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G. J. HECKMAN

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DUNKL OPERATORS

by G.J. HECKMAN

INTRODUCTION

Hypergeometric functions. It is known that almost all the special functions of one variable to be met with in mathematical physics may be obtained from the general hypergeometric function of Gauss by a suitable choice of parameters. These same functions appear as elements of representations of the simplest classical groups, namely the groups of rotations of the sphere and of the Lobacevskii plane. This connection lies in the nature of the matter, since the special functions make their appearance by way of considerations connected with this or that invariance of a problem under transformations of a space. Hence, it is natural to construct the theory of hypergeometric functions of several variables, relying on results and methods of the theory of the representations of compact or locally compact Lie groups. It is thus necessary so to construct the theory of hypergeometric functions that it should contain the theory of general spherical functions, connected with the representations of semisimple groups.

I.M. Gelfand, On some problems of functional analysis, Usp. Mat. Nauk 11 (6) (1956), 3-12.

For a finite reflection group W acting on a Euclidean vector space Dunkl introduced in [Du1] the remarkable operator

$$T(\xi, k) = \partial(\xi) + \sum_{\alpha} k_\alpha \alpha(\xi)\alpha(\cdot)^{-1}(1 - s_\alpha)$$

as perturbation in the parameter k_α (satisfying $k_{w\alpha} = k_\alpha \; \forall w, \forall \alpha$) of the differentiation $\partial(\xi)$ in the direction of a vector ξ. Here α runs over a set of equations for the reflection hyperplanes of W, and $s_\alpha \in W$ is the corresponding reflection. Dunkl operators act on polynomials (and many other function spaces) with the properties

$$\begin{aligned} T(\xi,k)T(\eta,k) &= T(\eta,k)T(\xi,k) \quad && \forall\xi,\eta \\ wT(\xi,k)w^{-1} &= T(w\xi,k) && \forall w,\forall\xi, \end{aligned}$$

and their simultaneous spectral theory has an exact solution [Du2, J]. In this lecture we have left these (rational) Dunkl operators aside, and instead focus on their trigonometric analogues (by lack of time, and because the latter seem to be more interesting). It will become clear that trigonometric Dunkl operators form the basic tool in the hypergeometr(ic theor)y for root systems.

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1. TRIGONOMETRIC DUNKL OPERATORS

Let $\mathfrak{a}$ be a Euclidean vector space of dimension n, and let $R \subset \mathfrak{a}^*$ be a possibly nonreduced root system. Let $R^\vee \subset \mathfrak{a}$ be the dual root system. Let $Q^\vee = \mathbb{Z}R^\vee \subset \mathfrak{a}$ be the coroot lattice. The dual lattice $P = (Q^\vee)^* \subset \mathfrak{a}^*$ is the weight lattice of R. Let us denote by $\mathfrak{h}$ the complexification of $\mathfrak{a} : \mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ with $\mathfrak{t} = i\mathfrak{a}$. Let $H = \mathrm{Hom}(P, \mathbb{C}^\times)$ be the complex torus with rational character lattice P. We have the polar decomposition

$$H = AT, \quad A = \mathrm{Hom}(P, \mathbb{R}_{>0}), \quad T = \mathrm{Hom}(P, S^1) \tag{1.1}$$

and $\mathfrak{h} = \mathrm{Lie}(H)$, $\mathfrak{a} = \mathrm{Lie}(A)$, $\mathfrak{t} = \mathrm{Lie}(T)$. Let $\mathbb{C}[H]$ be the algebra of regular functions (Laurent polynomials) on H. It has a $\mathbb{C}$-basis e^μ indexed by $\mu \in P$, and the multiplication is given by $e^\mu e^\nu = e^{\mu+\nu}$, $e^0 = 1$.

