$  of genus  $g$ .
- 2) Points  $p_1, \dots, p_m$  on  $X$  with  $m \geq 1$ .
- 3) A vector space  $V$  of dimension  $n$  over  $\mathbb{C}$ .
- 4) An algebraic subgroup  $G \subset \mathrm{GL}(V)$ , such that  $G/L(G)$  is topologically generated by  $2g + m - 1$  elements.

*Then there exists a differential equation over  $X$ , singular at most at the points  $p_1, \dots, p_m$ , regular singular at  $p_1, \dots, p_{m-1}$ , having a solution space, which is identified with  $V$  such that  $G \subset \mathrm{GL}(V)$  is the differential Galois group.*

*Proof.*— The function field of  $X$  will be denoted by  $K$ . For a point  $p \in X$  we denote the field of the meromorphic functions at  $p$  by  $K_p$ .

First one chooses small disjoint disks  $X_1, \dots, X_m$  around the points  $p_1, \dots, p_m$ . Put  $X_i^* = X_i \setminus \{p_i\}$ . In  $X_m^*$  one chooses a point  $c$ . The fundamental group  $\pi_1(X_0, c)$ , where  $X_0 = X \setminus \{p_1, \dots, p_m\}$ , is generated by  $a_1, b_1, \dots, a_g, b_g, \lambda_1, \dots, \lambda_m$  and has one relation  $a_1 b_1 a_1^{-1} b_1^{-1} \dots \lambda_1 \dots \lambda_m = 1$ . The element  $\lambda_m$  is a loop in  $X_m^*$  around  $p_m$  and the other  $\lambda_i$  are loops around  $p_1, \dots, p_{m-1}$ . The differential module over  $X$  is constructed by glueing certain connections  $M_0, \dots, M_m$  (with possibly singularities), living above the spaces  $X_0, \dots, X_m$ .

Let  $pr : G \rightarrow G/L(G)$  denote the canonical homomorphism. One chooses a homomorphism  $\rho : \pi_1(X_0, c) \rightarrow G \subset \mathrm{GL}(V)$ , such that the homomorphism  $pr\rho$  has Zariski dense image. Consider the algebraic group  $G' = pr^{-1}(\ll pr\rho(\lambda_m) \gg)$ , where  $\ll a \gg$  denotes the algebraic subgroup generated by the element  $a$ . The group  $G'$  contains  $L(G)$  and so  $G'/L(G')$  is topologically generated by the image of  $\rho(\lambda_m)$ . According to the Ramis' local theorem,  $G'$  is the differential Galois group of a differential equation over the field  $K_{p_m}$ . One can extend this very local object to a differential module  $M_m$ , living above  $X_m$ , with only  $p_m$  as singular point. The solution space at the point  $c \in X_m$  and the action of  $G'$  on this space can be identified with  $V$  and  $G' \subset \mathrm{GL}(V)$ . The topological monodromy corresponding to  $\lambda_m$  can be arranged to be  $\rho(\lambda_m) \in G'$ .

The usual solution of the Riemann-Hilbert problem (in weak form) provides a differential module  $M_0$  above  $X_0$  such that the monodromy action is equal to  $\rho$ . The restrictions of  $M_0$  to  $X_m^*$  and  $M_m$  to  $X_m^*$  are determined by their local monodromies. These are both equal to  $\rho(\lambda_m) \in G \subset \mathrm{GL}(V)$ . Thus we have a canonical way to glue  $M_0$  and  $M_m$  over the open subset  $X_m^*$ . For each point  $p_i$ ,  $i = 1, \dots, m-1$  one can consider the restriction of  $M_0$  to  $X_i^*$ . This restriction is determined by its local monodromy around the point  $p_i$ . Clearly, the restriction of  $M_0$  to  $X_i^*$  can be extended to a differential module  $M_i$  above  $X_i$  with a regular singular point at  $p_i$ .

The modules  $M_0, M_1, \dots, M_m$  (or rather the corresponding analytic vector bundles) are in this way glued to a vector bundle  $M$  above  $X$ . The connections can be written as  $\nabla : M_0 \rightarrow \Omega_X \otimes M_0$ ,  $\nabla : M_i \rightarrow \Omega_X(p_i) \otimes M_i$  for  $i = 1, \dots, m-1$  and  $\nabla : M_m \rightarrow \Omega_X(dp_m) \otimes M_m$  for a suitable integer  $d \geq 0$ . The connections also glue to a connection  $\nabla : M \rightarrow \Omega_X(p_1 + \dots + p_{m-1} + dp_m) \otimes M$ . Let  $M_*$  denote the set of all meromorphic sections of  $M$ . Then  $M_*$  is a vector space over  $K$  of dimension  $n$  with a connection  $\nabla : M_* \rightarrow \Omega_{K/C} \otimes M_*$ . Thus we have found a differential module over  $K$  with the correct singularities.  $K$  has a natural embedding in the field  $K_c$ . The differential module is trivial over  $K_c$  and its Picard-Vessiot field  $PV$  over  $K$  can be seen as a subfield

of  $K_c$ . The solution space is  $V \subset PV \subset K_c$ .

Finally we have to show that the differential Galois group  $H \subset \text{GL}(V)$  is actually  $G$ . By construction  $G' \subset H$  and also the image of  $\rho$  lies in  $H$ . This implies that  $G \subset H$ .

In order to conclude that  $G = H$  we can use the Galois correspondence. Thus we have finally to prove that an element  $f \in PV \subset K_c$ , which is invariant under  $G$ , belongs to  $K$ .

Since  $G'$  is by construction the differential Galois group of  $M_m$  above  $X_m$ , we conclude from the invariance of  $f$  under  $G'$  that  $f$  extends to a meromorphic solution of the differential equation above  $X_m$ . The invariance of  $f$  under the image of  $\rho$  implies that  $f$  extends to a meromorphic solution of the differential equation above  $X_0 \cup X_m$ . The points  $p_1, \dots, p_{m-1}$  are regular singular and any solution of the differential equation above  $X_i^*$ ,  $i = 1, \dots, m-1$  extends meromorphically to  $X_i$ . Thus  $f$  is meromorphic on  $X$  and belongs to  $K$ .

*Remark.*— For the case of  $X = \mathbf{P}^1$  and  $p_m = \infty$ , it is possible to refine the above reasoning to prove the following statement.

**Corollary 5.2.**— *Let  $G \subset \text{GL}(n, \mathbf{C})$  be an algebraic group such that  $G/L(G)$  is topologically generated by  $m-1$  elements. Then there are constant matrices  $A_1, \dots, A_{m-1}$  and there is a matrix  $A_\infty$  with polynomial coefficients (all matrices of order  $n \times n$ ) such that the matrix differential equation*

$$y' = \left( \frac{A_1}{z-p_1} + \dots + \frac{A_{m-1}}{z-p_{m-1}} + A_\infty \right) y$$

*has differential Galois group  $G \subset \text{GL}(n, \mathbf{C})$ .*

This result is close to the one of Mitschi-Singer. We note however that there seems to be no bound on the degrees of the coefficients of the matrix  $A_\infty$ .

## 6. THE FORMAL THEORY

In this section the differential field is  $\hat{K} := \mathbf{C}((z))$ . The classification of differential equations over  $\hat{K}$  is well known (Birkhoff, Turrittin, Malgrange, Levelt, et al.). We will present this classification in a somewhat different form, which clarifies the connection with the differential Galois group. For this purpose we need a *universal differential extension*  $\hat{R}$  of  $\hat{K}$  which can be defined as follows:

- (i) The  $\hat{K}$ -algebra  $\hat{R}$  has a differentiation extending the one of  $\hat{K}$ .
- (ii) Every differential ideal of  $\hat{R}$  is trivial, i.e., 0 or  $\hat{R}$ .
- (iii) Every homogeneous linear differential equation of order  $n$  over  $\hat{K}$  has a solution space in  $\hat{R}$ , which is a  $\mathbf{C}$ -vector space of dimension  $n$ .
- (iv)  $\hat{R}$  is minimal in the sense that  $\hat{R}$  is generated over  $\hat{K}$  by all the solutions and their

derivatives of all homogeneous linear equations over  $\hat{K}$ .

For any differential field one can show that there exists an universal differential extension (as above). Moreover one can show that this extension is unique up to differential isomorphism. The interesting thing is that one can write down this extension explicitly in the case of  $\hat{K}$ .

We introduce first some notations:  $\delta$  is the derivation  $z \frac{d}{dz}$ . We will write  $f'$  for  $z \frac{d}{dz} f$ . Put  $\mathcal{Q} = \cup_{m \geq 1} z^{-1/m} \mathbf{C}[z^{-1/m}]$ . The  $\hat{K}$ -algebra  $\hat{R}$  is represented by generators and relations as follows.