For $\alpha \in R$ let $s_\alpha : \lambda \mapsto \lambda - \lambda(\alpha^\vee)\alpha$ denote the corresponding reflection, and let $W = \langle s_\alpha; \alpha \in R\rangle \subset GL(\mathfrak{h}^*)$ be the Weyl group of R. By duality W also acts on $\mathfrak{h}$ and H. Fix a set of positive roots $R_+ \subset R$. Let $\alpha_1^\vee, \ldots, \alpha_n^\vee \in R_+^\vee$ be the set of simple coroots, and $s_1, \ldots, s_n \in W$ the corresponding simple reflections.

For $\alpha \in R$ let $H_\alpha = \{h \in H; e^\alpha(h) = 1\}$ and put $H_{\mathrm{reg}} = H \backslash \bigcup H_\alpha$. Let $\mathbb{C}[H_{\mathrm{reg}}]$ be the algebra of regular functions on H_{reg}. For $p \in S\mathfrak{h}$ let $\partial(p)$ denote the corresponding translation invariant differential operator on H, so $\partial(p)e^\mu = p(\mu)e^\mu$ for $p \in S\mathfrak{h}$ and $\mu \in P$. Denote by $\mathbb{D}[H_{\mathrm{reg}}]$ the algebra of differential operators on H with coefficients in $\mathbb{C}[H_{\mathrm{reg}}]$. Clearly $\mathbb{C}[H_{\mathrm{reg}}]$ is a natural left module for $\mathbb{D}[H_{\mathrm{reg}}]$. Let $\mathbb{D}[H_{\mathrm{reg}}] \otimes \mathbb{C}[W]$ be the algebra of *differential reflection* operators on H_{reg}. The algebra structure is the natural one making $\mathbb{C}[H_{\mathrm{reg}}]$ into a left module for $\mathbb{D}[H_{\mathrm{reg}}] \otimes \mathbb{C}[W]$.

DEFINITION 1.1.– *Let* $K = \{k \in \mathbb{C}^R; k = (k_\alpha)$ *with* $k_{w\alpha} = k_\alpha\ \forall w \in W, \forall \alpha \in R\}$ *be the linear space of* multiplicity *(or* coupling*)* parameters *for* R. *For* $\xi \in \mathfrak{h}$ *and* $k \in K$ *the (trigonometric)* Dunkl operator $T(\xi, k) \in \mathbb{D}[H_{\mathrm{reg}}] \otimes \mathbb{C}[W]$ *is defined by*

$$T(\xi,k) = \partial(\xi) - \rho(k)(\xi) + \sum_{\alpha>0} k_\alpha \alpha(\xi)(1-e^{-\alpha})^{-1} \otimes (1-s_\alpha) \tag{1.2}$$

with

$$\rho(k) = \tfrac{1}{2}\sum_{\alpha>0} k_\alpha \alpha \in \mathfrak{h}^*. \tag{1.3}$$

Note that the *divided difference* operator $(1-e^{-\alpha})^{-1}(1-s_\alpha)$ preserves the space $\mathbb{C}[H]$, and likewise does the Dunkl operator $T(\xi,k)$. Let R^0 be the set of unmultipliable roots in R, and put $k^0_\alpha = \frac{1}{2}k_{\frac{1}{2}\alpha} + k_\alpha$ for $\alpha \in R^0$ with the convention $k_\beta = 0$ if $\beta \notin R$. So k^0 is a multiplicity parameter for R^0. Taking $R^0_+ = R^0 \cap R_+$ we get $\rho(k^0) = \rho(k)$. It is easy to check (just by a rank one calculation) that

$$s_i T(\xi,k) - T(s_i\xi,k)s_i = -k^0_{\alpha_i}\alpha_i(\xi) \tag{1.4}$$

for each simple root $\alpha_i \in R^0_+$. The next theorem is the basic result of this section.

THEOREM 1.2.– *We have* $[T(\xi,k), T(\eta,k)] = 0\ \ \forall \xi,\eta \in \mathfrak{h},\ \forall k \in K$.

We will give two indications how to prove this result.

First proof: Verification by an (elementary but somewhat cumbersome) calculation along the same lines as Dunkl's original proof of commutativity in the rational case [Du1]. The only illuminating point is that this calculation admits a reduction to rank two. This is a basic feature in the theory of the Yang-Baxter equation. □

Second proof: It is easy to see that an element of $\mathbb{D}[H_{\mathrm{reg}}] \otimes \mathbb{C}[W]$ is zero as soon as it vanishes on $\mathbb{C}[H]$ (see [O5, Lemma 2.8]). Therefore it suffices to prove commutativity of the Dunkl operators as elements of $\mathrm{End}(\mathbb{C}[H])$. It also suffices to check the commutation relation for a Zariski dense set of multiplicity parameters, say $k_\alpha \geq 0$ $\forall \alpha \in R$. In this case define a hermitian inner product $(\cdot,\cdot)_k$ on $\mathbb{C}[H]$ by

$$(f,g)_k = |W|^{-1}\int_T f\,\overline{g}\prod_{\alpha>0}|e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}|^{2k_\alpha}dt \tag{1.5}$$

with dt the normalized Haar measure on T. Now it is easy to check that

$$(T(\xi,k)f,g)_k = (f,T(\overline{\xi},k)g)_k \tag{1.6}$$

for all $f,g \in \mathbb{C}[H]$ and $\xi \in \mathfrak{h}$. The bar denotes complex conjugation on $\mathfrak{h}$ with respect to the real form $\mathfrak{a}$.

One has the usual partial ordering $\leq$ on $\mathfrak{h}^*$ defined by $\mu \leq \nu$ iff $\nu - \mu \in \mathbb{N}R_+$. Let P_+ be the cone of dominant weights. For $\mu \in P$ we denote by μ_+ the unique dominant weight in $W\mu$. Define a new partial ordering $\leq_W$ on P by

$$\mu \leq_W \nu \text{ if either } \mu_+ < \nu_+ \text{ or } \mu_+ = \nu_+ \wedge \nu \leq \mu. \tag{1.7}$$

So μ_+ is the smallest and $w_0\mu_+$ is the largest element in the orbit $W\mu$. Here $w_0 \in W$ is the longest element. Now it is easy to check that the Dunkl operators are upper triangular with respect to the basis e^μ partially ordered by $\leq_W$.

Next define a new basis $E(\mu,k)$, $\mu \in P$, of $\mathbb{C}[H]$ by the conditions

$$E(\mu,k) = e^\mu + \cdots \tag{1.8}$$

$$(E(\mu,k),e^\nu)_k = 0 \quad \forall \nu \in P \ \text{ with } \ \nu <_W \mu. \tag{1.9}$$

Here the dots denote lower order terms with respect to $\leq_W$. Clearly the Dunkl operators are also upper triangular with respect to the basis $E(\mu,k)$ partially ordered by $\leq_W$. Since $(E(\mu,k),E(\nu,k))_k = 0$, $\forall \nu <_W \mu$, it follows from (1.6) that the Dunkl operators are diagonalized by the basis $E(\nu,k)$. Hence Dunkl operators commute on $\mathbb{C}[H]$. □

REMARK 1.3.– Let $\varepsilon : \mathbb{R} \to \{\pm 1\}$ be defined by $\varepsilon(x) = +1$ if $x > 0$ and $\varepsilon(x) = -1$ if $x \leq 0$. For $\mu \in P$ let $\widetilde{\mu} \in \mathfrak{h}^*$ be given by

$$\widetilde{\mu} = \mu + \tfrac{1}{2}\sum_{\alpha>0} k_\alpha \varepsilon(\mu(\alpha^\vee))\alpha. \tag{1.10}$$

Then a direct calculation yields

$$T(\xi,k)E(\mu,k) = \widetilde{\mu}(\xi)E(\mu,k). \tag{1.11}$$

Since $k_\alpha \geq 0$, $\forall \alpha \in R$, we have $\widetilde{\mu} \neq \widetilde{\nu}$ if $\mu,\nu \in P$ are distinct. Hence $E(\mu,k)$, $\mu \in P$, is in fact an orthogonal basis of $\mathbb{C}[H]$ with respect to $(\cdot,\cdot)_k$.