$\hat{R} = \hat{K}[\{z^a\}_{a \in \mathbf{C}}, \{e(q)\}_{q \in \mathcal{Q}}, l]$  and the relations are:

$z^{a+b} = z^a z^b$ ,  $e(q_1 + q_2) = e(q_1)e(q_2)$  and  $z^a = z^a \in \hat{K}$  for  $a \in \mathbf{Z}$ .

The differentiation on  $\hat{R}$  is given by:

$(z^a)' = az^a$ ,  $e(q)' = qe(q)$ ,  $l' = 1$ .

The intuitive interpretation of  $\hat{R}$  is:

$z^a$  is the function  $e^{a \log(z)}$ ,  $l$  is the function  $\log(z)$  and  $e(q)$  is the function  $\exp(\int q \frac{dz}{z})$ .

In sectors at the point  $z = 0$  this interpretation makes sense and can be combined with the lifting of certain formal Laurent series in  $\hat{K}$  to actual meromorphic functions on a sector.

*Some differential automorphisms of  $\hat{R}/\hat{K}$ :*

The formal monodromy  $\gamma$  is the differential automorphism of  $\hat{R}/\hat{K}$  defined by:

$\gamma(z^a) = e^{2\pi i a} z^a$  and  $\gamma(l) = l + 2\pi i$ ,

in particular the action of  $\gamma$  on the algebraic closure of  $\hat{K}$  and on  $\mathcal{Q}$  is defined.

$\gamma(e(q)) = e(\gamma q)$ .

The exponential torus  $\text{Hom}(\mathcal{Q}, \mathbf{C}^*)$  acts as a group of differential automorphisms of  $\hat{R}/\hat{K}$ . For an element  $c$  of this group, one defines the differential automorphism  $\tilde{c}$  by  $\tilde{c}z^a = z^a$ ,  $\tilde{c}l = l$ ,  $\tilde{c}e(q) = c(q)e(q)$ .

It is possible to write down all the differential automorphisms of  $\hat{R}/\hat{K}$ . This has the structure of an affine group scheme over  $\mathbf{C}$ . The subgroup generated by the formal monodromy and the exponential torus is a dense subgroup in a certain sense. For our purposes the formal monodromy and the exponential torus will suffice.

*The triple  $(V, \{V_q\}, \gamma_V)$  associated to a differential equation over  $\hat{K}$ .*

Let  $L(y) = 0$  be a homogeneous differential equation of order  $n$  over  $\hat{K}$ . Put  $V := \{r \in \hat{R} \mid L(r) = 0\}$ , the solution space of  $L$  in  $\hat{R}$ . Then  $V$  is a vector space over  $\mathbf{C}$  of dimension  $n$ . The set  $V$  is invariant under the actions of the formal monodromy and the exponential torus. Let  $\gamma_V$  denote the restriction of  $\gamma$  to  $V$ . Put  $V_q := V \cap \hat{K}[\{z^a\}_{a \in \mathbf{C}}, l]e(q)$ . Then for almost all  $q \in \mathcal{Q}$  the vector space  $V_q = 0$ . One has further  $V = \oplus_{q \in \mathcal{Q}} V_q$  and  $\gamma_V(V_q) = V_{\gamma q}$  for all  $q \in \mathcal{Q}$ . This defines the triple  $(V, \{V_q\}, \gamma_V)$  associated to the differential equation over  $\hat{K}$ .

The important observation is:

*The differential equation over  $\hat{K}$  is equivalent to the associated triple  $(V, \{V_q\}, \gamma_V)$ .*

We will make this statement more precise by considering the category  $\mathbf{Gr}_1$ , whose objects are the triples  $(V, \{V_q\}, \gamma_V)$  satisfying:

- (i)  $V$  is a finite dimensional vector space over  $\mathbf{C}$ ,
- (ii)  $\{V_q\}_{q \in \mathcal{Q}}$  is a family of subspaces such that  $V = \oplus V_q$ .
- (iii)  $\gamma_V$  is a  $\mathbf{C}$ -linear automorphism of  $V$  such that  $\gamma_V(V_q) = V_{\gamma q}$  for all  $q \in \mathcal{Q}$ .

One can give  $\mathbf{Gr}_1$  (in an obvious way) the structure of a (neutral) Tannakian category. Let  $\text{Diff}_{\hat{K}}$  denote the Tannakian category of the differential modules over  $\hat{K}$ .

**Proposition 6.1.**— *The above construction is an equivalence between the Tannakian categories  $\text{Diff}_{\hat{K}}$  and  $\mathbf{Gr}_1$ .*

**Corollary 6.2.**— *Let the triple  $(V, \{V_q\}, \gamma_V)$  be associated to the differential equation  $L(y) = 0$  over  $\hat{K}$ . Then the differential Galois group  $G$  of  $L$  can be identified with the algebraic subgroup of  $\text{GL}(V)$  generated by  $\gamma_V$  and the action of the exponential torus on  $V$ .*

*Examples 6.3 : The Airy equation*

The Airy equation  $y'' = zy$  has a singular point at  $z = \infty$ . The triple associated to the equation is  $V = V_{z^{3/2}} \oplus V_{-z^{3/2}}$ . The two spaces  $V_{z^{3/2}}, V_{-z^{3/2}}$  have both dimension 1. After a suitable choice of bases for the two spaces the formal monodromy has the matrix  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and the exponential torus has the form  $\left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbf{C}^* \right\}$ . The differential Galois group is therefore the infinite Dihedral group  $D_\infty \subset \text{SL}(2)$ .

The information in this section on the formal differential Galois group suffices to give a proof of the “easy halves” of the theorems.

**Proposition 6.4.** The easy implications for the local and the global theorem.— (1) *Let  $G$  be the differential Galois group of an equation over the field  $\mathbf{C}(\{z\})$ . Then  $G/L(G)$  is topologically generated by one element.*

(2) *Let  $X$  be a compact Riemann surface of genus  $g$ ,  $S \subset X$  a finite subset with cardinality  $m \geq 0$  and  $G$  the differential Galois group of a differential equation over  $X$  with singularities in  $S$ . Then the natural homomorphism  $\pi_1(X \setminus S, *) \rightarrow G/L(G)$  has dense image.*

*Proof.*— (1) The group  $G/L(G)$  is according to section 2 also the differential Galois group of some equation  $y' = Ay$  over  $\mathbf{C}(\{z\})$ . The differential Galois group  $H$  of this equation over  $\mathbf{C}((z))$  is a subgroup of  $G/L(G)$ . The latter group has a trivial maximal torus

and thus the exponential torus is not present in  $H$ . Therefore the differential equation is regular singular and is equivalent to  $zy' = By$  where  $B$  is a constant matrix. It is well known that the differential Galois group (either over  $\mathbf{C}(\{z\})$  or over  $\mathbf{C}((z))$ ) is the algebraic subgroup of  $\mathrm{GL}(n)$  generated by  $e^{2\pi i B}$ .

(2) Let  $K$  denote the function field of  $X$ . The group  $G/L(G)$  is the differential Galois group of a matrix equation  $y' = Ay$  over  $X$  which has its singularities in the set  $S$ . As in (1), it follows that the equation has only regular singularities. Take a point  $c \in X \setminus S$  and let  $V$  be the space of the local solutions of the equation. The Picard-Vessiot field  $L$  can be identified with the subfield of  $K_c$  generated by  $K$  and (the coordinates of the elements of)  $V$ . The monodromy of  $y' = Ay$  on  $X \setminus S$  induces a homomorphism  $\pi_1(X \setminus S, c) \rightarrow \mathrm{GL}(V)$ . The image of this homomorphism lies in  $G/L(G)$ . Let  $f \in L$  be invariant under this image. Then  $f$  extends to a meromorphic function on  $X \setminus S$ . Since the singularities of  $y' = Ay$  are regular singular,  $f$  extends to a meromorphic function on all of  $X$ . Therefore  $f \in K$ . From the Galois correspondence we draw the conclusion that the image of  $\pi_1(X \setminus S, c)$  in  $G/L(G)$  is Zariski dense.