Due to the commutativity of the Dunkl operators we can extend the map $\mathfrak{h} \to \mathbb{D}[H_{\text{reg}}] \otimes \mathbb{C}[W]$, $\xi \mapsto T(\xi,k)$, in a unique way to an algebra homomorphism $S\mathfrak{h} \to$

$\mathbb{D}[H_{\text{reg}}] \otimes \mathbb{C}[W]$. The image of $p \in S\mathfrak{h}$ will be denoted by $T(p,k)$. The next definition is due independently to Drinfeld [Dr] and Lusztig [Lu].

DEFINITION 1.4.– *The* degenerate *(or* graded*)* Hecke algebra $\mathbb{H} = \mathbb{H}(R_+, k)$ *is the unique associative algebra over* $\mathbb{C}$ *satisfying*

(1) $\mathbb{H} = S\mathfrak{h} \otimes \mathbb{C}[W]$ *as a vector space over* $\mathbb{C}$,

(2) $S\mathfrak{h} \to \mathbb{H}$, $p \mapsto p \otimes 1$ *and* $\mathbb{C}[W] \to \mathbb{H}$, $w \mapsto 1 \otimes w$ *are algebra homomorphisms (often we will identify* $S\mathfrak{h}$ *and* $\mathbb{C}[W]$ *with their images in* $\mathbb{H}$ *via these maps),*

(3) $p \cdot w = p \otimes w \ \forall p \in S\mathfrak{h}, \forall w \in W$,

(4) $s_i \cdot \xi - s_i\xi \cdot s_i = -k^0_{\alpha_i}\alpha_i(\xi) \ \forall \xi \in \mathfrak{h}$, *and for* α_i *a simple root of* R^0_+.

PROPOSITION 1.5.– *In the degenerate Hecke algebra* $\mathbb{H}$ *we have*

(1) $w \cdot \xi \cdot w^{-1} = w\xi + \sum_{\alpha \in R^0_+ \cap wR^0_-} k^0_\alpha \alpha(w\xi) s_\alpha \ \forall \xi \in \mathfrak{h}, \forall w \in W$,

(2) $s_i \cdot p - s_i p \cdot s_i = -k^0_{\alpha_i}(p - s_i p)/\alpha_i^\vee \ \forall p \in S\mathfrak{h}$,

(3) *the center* $Z(\mathbb{H})$ *of* $\mathbb{H}$ *is equal to* $S\mathfrak{h}^W$.

Proof: (1) Use induction on the length $l(w)$ of $w \in W$. If $w = s_i v$ with $l(v) < l(w)$ then one has $R^0_+ \cap wR^0_- = s_i(R^0_+ \cap vR^0_-) \cup \{\alpha_i\}$. Using the induction hypothesis and relation (4) of Definition 1.4 one obtains the desired formula.

(2) By induction on the degree of p.

(3) From (1) it follows that $Z(\mathbb{H}) \subset S\mathfrak{h}$, and then (2) gives $Z(\mathbb{H}) = S\mathfrak{h}^W$. □

Combination of relation (1.4) and Theorem 1.2 with Definition 1.4 gives the following conclusion.

CONCLUSION 1.6.– *The maps* $p \mapsto T(p,k)$, $w \mapsto w$ *define a homomorphism of the degenerate Hecke algebra* $\mathbb{H}$ *into the algebra* $\mathbb{D}[H_{\text{reg}}] \otimes \mathbb{C}[W]$ *of differential reflection operators on* H_{reg}. *In turn this defines a representation of* $\mathbb{H}$ *on* $\mathbb{C}[H_{\text{reg}}]$ *leaving the subspace* $\mathbb{C}[H]$ *invariant.*

REMARK 1.7.– Trigonometric Dunkl operators were originally introduced in [He2] in the different form

$$S(\xi,k) = \partial(\xi) + \tfrac{1}{2}\sum_{\alpha>0} k_\alpha \alpha(\xi)\frac{1+e^{-\alpha}}{1-e^{-\alpha}} \otimes (1 - s_\alpha) \tag{1.12}$$

satisfying the properties

$$wS(\xi,k)w^{-1} = S(w\xi,k) \quad \forall w \in W, \forall \xi \in \mathfrak{h} \tag{1.13}$$

and

$$[S(\xi,k),S(\eta,k)] = -\tfrac{1}{4}\sum_{\alpha,\beta>0} k_\alpha k_\beta\{\alpha(\xi)\beta(\eta)-\alpha(\eta)\beta(\xi)\}s_\alpha s_\beta \tag{1.14}$$

for all $\xi,\eta \in \mathfrak{h}$. Subsequently Cherednik [Ch1] suggested the formula for $T(\xi,k)$ and found Theorem 1.2. Comparison of (1.2) and (1.12) yields the relation

$$T(\xi,k) = S(\xi,k) - \tfrac{1}{2}\sum_{\alpha>0} k_\alpha \alpha(\xi)s_\alpha. \tag{1.15}$$

With this in mind it is easy to see that (1.14) is just an equivalent form of Theorem 1.2. The second proof of Theorem 1.2 given here is due to the author (unpublished and reproduced in [O5]). The connection between Dunkl operators and the degenerate Hecke algebra is due to Cherednik [Ch2, O5].

2. THE HYPERGEOMETRIC SYSTEM

In this section we explain the intimate connection between Dunkl operators and the hypergeometric theory for root systems as introduced in [HO1, He1, O1, O2]. By Proposition 1.5 the center of the degenerate Hecke algebra equals $S\mathfrak{h}^W$. Hence for $p \in S\mathfrak{h}^W$ the Dunkl operator

$$T(p,k) = \sum_w D(w,p,k)\otimes w \in \mathbb{D}[H_{\mathrm{reg}}]\otimes \mathbb{C}[W] \tag{2.1}$$

commutes with all elements from W, and therefore

$$D(p,k) := \sum_w D(w,p,k) \in \mathbb{D}[H_{\mathrm{reg}}]^W. \tag{2.2}$$

Clearly $D(p,k)$ is the unique element in $\mathbb{D}[H_{\mathrm{reg}}]^W$ which has the same restriction to $\mathbb{C}[H]^W$ as the Dunkl operator $T(p,k)$. In particular $D(p,k)$ preserves the space $\mathbb{C}[H]^W$. It is also clear that

$$D(p,k)D(q,k) = D(pq,k) \quad \forall p,q \in S\mathfrak{h}^W \tag{2.3}$$

and so $\{D(p,k); p \in S\mathfrak{h}^W\}$ is a commutative algebra of differential operators.

DEFINITION 2.1.– *Fix $\lambda \in \mathfrak{h}^*$. The system of differential equations*

$$D(p,k)f = p(\lambda)f, \quad p \in S\mathfrak{h}^W \tag{2.4}$$

is called the hypergeometric system *associated with the root system R, and with spectral parameter λ.*

THEOREM 2.2.– *If $\xi_1, \dots, \xi_n$ is an orthonormal basis of $\mathfrak{a}$ then*

$$D(\sum_i \xi_i^2, k) = L(k) + \big(\rho(k), \rho(k)\big) \tag{2.5}$$

with

$$L(k) = \sum_i \partial(\xi_i)^2 + \sum_{\alpha>0} k_\alpha \frac{1+e^{-\alpha}}{1-e^{-\alpha}} \partial(\alpha) \tag{2.6}$$

and $\partial(\alpha) = \frac{1}{2}(\alpha, \alpha)\partial(\alpha^\vee)$.