## 7. THE ANALYTIC LOCAL THEORY

In this section the differential field  $K$  is  $\mathbf{C}(\{z\})$ . For notational convenience we take the derivation  $\delta$  to be  $f \mapsto \delta(f) = f' = z \frac{df}{dz}$ . For the description of the differential modules over  $K$  we introduce a second category  $\mathbf{Gr}_2$ . First we will need some notation. Let  $d \in \mathbf{R}$  and  $q \in \mathcal{Q}$ ,  $q \neq 0$ . Then  $d$  (or rather  $e^{id}$ ) is considered as a direction at the point  $z = 0$ . The *front* of  $q$ , in notation  $Fr(q)$ , consists of the  $d$ 's such that the function  $h := e^{\int q \frac{d}{dz}}$  has maximal decrease in the direction  $d$ . In other words  $|h(re^{id})|$  has maximal decrease when  $r > 0$  tends to zero. For the example  $q = z^{-1}$  one finds  $|h(re^{id})| = e^{r^{-1} \cos(d)}$  and the front  $Fr(z^{-1})$  is thus  $\pi + 2\pi\mathbf{Z}$ . For any  $q \in \mathcal{Q}$ , one denotes the projection  $V \rightarrow V_q$  (along the other  $V_{q'}$ 's) by  $pr_q$ . Using this terminology we can define an object  $(V, \{V_q\}, \gamma_V, St_{V,d})$  of  $\mathbf{Gr}_2$  as follows:

- (1)  $(V, \{V_q\}, \gamma_V)$  is an object of  $\mathbf{Gr}_1$ .
- (2) For every  $d \in \mathbf{R}$  there is given an automorphism  $St_{V,d}$  of  $V$  of the form

$$id + \sum_{q_1, q_2; d \in Fr(q_2 - q_1)} N_{q_2, q_1},$$

where  $N_{q_2, q_1}$  is a  $\mathbf{C}$ -linear map  $N_{q_2, q_1} : V \xrightarrow{pr_{q_2}} V_{q_2} \rightarrow V_{q_1} \subset V$ .

- (3) The  $St_{V,d}$  should satisfy the relation  $\gamma_V^{-1} St_{V,d} \gamma_V = St_{V, d+2\pi}$ .

The  $St_{V,d}$  are seen to be unipotent maps.

If  $(V, \{V_q\}, \gamma_V)$  is fixed then there are modulo  $2\pi$  only finitely many  $d$ 's such that there exists  $q_2, q_1$  with  $d \in Fr(q_2 - q_1)$  and  $V_{q_2} \neq 0$ ,  $V_{q_1} \neq 0$ . The possibilities for the  $St_{V,d}$  form thus a finite dimensional vector space over  $\mathbf{C}$ .

Morphisms between two objects are  $\mathbf{C}$ -linear maps preserving all the data. The tensor product of two objects  $V = (V, \{V_q\}, \gamma_V, St_{V,d})$  and  $W = (W, \{W_q\}, \gamma_W, St_{W,d})$  is defined as the vector space  $V \otimes W$  with the data:

$$(V \otimes W)_q = \bigoplus_{q_1, q_2: q_1 + q_2 = q} V_{q_1} \otimes W_{q_2}.$$

$$\gamma_{V \otimes W} = \gamma_V \otimes \gamma_W \text{ and}$$

$$St_{V \otimes W, d} = St_{V, d} \otimes St_{W, d}.$$

One can show that  $\mathbf{Gr}_2$  is a neutral Tannakian category over  $\mathbf{C}$ . Let  $\text{Diff}_K$  denote the Tannakian category of the differential modules over  $K = \mathbf{C}(\{z\})$ . The main results of Martinet and Ramis on multisummation can now be phrased as:

**Theorem 7.1.**— *The multisummation operator provides an equivalence between the Tannakian categories  $\text{Diff}_K$  and  $\mathbf{Gr}_2$ .*

**Corollary 7.2.**— *Let  $L$  be a linear homogeneous differential equation over  $K$  and let  $(V, \{V_q\}, \gamma_V, St_{V,d})$  denote the object of  $\mathbf{Gr}_2$  associated to  $L$ . Then the differential Galois group of  $L$  can be identified with the Zariski closure of the subgroup of  $\text{GL}(V)$  generated by:*

- (1) *The formal monodromy  $\gamma_V$ .*
- (2) *The exponential torus.*
- (3) *The collection  $\{St_{V,d}\}$ , called the Stokes matrices of  $L$ .*

*Remark.*— It must be stressed that the theorem and the corollary are at the heart of the theory of multisummation and form moreover the essential part of the proof of Ramis' local theorem.

### *The category $\mathbf{Gr}_3$*

The categories  $\mathbf{Gr}_1$  and  $\mathbf{Gr}_2$  were introduced in [P1] in order to obtain a nice and compact formulation of the work of Martinet and Ramis. A further reformulation, due to Ramis, will simplify the proof of the local theorem even further. The category  $\mathbf{Gr}_3$  is defined by:

the objects are tuples  $(V, \{V_q\}, \gamma_V, st_{V,d})$  with

- (1)  $(V, \{V_q\}, \gamma_V)$  is an object of  $\mathbf{Gr}_1$ .
- (2) For every  $d \in \mathbf{R}$  there is given a  $st_{V,d} \in \bigoplus_{q_1, q_2: d \in Fr(q_2 - q_1)} \text{Hom}(V_{q_1}, V_{q_2})$ .
- (3) The  $st_{V,d}$  should satisfy  $\gamma_V^{-1} st_{V,d} \gamma_V = st_{V, d+2\pi}$ .

We identify, as before,  $\text{Hom}(V_{q_1}, V_{q_2})$  with a linear subspace of  $\text{End}(V)$ . A morphism  $f : (V, \{V_q\}, \gamma_V, st_{V,d}) \rightarrow (W, \{W_q\}, \gamma_W, st_{W,d})$  is a linear map  $V \rightarrow W$  satisfying  $f(V_q) \subset W_q$ ,  $f \circ \gamma_V = \gamma_W \circ f$ ,  $f \circ st_{V,d} = st_{W,d} \circ f$ . The tensor product of two objects  $(V, \{V_q\}, \gamma_V, st_{V,d})$ ,  $(W, \{W_q\}, \gamma_W, st_{W,d})$  is the vector space  $V \otimes W$  with the data  $(V \otimes W)_q = \bigoplus_{q_1, q_2: q_1 + q_2 = q} V_{q_1} \otimes W_{q_2}$ ,  $\gamma_{V \otimes W} = \gamma_V \otimes \gamma_W$  and  $st_{V \otimes W, d} = st_{V,d} \otimes id_W + id_V \otimes st_{W,d}$ .

It is easily seen that  $\mathbf{Gr}_3$  is again a neutral Tannakian category. In fact, the Tannakian categories  $\mathbf{Gr}_2$  and  $\mathbf{Gr}_3$  are isomorphic.

**Lemma 7.3.**— (1) *The exponential map induces an equivalence of Tannakian categories  $\mathbf{Exp} : \mathbf{Gr}_3 \rightarrow \mathbf{Gr}_2$ .*

(2) *Let the object  $(V, \{V_q\}, \gamma_V, st_{V,d})$  be associated to a differential equation over  $\mathbf{C}(\{z\})$ . Then the differential Galois group  $G \subset \mathrm{GL}(V)$  is the smallest algebraic subgroup such that:*

- (a) *The exponential torus and  $\gamma_V$  belong to  $G$ .*
- (b) *All  $st_{V,d}$  belong to the Lie algebra of  $G$ .*

*Proof.*— The exponential map associates to  $(V, \{V_q\}, \gamma_V, st_{V,d})$  the object  $(V, \{V_q\}, \gamma_V, St_{V,d})$  with  $St_{V,d} = \exp(st_{V,d})$ . It is easily seen that this results into an equivalence of Tannakian categories. The second part of the lemma is a reformulation of 7.2.

### Multisummation and the Stokes matrices $St_{V,d}$

Consider a differential equation  $L(y) = g$  over  $K$  and with  $g \in K$ . Let  $\hat{f} \in \hat{K}$  be a formal solution of the equation. The *main theorem of the asymptotic theory of differential equations* states that for a sector  $S$  at 0 with small enough opening, there exists a meromorphic function  $f$  on  $S$  with asymptotic expansion  $\hat{f}$  and still satisfying  $L(f) = g$ . In general this  $f$  is not unique. Multisummation produces a unique choice for the  $f$  above. For  $d \in \mathbf{R}$  one considers the direction  $e^{id}$  at 0. There are, modulo  $2\pi\mathbf{Z}$ , finitely many *singular directions*  $d$  for the differential equation  $L(y) = g$ . For a direction  $d$ , which is not singular, there is a multisummation operator  $S_d$  which maps a formal solution  $\hat{f}$  to a solution  $S_d(\hat{f})$  of the equation, which lives on a certain sector with bisector  $d$  and has  $\hat{f}$  as asymptotic expansion. In general  $S_d(\hat{f})$  depends on the choice of  $d$ . This is usually called the *Stokes phenomenon*. However  $S_d(\hat{f})$  does not depend on  $d \in (a, b)$  if this interval does not contain any singular direction.

Let  $d$  be a singular direction. Then we define  $S_{d+}(\hat{f})$  and  $S_{d-}(\hat{f})$  as  $S_{d+\epsilon}(\hat{f})$  and  $S_{d-\epsilon}(\hat{f})$  for  $\epsilon > 0$  and small enough. The difference between  $S_{d+}(\hat{f})$  and  $S_{d-}(\hat{f})$  is measured by the Stokes matrix  $St_{V,d}$ . We will make this more precise.

Let  $M$  be a differential module over  $K = \mathbf{C}(\{z\})$ . The solution space  $V$  is defined as  $V = \ker(\partial, \hat{R} \otimes_K M)$ . The ring  $\hat{R}$  is given a basis over  $\hat{K} = \mathbf{C}((z))$ , e.g.  $z^a l^n e(q)$  with  $a \in \mathbf{C}$  and  $0 \leq \mathrm{Re}(a) < 1$ ;  $n \in \mathbf{Z}, n \geq 0$  and  $q \in \mathcal{Q}$ . Further  $M$  can be given a basis over  $K$ . Combining this one obtains a basis  $\{B_i\}_{i \in I}$  of  $\hat{R} \otimes_K M$  over  $\hat{K}$ . The elements  $v \in V$  can be written as finite sums  $\sum \hat{f}_i B_i$  with  $\hat{f}_i \in \hat{K}$ . The  $\hat{f}_i$  are formal solutions of a set of (inhomogeneous) linear differential equations over  $K$ . For a direction  $d$ , which is not singular, one defines  $S_d(v)$  as  $\sum S_d(\hat{f}_i) \mathrm{int}_d(B_i)$ , where  $\mathrm{int}_d(B_i)$  is the interpretation of  $B_i$  (this involves  $z^a$  and the logarithm) in the direction  $d$ . There results a  $\mathbf{C}$ -linear bijection  $S_d : V \rightarrow \mathrm{Sol}_d$ , where  $\mathrm{Sol}_d$  is the vector space of the solutions of

$M$  on a suitable sector containing the direction  $d$ . For *any* direction  $d$ , there are two isomorphisms  $S_{d+}, S_{d-} : V \rightarrow \text{Sol}_d$ . One defines  $St_{V,d} = S_{d-}^{-1} S_{d+} \in \text{GL}(V)$ . Clearly  $St_{V,d}$  is the identity if  $d$  is not a singular direction. It is not difficult to prove that the object  $(V, \{V_q\}, \gamma_V, St_{V,d})$  belongs to the category  $\mathbf{Gr}_2$ , and that multisummation produces in fact a functor  $F_2 : \text{Diff}_K \rightarrow \mathbf{Gr}_2$  of Tannakian categories. The proof of 7.1, i.e., proving that  $F_2$  is an equivalence of categories, is rather involved (see [P3] and [R2]).

### The relation between the formal and the topological monodromy

Let the differential module  $M$  over  $K$  be given with  $F_2(M) = (V, \{V_q\}, \gamma_V, St_{V,d})$ . Let  $0 \leq d_1 < d_2 < \dots < d_m < 2\pi$  denote the singular directions (in fact for the differential module  $\text{Hom}(M, M)$ ) lying in  $[0, 2\pi)$ . The topological monodromy  $\gamma_M$  of  $M$  is easily seen to satisfy (up to conjugation) the formula

$$\gamma_M = \gamma_V \cdot St_{V,d_1} St_{V,d_2} \cdots St_{V,d_m}.$$

#### Examples 7.4 : The Airy equation, continued

We consider again the Airy equation  $y'' = zy$ . There are two singular directions  $d = 0, \pi$  modulo  $2\pi\mathbf{Z}$ . The topological monodromy is trivial and the formal monodromy is not trivial. The formula for the topological monodromy implies that both Stokes matrices  $St_{V,0}$  and  $St_{V,\pi}$  are different from 1. The two matrices have the form  $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$  with respect to the decomposition  $V = V_{z^{3/2}} \oplus V_{-z^{3/2}}$ . It follows from 7.2 that the differential Galois group of the Airy equation over  $\mathbf{C}(z)$  is  $\text{SL}(2)$ .

The same argument shows that the differential Galois group of any equation  $y'' = ry$  with  $r \in \mathbf{C}[z]$  a polynomial of odd degree is  $\text{SL}(2)$ .

## 8. THE PROOF OF THE LOCAL THEOREM

The following lemma will be a guide for the construction of a suitable object in the category  $\mathbf{Gr}_3$ .

### The data

$V$  is a finite dimensional vector space over  $\mathbf{C}$  and  $G \subset \text{GL}(V)$  is an algebraic subgroup.  $T$  denotes a maximal torus and  $\underline{g}, \underline{t} \subset \text{End}(V)$  are the Lie algebras of  $G$  and  $T$ .

The action of  $T$  on  $V$  yields a decomposition  $V = \bigoplus_{i=1}^s V_{\chi_i}$ , where the  $\chi_i$  are distinct characters of  $T$  and the non-trivial spaces  $V_{\chi_i}$  are defined as  $\{v \in V \mid tv = \chi_i(t)v \text{ for all } t \in T\}$ .

For each  $i, j$  one identifies  $\text{Hom}(V_{\chi_i}, V_{\chi_j})$  with a linear subset of  $\text{End}(V)$  by identifying

$L \in \text{Hom}(V_{\chi_i}, V_{\chi_j})$  with  $V \xrightarrow{pr_i} V_{\chi_i} \xrightarrow{L} V_{\chi_j} \subset V$ , where  $pr_i$  denotes the projection onto  $V_{\chi_i}$ , along  $\oplus_{k \neq i} V_{\chi_k}$ .

The adjoint action of  $T$  on  $\underline{g}$  yields a decomposition  $\underline{g} = \underline{g}_0 \oplus \sum_{\alpha \neq 0} \underline{g}_\alpha$ . By definition, the adjoint action of  $T$  on  $\underline{g}_0$  is the identity and is multiplication by the character  $\alpha \neq 0$  on the spaces  $\underline{g}_\alpha$ . (We note that here the additive notation for characters is used. In particular,  $\alpha \neq 0$  means that  $\alpha$  is not the trivial character.)

Any  $B \in \underline{g}$  can be written as  $\sum B_{i,j}$  with  $B_{i,j} \in \text{Hom}(V_{\chi_i}, V_{\chi_j})$ . The adjoint action of  $t \in T$  on  $B$  has the form  $Ad(t)B = \sum \chi_i^{-1} \chi_j(t) B_{i,j}$ . It follows that the  $\alpha \neq 0$  with  $\underline{g}_\alpha \neq 0$  have the form  $\chi_i^{-1} \chi_j$ . In particular  $\sum_{i,j; \chi_i^{-1} \chi_j = \alpha} B_{i,j} \in \underline{g}_\alpha \subset \underline{g}$ . Let  $L(G)$  denote, as before, the subgroup of  $G$  generated by all the conjugates of  $T$ .

*Lemma 8.1 (Ramis).—* (1)  $L(G)$  is a normal algebraic subgroup of  $G$  (and of  $G^\circ$ ).

(2) The Lie algebra of  $L(G)$  is generated by the subspaces  $\underline{t}$  and  $\{\underline{g}_\alpha\}_{\alpha \neq 0}$ .

*Proof.*— The first statement follows from [Bo], Ch 2, Prop.(7.5) p.190 and Thm.(7.6) p.192. Consider some  $\alpha \neq 0$  and a non-zero element  $\xi \in \underline{g}_\alpha$ . From the definition of  $\underline{g}_\alpha$  and  $\alpha \neq 0$  it follows that there is an ordering, denoted by  $V_1, \dots, V_s$ , of the spaces  $\{V_{\chi_i}\}$ , such that  $\xi$  maps each  $V_i$  into some  $V_j$  with  $j > i$ . In particular,  $\xi$  is nilpotent and  $\mathbf{C}\xi$  is an algebraic Lie algebra corresponding to the algebraic subgroup  $\{\exp(c\xi) | c \in \mathbf{C}\}$  of  $G$ . Let  $\underline{h}$  denote the Lie algebra generated by the algebraic Lie algebras  $\underline{t}$  and  $\mathbf{C}\xi$  for all  $\xi \in \underline{g}_\alpha$  with  $\alpha \neq 0$ . Then  $\underline{h}$  is an algebraic Lie algebra. ([Bo], loc. cit.).

Take an element  $t \in T$  such that all  $\chi_i(t)$  are distinct. Then clearly  $t \cdot \exp(\xi)$  is semi-simple and lies therefore in a conjugate of the maximal torus  $T$ . Thus  $\exp(\xi) = t^{-1} \cdot (t \cdot \exp(\xi)) \in L(G)$  and  $\xi$  lies in the Lie algebra of  $L(G)$ . This proves that the Lie algebra  $\underline{h}$  is a subset of the Lie algebra of  $L(G)$ .

On the other hand,  $\underline{h}$  is easily seen to be an ideal in  $\underline{g}$ . The connected normal algebraic subgroup  $H \subset G^\circ$  corresponding to  $\underline{h}$  contains  $T$  and therefore  $L(G)$ . This proves the other inclusion.