Proof: For homogeneous $p \in S\mathfrak{h}^W$ the leading symbol of $D(p,k)$ is equal to $\partial(p)$. Moreover by (1.6) the adjoint of $D(p,k)$ with respect to $(\cdot,\cdot)_k$ on $\mathbb{C}[H]^W$ is equal to $D(\overline{p}, k)$. Finally, by (1.11) the constant term $D(p,k)1$ is equal to $p\big(\rho(k)\big)$. Therefore, the proposition follows, since $L(k) \in \mathbb{D}[H_{\text{reg}}]^W$ is the unique second order differential operator with leading symbol $\sum_i \partial(\xi_i)^2$ which is symmetric with respect to $(\cdot,\cdot)_k$ and with constant term $L(k)1 = 0$ (as follows from the next theorem). □

REMARK 2.3.– Suppose $\mathfrak{g}$ is a real semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, $\mathfrak{a} \subset \mathfrak{s}$ a maximal abelian subspace, $\sum = \sum(\mathfrak{g}, \mathfrak{a})$ the restricted root system, and m_α the corresponding root multiplicities. If we put

$$R = 2\sum, \; k_{2\alpha} = \tfrac{1}{2} m_\alpha \tag{2.7}$$

then the radial part of the Laplace operator on the symmetric space G/K with respect to the left action by K has the form (2.6). For these particular multiplicity parameters the commuting algebra $\{D(p,k); p \in S\mathfrak{h}^W\}$ therefore represents the radial parts of the algebra $\mathbb{D}[G/K]$ of all invariant differential operators on G/K. For more details and variations see [HS].

THEOREM 2.4.– *If we put $\delta(k)^{\frac{1}{2}} = \prod_{\alpha>0} |e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}|^{k_\alpha}$, then*

$$\begin{aligned} &\delta(k)^{\frac{1}{2}} \circ \{L(k) + \big(\rho(k), \rho(k)\big)\} \circ \delta(k)^{-\frac{1}{2}} \\ &= \sum_i \partial(\xi_i)^2 + \sum_{\alpha>0} \frac{k_\alpha(1-k_\alpha-2k_{2\alpha})(\alpha,\alpha)}{(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha})^2} \end{aligned} \tag{2.8}$$

Proof: This is a rather straightforward calculation. See [HS, Part I, Ch 2] where the proof is spelled out. $\square$

REMARK 2.5.– The differential operator in the right hand side of (2.8) is the Schrödinger operator of the periodic Calogero-Moser system. For the root system R of type A_n this system describes the motion of $n+1$ points on the circle $\mathbb{R}/2\pi\mathbb{Z}$ with a potential proportional to the sum of the inverse squares of the pairwise distances. Conjugation of the commuting family $\{D(p,k); p \in S\mathfrak{h}^W\}$ with the function $\delta(k)^{\frac{1}{2}}$ yields the quantum complete integrability of this system, and via a classical limit also the classical complete integrability. For R a classical root system the classical integrability was obtained by Moser [Mo] for type A_n and by Olshanetsky and Perelomov [OP] for the other classical types by realizing the system as a Lax pair. However, the only known proof of classical integrability valid also for exceptional root systems is the one sketched above through quantum integrability and a classical limit. For more details see [HS, Part I, Ch 2].