We start the proof of Ramis' local theorem by considering the data:

$V$  is a finite dimensional vector space over  $\mathbf{C}$  and  $G \subset \text{GL}(V)$  is an algebraic group such that  $G/L(G)$  is topologically generated by one element  $a$ . It suffices to construct an object  $(V, \{V_q\}, \gamma_V, st_{V,d})$  in the category  $\mathbf{Gr}_3$  with group  $G$ , i.e.,  $G$  is the smallest algebraic subgroup of  $\text{GL}(V)$  such that:

- (1)  $G$  contains  $\gamma_V$  and the exponential torus.
- (2) The Lie algebra  $\underline{g}$  of  $G$  contains all  $st_{V,d}$ .

\* Choose a maximal torus  $T \subset G$  and a representative  $A \in G$  of  $a \in G/L(G)$ . Since  $T$  is also a maximal torus of  $L(G)$ , there exists  $B \in L(G)$  such that  $ATA^{-1} = BTB^{-1}$ . After replacing  $A$  by  $B^{-1}A$  we may suppose that  $ATA^{-1} = T$ .

- \* Let  $V = V_{\chi_1} \oplus \cdots \oplus V_{\chi_s}$  denote the decomposition of  $V$  with respect to the action of  $T$ .
- \* The element  $A \in G$  permutes the spaces  $V_{\chi_i}$ . More precisely,  $A(V_{\chi_i}) = V_{\chi_j}$  where the character  $\chi_j$  is given by  $\chi_j(t) = \chi_i(A^{-1}tA)$ . We will write  $\tilde{A}\chi_i = \chi_j$ . Indeed, for  $t \in T, v \in V_{\chi_i}$  one has

$$tAv = A(A^{-1}tA)v = A\chi_i(A^{-1}tA)v = \chi_i(A^{-1}tA)Av.$$

- \* We recall that  $\mathcal{Q} = \bigcup_{m \geq 1} z^{-1/m} \mathbb{C}[z^{-1/m}]$  carries an action of  $\gamma$  given by  $\gamma z^\lambda = e^{2\pi i \lambda} z^\lambda$ .

*Lemma 8.2.— There are elements  $q_1, \dots, q_s \in \mathcal{Q}$  such that*

- (1) *If  $\tilde{A}\chi_i = \chi_j$  then  $\gamma(q_i) = q_j$ .*
  - (2) *If  $n_1\chi_1 + \cdots + n_s\chi_s = 0$  (here with additive notation for characters) for some  $n_1, \dots, n_s \in \mathbb{Z}$ , then  $n_1q_1 + \cdots + n_sq_s = 0$ .*
- Let  $N \geq 1$  be any integer, then there exists  $q_1, \dots, q_s$  satisfying (1) and (2) and such that for any  $i \neq j$  the degrees of  $q_i - q_j$  in  $z^{-1}$  are  $\geq N$ .*

*Proof.*— Conditions (1) and (2) can be translated into: the  $\mathbb{Z}$ -module  $\mathbb{Z}\chi_1 + \cdots + \mathbb{Z}\chi_s$  can be embedded into  $\mathcal{Q}$  such that the action of  $\tilde{A}$  is compatible with the action of  $\gamma$  on  $\mathcal{Q}$ .

Let  $\bar{\mathcal{Q}}$  denote the algebraic closure of  $\mathcal{Q}$ . Consider  $M := \bar{\mathcal{Q}} \otimes (\mathbb{Z}\chi_1 + \cdots + \mathbb{Z}\chi_s) \subset \mathbb{Z}\chi_1 + \cdots + \mathbb{Z}\chi_s$  with the induced  $\tilde{A}$  action. Since a power of  $\tilde{A}$  acts trivially on  $M$  we may present this action by  $M = \bigoplus_{\zeta_j} M_{\zeta_j}$ , where  $\zeta_j$  runs in a finite set of all roots of unity and  $\tilde{A}$  acts on  $M_{\zeta_j}$  as multiplication by  $\zeta_j$ .

Further  $\mathcal{Q} = \bigoplus_{\lambda \in \mathbb{Q}, \lambda < 0} \mathbb{C}z^\lambda$  and  $\gamma$  acts on  $\mathbb{C}z^\lambda$  as multiplication by  $e^{2\pi i \lambda}$ . Define  $\lambda_j \in \mathbb{Q}, \lambda_j < 0$  by  $e^{2\pi i \lambda_j} = \zeta_j$  and  $\lambda_j$  is maximal. Choose for every  $j$  an embedding of  $\bar{\mathcal{Q}}$ -vector spaces  $M_{\zeta_j} \subset z^{\lambda_j} z^{-N} \mathbb{C}[z^{-1}]$ . Then the resulting embedding  $M \subset \mathcal{Q}$  has the required properties. Moreover, any non-zero element of  $M$  is mapped to an element of degree  $\geq N$  in the variable  $z^{-1}$ .

We continue now the proof of the Ramis' local theorem. Define the formal structure  $(V, \{V_q\}, \gamma_V)$  (i.e., an object of  $\mathbf{Gr}_1$ ) by  $V_{\chi_i} = V_{q_i}$  and  $\gamma_V = A$ . As before  $\underline{g}$  denotes the Lie algebra of  $G$  (or  $G^\circ$ ). The decomposition of  $\underline{g}$  with respect to the adjoint action of the torus  $T$  has already been made explicit, namely

$$\underline{g}_\alpha = \bigoplus_{i,j: \chi_i^{-1}\chi_j = \alpha} \underline{g} \cap \text{Hom}(V_{q_i}, V_{q_j}).$$

For the definition of the structure of an object from the category  $\mathbf{Gr}_3$  on  $V$  we may choose arbitrary elements  $st_{V,d} \in \underline{g}_\alpha$ , where  $\alpha = \chi_i^{-1}\chi_j$ ,  $d \in Fr(q_i - q_j)$ ,  $0 \leq d < 2\pi$ . The number of directions  $d$  modulo  $2\pi$  in  $Fr(q_i - q_j)$  is by construction sufficiently large to ensure a choice of the set  $\{st_{V,d}\}$  such that these elements generate the vector space  $\bigoplus_{\alpha \neq 0} \underline{g}_\alpha$ .

Finally, we verify that the algebraic group  $\tilde{G}$ , associated to the object  $(V, \{V_q\}, \gamma_V, st_{V,d})$ , is equal to  $G$ . By construction  $\tilde{G} \subset G$  and  $\tilde{G}$  is the smallest algebraic group with:

- (a) The exponential torus and  $\gamma_V$  lie in  $\tilde{G}$ .

(b) The Lie algebra of  $\tilde{G}$  contains all  $st_{V,d}$ .

By construction, the exponential torus is equal to  $T$  and its Lie algebra  $\underline{t}$  lies in  $\tilde{G}$ . Again by construction, the  $\underline{g}_\alpha$  (with  $\alpha \neq 0$ ) belong to the Lie algebra of  $\tilde{G}$ . Thus  $\tilde{G}$  contains  $L(G)$ . The choice of  $\gamma_V$  implies that  $\tilde{G} = G$ .

## 9. THE CONSTRUCTIVE INVERSE PROBLEM

In this section we will present a constructive proof of the following “gem” which is at the heart of the work of Mitschi and Singer.

**Theorem 9.1** (Mitschi-Singer).— *The field  $C$  is supposed to be algebraically closed and of characteristic 0. Every connected semi-simple linear algebraic group is the differential Galois group of an equation  $y' = (A_0 + A_1 z)y$  over  $C(z)$ , where  $A_0, A_1$  are constant matrices.*

The proof given here uses the language of Tannakian categories (see [De-M]). For a differential module  $M$  (over  $C(z)$ ) one denotes by  $\{\{M\}\}$  the full Tannakian subcategory of  $\text{Diff}_{C(z)}$  generated by  $M$ . By definition the objects of  $\{\{M\}\}$  are the differential modules isomorphic to a finite direct sum of subquotients  $M_1/M_2$  of some  $M \otimes \cdots \otimes M \otimes M^* \otimes \cdots \otimes M^*$  (i.e.,  $M_2 \subset M_1$  are (differential) submodules of such a tensor product). For any linear algebraic group  $L$  (over  $C$ ) one denotes by  $\text{Repr}_L$  the Tannakian category of the finite dimensional representations of  $L$  (over  $C$ ). The differential Galois group of  $M$  is  $L$  if there is an equivalence between the Tannakian categories  $\{\{M\}\}$  and  $\text{Repr}_L$ .

Let a finite dimensional vector space  $V$  over  $C$  and an algebraic subgroup  $G \subset \text{GL}(V)$  be given. Our first aim is to produce a differential module  $M = (C(z) \otimes V, \partial)$  and a functor of Tannakian categories  $\text{Repr}_G \rightarrow \{\{M\}\}$ .

### 9.2. The functor $\text{Repr}_G \rightarrow \{\{M\}\}$

The Lie algebra of  $G$  will be written as  $\underline{g} \subset \text{End}(V)$ . One chooses a matrix  $A(z) \in C(z) \otimes \underline{g} \subset C(z) \otimes \text{End}(V)$ . To this choice there corresponds a differential equation  $y' = A(z)y$  and a differential module  $M := (C(z) \otimes V, \partial)$  with  $\partial$  defined by  $\partial(v) = -A(z)v$  for all  $v \in V$ . Let a representation  $\rho : G \rightarrow \text{GL}(T)$  be given. The induced maps  $\underline{g} \rightarrow \text{End}(T)$  and  $C(z) \otimes \underline{g} \rightarrow C(z) \otimes \text{End}(T)$  are also denoted by  $\rho$ . One associates to  $(T, \rho)$  the differential module  $(C(z) \otimes T, \partial)$  with  $\partial(t) = -\rho(A(z))t$  for all  $t \in T$ . The corresponding differential equation is  $y' = \rho(A(z))y$ . In this way one obtains a functor of Tannakian categories  $\text{Repr}_G \rightarrow \text{Diff}_{C(z)}$ . We claim that every  $(C(z) \otimes T, \partial)$ , as above, lies in fact in  $\{\{M\}\}$ .

Indeed, consider  $\{\{V\}\}$ , the full subcategory of  $\text{Repr}_G$  generated by  $V$  (the definition is similar to the definition of  $\{\{M\}\}$ ). It is known that  $\{\{V\}\} = \text{Repr}_D$  (see

[De-M]). The representation  $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$  is mapped to the differential module  $M \otimes \cdots \otimes M \otimes M^* \otimes \cdots \otimes M^*$ . Let  $V_2 \subset V_1$  be  $G$ -invariant subspaces of  $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$ . Then  $V_2$  and  $V_1$  are invariant under the action of  $\underline{g}$ . Since  $A(z) \in C(z) \otimes \underline{g}$ , one has that  $C(z) \otimes V_2$  and  $C(z) \otimes V_1$  are (differential) submodules of  $M \otimes \cdots \otimes M \otimes M^* \otimes \cdots \otimes M^*$ . It follows that the differential module  $(C(z) \otimes V_1/V_2, \partial)$  lies in  $\{\{M\}\}$ . This proves the claim.

The next step is to make the differential Galois group  $L$  of  $M$  and the equivalence  $\{\{M\}\} \rightarrow \text{Repr}_L$  as concrete as we can.

### 9.3. Differential modules over $O$

Fix some  $c \in C$  and let  $O$  denote the localization of  $C[z]$  at  $(z - c)$ . A differential module  $(N, \partial)$  over  $O$  is a finitely generated  $O$ -module equipped with a  $C$ -linear map  $\partial$  satisfying  $\partial(fn) = f'n + f\partial(n)$  for all  $f \in O$  and  $n \in N$ . It is an exercise to show that  $N$  has no torsion elements. It follows that  $N$  is a free, finitely generated  $O$ -module. Let  $\text{Diff}_O$  denote the category of the differential modules over  $O$ . The functor  $N \mapsto C(z) \otimes_O N$  induces an equivalence of  $\text{Diff}_O$  with a full subcategory of  $\text{Diff}_{C(z)}$ . It is easily seen that this full subcategory has as objects the differential modules over  $C(z)$  which are regular at  $z = c$ .

Let  $\hat{O} = C[[z - c]]$  denote the completion of  $O$ . For any differential module  $N$  over  $O$  of rank  $n$ , one writes  $\hat{N}$  for  $\hat{O} \otimes N$ . We note that  $\hat{N}$  is a differential module over  $\hat{O}$  and that  $\hat{N}$  is in fact a trivial differential module. The space  $\ker(\partial, \hat{N})$  is a vector space over  $C$  of dimension  $n$ , which will be called  $\text{Sol}_c(N)$ , the solution space of  $N$  over  $C[[z - c]]$  (and also over  $C((z - c))$ ). The canonical map  $\text{Sol}_c(N) \rightarrow \hat{N}/(z - c)\hat{N} = N/(z - c)N$  is an isomorphism. Let  $\text{Vect}_C$  denote the Tannakian category of the finite dimensional vector spaces over  $C$ . The above construction  $N \mapsto N/(z - c)N$  is a fibre functor of  $\text{Diff}_O \rightarrow \text{Vect}_C$ . For a fixed object  $M$  of  $\text{Diff}_O$  one can consider the restriction  $\omega : \{\{M\}\} \rightarrow \text{Vect}_C$ , which is again a fibre functor. The differential Galois group  $L$  of  $M$  is defined in [De-M] as  $\underline{\text{Aut}}^\otimes(\omega)$ . By definition  $L$  acts on  $\omega(N)$  for every object  $N$  of  $\{\{M\}\}$ . Thus we find a functor  $\{\{M\}\} \rightarrow \text{Repr}_L$ , which is an equivalence of Tannakian categories. The composition  $\{\{M\}\} \rightarrow \text{Repr}_L \rightarrow \text{Vect}_C$  (where the last arrow is the forgetful functor) is the same as  $\omega$ . (We note that the above remains valid if we replace  $O$  by any localization of  $C[z]$ .)

### 9.4. Some observations

(9.4.1) Let  $G$  again be an algebraic subgroup of  $\text{GL}(V)$ , let  $A(z) \in C(z) \otimes \underline{g}$  be chosen. Suppose that the matrix  $A(z)$  has no poles at  $z = c$ . Then we have functors of Tannakian categories  $\text{Repr}_G \rightarrow \{\{M\}\}$  and  $\{\{M\}\} \rightarrow \text{Repr}_L$ . The last functor is made by considering  $M$  as differential module over  $O$ . The composition of the two functors

maps a representation  $(W, \rho)$  of  $G$  to a representation of  $L$  on  $O/(z - c)O \otimes W$  which is canonically isomorphic to  $W$ . It follows that  $L$  is an algebraic subgroup of  $G$ .

(9.4.2) Suppose that  $L$  is a proper subgroup of  $G$ , then there exists a representation  $(W, \rho)$  of  $G$  and a line  $\tilde{W} \subset W$  such that  $L$  stabilizes  $\tilde{W}$  and  $G$  does not. The differential module  $(C(z) \otimes W, \partial)$  has as image in  $\text{Repr}_L$  the space  $W$  with its  $L$ -action. The  $L$ -invariant subspace  $\tilde{W}$  corresponds with a one-dimensional (differential) submodule  $C(z)w \subset C(z) \otimes W$ . After multiplication of  $w$  by an element in  $C(z)$ , we may suppose that  $w \in C[z] \otimes W$  and that the coordinates of  $w$  with respect to a basis of  $W$  have g.c.d. 1. Let us write  $\frac{d}{dz}$  for the differentiation on  $C(z) \otimes W$ , given by  $\frac{d}{dz}fa = f'a$  for  $f \in C(z)$  and  $a \in W$ . Then one finds the equation

$$\left[ \frac{d}{dz} - \rho(A(z)) \right] w = cw \text{ for some } c \in C(z).$$

(9.4.3) The idea for the rest of the proof is to make a choice for  $A(z)$  which contradicts the equation for  $w$  above. For a given proper algebraic subgroup  $H$  of  $G$  one can produce a suitable  $A(z)$  which contradicts the statement that  $L$  lies in a conjugate of  $H$ . In general however one has to consider infinitely many (conjugacy classes) of proper algebraic subgroups of  $G$ . This will probably not lead to a *construction* of the matrix  $A(z)$ . In the sequel we will make two restrictions, namely  $A(z)$  is a polynomial matrix (i.e.,  $A(z) \in C[z] \otimes \mathfrak{g}$ ) and that  $G$  is connected and semi-simple. As we will see the first restriction implies that the differential Galois group is a *connected* algebraic subgroup of  $G$ . The second restriction implies that  $G$  has finitely many conjugacy classes of maximal proper connected subgroups.

(9.4.4) *The differential Galois group  $M$  of the differential equation  $y' = (A_0 + A_1z + \dots + A_mz^m)y$ , where  $A_0, \dots, A_m \in \text{End}(W)$  and  $W$  is a finite dimensional vector space over  $C$ , is connected.*

Indeed, let  $E$  denote the Picard-Vessiot ring. Put  $M^\circ =$  the component of the identity of  $M$  and consider  $F = E^{M^\circ}$ . This is a finite Galois extension  $F$  of  $C(z)$  with Galois group  $M/M^\circ$ . The extension  $C(z) \subset F$  can be ramified only above the singular points of the differential equation. The only singular point of the differential equation is  $\infty$ . It follows that  $C(z) = F$  and by Galois correspondence  $M = M^\circ$ .

(9.4.5) A faithful representation  $\rho : G \rightarrow \text{GL}(W)$ , in other words a faithful  $G$ -module  $W$ , will be called a *Chevalley module* if:

- (a)  $G$  leaves no line in  $W$  invariant.
- (b) Any proper connected closed subgroup of  $G$  has an invariant line.

We will postpone the proof that a connected semi-simple  $G$  has a Chevalley module.

9.5. The choices for  $A_0$  and  $A_1$  in  $\underline{g} \subset \text{End}(V)$ 

The connected semi-simple group  $G$  is given as an algebraic subgroup  $G \subset \text{GL}(V)$ , where  $V$  is a finite dimensional vector space over  $C$ . We recall that  $G$  is semi-simple if and only if its Lie algebra  $\underline{g}$  is semi-simple.

For the construction of the equation we will need *the root space decomposition of  $\underline{g}$* . This decomposition reads (see [J] and [F-H]):  $\underline{g} = \underline{h} \oplus (\oplus_{\alpha} \underline{g}_{\alpha})$ , where  $\underline{h}$  is a Cartan subalgebra and the one dimensional spaces  $\underline{g}_{\alpha} = CX_{\alpha}$  are the eigenspaces for the adjoint action of  $\underline{h}$  on  $\underline{g}$  corresponding to the non-zero roots  $\alpha : \underline{h} \rightarrow C$ . More precisely, the adjoint action of  $\underline{h}$  on  $\underline{h}$  is zero and for any  $\alpha \neq 0$  one has  $[h, X_{\alpha}] = \alpha(h)X_{\alpha}$  for all  $h \in \underline{h}$ .

We fix a Chevalley module  $\rho : G \rightarrow \text{GL}(W)$ . The induced (injective) morphism of Lie algebras  $\underline{g} \rightarrow \text{End}(W)$  is also denoted by  $\rho$ . The action of  $\underline{h}$  on  $W$  gives a decomposition of  $W = \oplus W_{\beta}$  into eigenspaces for a collection of linear maps  $\beta : \underline{h} \rightarrow C$ . (The  $\beta$ 's are called the weights of the representation.)

For  $A_0$  one chooses  $\sum_{\alpha \neq 0} X_{\alpha}$ . For  $A_1$  one chooses an element in  $\underline{h}$  satisfying conditions (a), (b) and (c) below.

(a) The  $\alpha(A_1)$  are distinct and different from 0 (for the non-zero roots  $\alpha$  of  $\underline{g}$ ).

(b) The  $\beta(A_1)$  are distinct and different from 0 (for the non-zero weights  $\beta$  of the representation)

It is clear that  $A_1$  with these properties exists. Choose such an  $A_1$ . We want  $A_1$  to satisfy the more technical condition:

(c) If the integer  $m$  is an eigenvalue of the operator  $\sum_{\alpha \neq 0} \frac{1}{-\alpha(A_1)} \rho(X_{-\alpha}) \rho(X_{\alpha})$  on  $W$ , then  $m = 0$ .

If  $A_1$  does not yet satisfy this last condition then a suitable multiple  $cA_1$ , with  $c \in C^*$ , satisfies all three conditions.

**Claim:** *Let  $A_0, A_1 \in \underline{g} \subset \text{End}(V)$  be chosen as above, then the action of the differential Galois group of  $y' = (A_0 + A_1 z)y$  on the solution space can be identified with  $G \subset \text{GL}(V)$ .*

*Three equations*

Then equation (9.4.2) reads now  $[\frac{d}{dz} - (\rho(A_0) + \rho(A_1)z)]w = cw$  with  $c \in C(z)$ . Clearly  $c \in C[z]$  and by comparing the degrees one finds that the degree of  $c$  is at most 1. More explicitly, one has

$$\left[ \frac{d}{dz} - (\rho(A_0) + \rho(A_1)z) \right] w = (c_0 + c_1 z)w \text{ with } w = w_m z^m + \cdots + w_1 z + w_0,$$

with all  $w_i \in W$ ,  $w_m \neq 0$  and  $c_0, c_1 \in C$ . Comparing the coefficients of  $z^{m+1}, z^m, z^{m-1}$  one obtains the relations

$$\rho(A_1)(w_m) = -c_1 w_m$$

$$\rho(A_0)(w_m) + \rho(A_1)(w_{m-1}) = -c_0 w_m - c_1 w_{m-1}$$

$$-m w_m + \rho(A_0)(w_{m-1}) + \rho(A_1)(w_{m-2}) = -c_0 w_{m-1} - c_1 w_{m-2}.$$

The eigenspaces for the action of  $\rho(A_1)$  on  $W$  will be denoted by  $W_b$  with  $b = \beta(A_1)$  the corresponding eigenvalue of  $\rho(A_1)$ . Any element  $\tilde{w} \in W$  is written as  $\tilde{w} = \sum_b \tilde{w}_b$ , with  $\tilde{w}_b \in W_b$ . The relation  $[A_1, X_\alpha] = \alpha(A_1)X_\alpha$  implies that  $\rho(X_\alpha)(W_d) \subset W_{d+\alpha(A_1)}$ . This implies that  $A_0$  has the property  $\rho(A_0)(W_d) \subset \oplus_{b \neq d} W_b$ .

We analyze now the three equations. The first equation can only be solved with  $w_m \in W_d$ ,  $w_m \neq 0$  and  $d = -c_1$ . The second equation, which can be read as  $c_0 w_m = -\rho(A_0)(w_m) + (-\rho(A_1) - c_1)w_{m-1}$ , imposes  $c_0 = 0$ . Indeed, the two right hand side terms  $-\rho(A_0)(w_m)$  and  $(-\rho(A_1) - c_1)w_{m-1}$  have no component in the eigenspace  $W_d$  for  $\rho(A_1)$  to which  $w_m$  belongs. Further

$$w_{m-1} = \sum_{b \neq d} \frac{1}{-b + d} \rho(A_0)(w_m)_b + v_d = \sum_{\alpha \neq 0} \frac{1}{-\alpha(A_1)} \rho(X_\alpha)(w_m) + v_d,$$

for some  $v_d \in W_d$ .

The third equation can be read as

$$-m w_m + \rho(A_0)(w_{m-1}) = (-\rho(A_1) - c_1)w_{m-2}.$$

A necessary condition for this equation to have a solution  $w_{m-2}$  is that the left hand side has 0 as component in  $W_d$ . The component in  $W_d$  of the left hand side is easily calculated to be

$$-m w_m + (\rho(A_0)(w_{m-1}))_d = (-m + \sum_{\alpha \neq 0} \frac{1}{-\alpha(A_1)} \rho(X_{-\alpha})\rho(X_\alpha))(w_m).$$

Since this is zero,  $m$  is an eigenvalue of the operator  $\sum_{\alpha \neq 0} \frac{1}{-\alpha(A_1)} \rho(X_{-\alpha})\rho(X_\alpha)$ . It follows from our assumption on  $A_1$  that  $m = 0$ .

This leaves us with the equation  $[\frac{d}{dz} - (\rho(A_0) + \rho(A_1)z)]w = c_1 z w$  and  $w \in W$ . Since  $\frac{d}{dz}w = 0$ , one finds that  $Cw$  is invariant under  $\rho(A_0)$  and  $\rho(A_1)$ . The Lie algebra  $\underline{g}$  is generated by  $A_0$  and  $A_1$ . Thus  $Cw$  is invariant under  $\underline{g}$  and under  $G$ . Our assumptions on the  $G$ -module  $W$  imply that  $w = 0$ . The proof of theorem 9.1 is completed by a proof of the existence of a Chevalley module.

*Lemma 9.6 (Mitschi and Singer [M-S3]).— Every connected semi-simple linear algebraic group has a Chevalley module.*

*Proof.*— Let the connected semi-simple closed subgroup  $G \subset \text{GL}(V)$  be given. Chevalley's theorem (see [H]) states that for any proper algebraic subgroup  $M$  there is a  $G$ -module  $E$  and a line  $L \subset E$  such that  $M$  is the stabilizer of that line. Since  $G$  is semi-simple,  $E$  is a direct sum of irreducible modules. The projection of  $L$  to one of these irreducible

components is again a line. Thus we find that  $M$  stabilizes a line in some irreducible  $G$ -module  $E$  of dimension greater than one. Any subgroup of  $G$ , conjugated to  $M$ , stabilizes also a line in  $E$ . Dynkin's theorem [D] implies that there are only finitely many conjugacy classes of maximal connected proper algebraic subgroups of  $G$ . One chooses an irreducible  $G$ -module  $W_i$ ,  $i = 1, \dots, m$  for each class and one chooses an irreducible faithful module  $W_0$ . Then  $W = W_0 \oplus \dots \oplus W_m$  has the required properties.

**Examples 9.7.** Chevalley modules for  $SL(2)$  and  $SL(3)$ .— (1) The standard action of  $SL(2, C)$  on  $C^2$  is a Chevalley module.

(2) The standard action of  $SL(3, C)$  on  $C^3$  will be called  $V$ . The induced representation on

$$W = V \oplus (\Lambda^2 V) \oplus (V \otimes_s V) = V \oplus (V \otimes V)$$

is a Chevalley module. Here  $\Lambda^2 V$  is the second exterior power and  $V \otimes_s V$  is the second symmetric power. Indeed, let  $M$  be a maximal proper connected subgroup of  $SL(3)$ . Then  $M$  leaves a line in  $V$  invariant or leaves a plane in  $V$  invariant or  $M$  is conjugated with  $PSL(2) \subset SL(3)$ . In the second case  $M$  leaves a line in  $\Lambda^2 V$  invariant and in the third case  $M$  leaves a line in  $V \otimes_s V$  invariant. Further  $SL(3)$  leaves no line in  $W$  invariant.

## 10. OTHER FORMULATIONS

Ramis' first formulation of the inverse problem for the differential field  $C(\{z\})$  was the following theorem.

**Theorem 10.1.**— *A linear algebraic group  $G$  over  $C$  is a differential Galois group over  $C(\{z\})$  if and only if  $G$  has a local Galois structure.*

For the rather complicated definition of a local Galois structure  $(T, a, \mathcal{N})$  we refer to [R1]. The next proposition simplifies matters.

**Proposition 10.2** (Mitschi and Singer [M-S3]).— *Let  $G$  be a linear algebraic group over an algebraically closed field of characteristic 0. Let  $R_u = R_u(G)$  denote the unipotent radical of  $G$  and let  $G^\circ$  be the component of the identity of  $G$ . The following statements are equivalent:*

(1)  $G$  has a local Galois structure.

(2) (a)  $G/G^\circ$  is cyclic, (b) the dimension of  $R_u/(R_u, G^\circ)$  is  $\leq 1$ , and (c) the action of  $G/G^\circ$  on  $R_u/(R_u, G^\circ)$  by conjugation, is trivial.

Let again  $G$  be a linear algebraic group over an algebraically closed field of characteristic 0. A further investigation of Ramis leads to the definition of the linear algebraic group

$S(G) = R_u/(R_u, G^\circ) \cdot G/G^\circ$ , which is the semi-direct product of  $R_u/(R_u, G^\circ)$  and  $G/G^\circ$ , with respect to the action (by conjugation) of  $G/G^\circ$  on  $R_u/(R_u, G^\circ)$ .

One easily sees that condition (2) in the last proposition is equivalent to  $S(G)$  is topologically (for the Zariski topology) generated by one element.

The connection with the linear algebraic group  $V(G) := G/L(G)$  ( $L(G)$  is defined in section 3) is the following.

**Proposition 10.3** (Ramis [R2]).— *There is an isomorphism*

$$S(G) \rightarrow V(G)/(V(G)^\circ, V(G)^\circ).$$

*Proof.*— (1) We start by proving that a reductive group  $M$  has the property  $L(M) = M^\circ$ . We recall that in characteristic 0, reductive for a linear algebraic group is equivalent to the complete reducibility of any finite dimensional representation. It follows at once that  $N := M/L(M)$  is also reductive. Thus the unipotent radical of  $N$  is trivial. By construction  $N^\circ$  has a trivial maximal torus and is therefore unipotent and equal to the unipotent radical of  $N$ . Thus  $N^\circ = 1$ . Since  $L(M)$  is connected, the statement follows.

(2) Let  $G$  be a connected linear algebraic group and  $R_u$  its unipotent radical. Then  $G$  is the semi-direct  $R_u \cdot M$ , where  $M$  is some reductive subgroup of  $G$ , called a Levi subgroup or Levi-factor (see [H] p. 184). We note that a maximal torus of  $M$  is also a maximal torus of  $G$ . Let  $\tau_1 : G = R_u \cdot M \rightarrow V(G) := G/L(G)$  denote the canonical map. Since  $L(M) = M$ , the kernel of  $\tau_1$  is the smallest normal subgroup of  $G$  containing  $M$ . The kernel of the natural map  $\tau_2 : G = R_u \cdot M \rightarrow V(G)/(V(G), V(G))$  is the smallest normal subgroup containing  $M$  and  $(G, G)$ . This is the same as the smallest normal subgroup containing  $M$  and  $(R_u, G)$ , since  $G = R_u \cdot M$ . Thus the map  $\tau_2$  induces an isomorphism  $R_u/(R_u, G) \rightarrow V(G)/(V(G), V(G))$ .

(3) Let  $G$  be any linear algebraic group. We consider  $V(G) := G/L(G)$  and  $S'(G) := V(G)/(V(G)^\circ, V(G)^\circ)$ . The component of the identity  $S'(G)^\circ$  is easily identified with  $S'(G)^\circ$  and according to (2) isomorphic to the unipotent group  $R_u/(R_u, G^\circ)$ . Further  $S'(G)/S'(G)^\circ$  is canonically isomorphic to  $G/G^\circ$ . Using that  $S'(G)^\circ$  is unipotent, one can construct a left inverse for the surjective homomorphism  $S'(G) \rightarrow G/G^\circ$  (see lemma 10.4). Thus  $S'(G)$  is isomorphic to the semi-direct product of  $R_u/(G^\circ, R_u)$  and  $G/G^\circ$ , given by the action of  $G/G^\circ$  on  $R_u/(R_u, G^\circ)$ , defined by conjugation. Therefore  $S(G)$  and  $S'(G)$  are isomorphic.

Finally we will need a lemma, which is probably well known.

*Lemma 10.4.*— *Let  $H$  be a linear algebraic group such that  $H^\circ$  is unipotent. Then*

- (a)  *$H$  is a semi-direct product of  $H^\circ$  and  $H/H^\circ$ .*
- (b)  *$H$  and  $H/(H^\circ, H^\circ)$  have the same minimal number of topological (for the Zariski topology) generators.*

*Proof.*— (a) Define the closed normal subgroups  $H_k^\circ$  of  $H$  by  $H_0^\circ = H^\circ$  and  $H_{k+1}^\circ = (H^\circ, H_k^\circ)$  for  $k \geq 0$ . Since  $H^\circ$  is a unipotent group  $H_k^\circ = \{1\}$  for large  $k$ . Let  $l(H)$  denote the smallest integer  $m \geq 0$  such that  $H_k^\circ = \{1\}$  for all  $m > k$ . By induction on  $l(H)$  we will prove the existence of a homomorphism  $s : H/H^\circ \rightarrow H$  such that  $H/H^\circ \rightarrow H \rightarrow H/H^\circ$  is the identity. For  $l(H) = 0$  the statement is trivial. Suppose that  $l(H) = n \geq 1$ . There is a map  $s : H/H^\circ \rightarrow H$  such that  $s$  is a left inverse of  $H \rightarrow H/H^\circ$  and  $H/H^\circ \rightarrow H \rightarrow H/H_n^\circ$  is a homomorphism. Consider the map  $f : H/H^\circ \times H/H^\circ \rightarrow H_n^\circ$ , given by  $f(a, b) = s(a)s(b)s(ab)^{-1}$ . The group  $H_n^\circ$  is isomorphic to a finite dimensional vector space over the base field. The group  $H/H^\circ$  acts on this vector space by conjugation and  $f$  is a 2-cocycle for this  $H/H^\circ$ -module. Since the base field is supposed to have characteristic 0, this 2-cocycle is trivial. Thus for a suitable map  $r : H/H^\circ \rightarrow H_n^\circ$ , the map  $a \mapsto s(a)r(a)$  is a homomorphism. This proves the induction step.

(b) It suffices to show the following:

“Let  $a_1, \dots, a_n \in H$  be such that their images in  $H/(H^\circ, H^\circ)$  are topological generators, then  $a_1, \dots, a_n$  are topological generators of  $H$  itself”.

We will prove the above statement by induction on  $l(H)$ . The cases  $l(H) = 0, 1$  are trivial. Suppose  $l(H) = n > 1$  and let  $M$  denote the closed subgroup of  $H$  generated by  $a_1, \dots, a_n$ . The induction hypothesis implies that the natural homomorphism  $M \rightarrow H/H_n^\circ$  is surjective. It suffices to show that  $M \supset H_n^\circ$ . Take  $a \in H^\circ$ ,  $b \in H_{n-1}^\circ$  and consider the element  $aba^{-1}b^{-1} \in H_n^\circ$ . One can write  $a = m_1A$ ,  $b = m_2B$  with  $m_1, m_2 \in M$  and  $A, B \in H_n^\circ$ . Since  $H_n^\circ$  lies in the center of  $H^\circ$ , one has  $aba^{-1}b^{-1} = m_1Am_2BA^{-1}m_1^{-1}B^{-1}m_2^{-1} = m_1m_2m_1^{-1}m_2^{-1} \in M$ .

**Conclusions :** The groups  $S(G)$ ,  $V(G) := G/L(G)$  and  $V(G)/(V(G)^\circ, V(G)^\circ)$  have the same minimal number of topological generators. The theorems 3.1 and 3.4 can therefore also be formulated with  $S(G)$  or  $V(G)/(V(G)^\circ, V(G)^\circ)$  instead of  $G/L(G)$ . Moreover the invariant  $d(G)$  in theorem 3.7 coincides with the minimal number of topological generators of  $G/L(G)$ .

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