EXAMPLE 2.6.– In case R has rank one and x is a coordinate on $H = \mathbb{C}^\times$ (so $\mathbb{C}[H] = \mathbb{C}[x, x^{-1}]$ and $\theta = x\frac{d}{dx}$ is a basis for $\mathfrak{a} = \mathbb{R}$) the hypergeometric equation (2.4) takes the form

$$\{\theta^2 + \left(k_1\frac{1+x^{-1}}{1-x^{-1}} + 2k_2\frac{1+x^{-2}}{1-x^{-2}}\right)\theta + (\tfrac{1}{2}k_1 + k_2)^2 - \lambda^2\}f = 0. \tag{2.9}$$

Note that the equation has Weyl group symmetry $x \mapsto x^{-1}$, and in the new coordinate $z = \frac{1}{2} - \frac{1}{4}(x + x^{-1})$ this becomes the Gauss hypergeometric equation

$$\{z(1-z)\frac{d^2}{dz^2} + \left(c - (1+a+b)z\right)\frac{d}{dz} - ab\}f = 0 \tag{2.10}$$

with parameters

$$a = \lambda + \tfrac{1}{2}k_1 + k_2,\ b = -\lambda + \tfrac{1}{2}k_1 + k_2,\ c = \tfrac{1}{2} + k_1 + k_2. \tag{2.11}$$

Following Harish-Chandra we substitute a formal series of the form

$$\sum_{\nu \le \mu} c_\nu e^\nu, \quad c_\mu = 1 \tag{2.12}$$

into the hypergeometric system (2.4). The leading exponents $\mu \in \mathfrak{h}^*$ for which such solutions exist satisfy (using Remark 1.3) the indicial equation

$$p(\mu + \rho(k)) = p(\lambda)\ \forall p \in S\mathfrak{h}^W \iff \mu \in W\lambda - \rho(k). \tag{2.13}$$

Therefore, the hypergeometric system (2.4) has the asymptotically free solution

$$\Phi(\lambda,k;\cdot) = \sum_{\kappa\leq 0} \Gamma_\kappa(\lambda,k)e^{\lambda-\rho(k)+\kappa}, \quad \Gamma_0(\lambda,k)=1 \tag{2.14}$$

with $\Gamma_\kappa(\lambda,k)$ satisfying the recurrence relations (using only the eigenvalue equation for the second order operator $L(k)$)

$$-(2\lambda+\kappa,\kappa)\Gamma_\kappa(\lambda,k) = 2\sum_{\alpha>0} k_\alpha \sum_{j\geq 1}(\lambda-\rho(k)+\kappa+j\alpha,\alpha)\Gamma_{\kappa+j\alpha}(\lambda,k). \tag{2.15}$$

These recurrence relations can be uniquely solved unless

$$(2\lambda+\kappa,\kappa)=0 \quad \text{for some } \kappa<0. \tag{2.16}$$

The series (2.14) converges absolutely and uniformly on compact sets in $\mathfrak{h}^*\times K\times A_+$ which avoid these hyperplanes. Here $A_+ = \{a\in A; e^\alpha(a)>1\ \forall\alpha>0\}$. It can be shown [HS, Part I, Ch 4] that

$$e^{-\lambda+\rho(k)}\Phi(\lambda,k;\cdot) \tag{2.17}$$

defines a meromorphic function on $\mathfrak{h}^*\times K\times A_+T$ with simple poles along the hyperplanes

$$\lambda(\alpha^\vee)\in\mathbb{N}+1 \quad \text{for some } \alpha\in R_+. \tag{2.18}$$

In other words, the hyperplanes (2.16) with κ not a multiple of a root give only apparent singularities.

DEFINITION 2.7.– *The meromorphic function $\widetilde{c}$ on $\mathfrak{h}^*\times K$ is defined by*

$$\widetilde{c}(\lambda,k) = \prod_{\alpha>0}\frac{\Gamma\big(\lambda(\alpha^\vee)+\frac{1}{2}k_{\frac{1}{2}\alpha}\big)}{\Gamma\big(\lambda(\alpha^\vee)+\frac{1}{2}k_{\frac{1}{2}\alpha}+k_\alpha\big)} \tag{2.19}$$

with the convention $k_\beta=0$ if $\beta\notin R$.

THEOREM 2.8.– *The function $\widetilde{F}(\lambda,k;h)$ given by*

$$\widetilde{F}(\lambda,k;h) = \sum_w \widetilde{c}(w\lambda,k)\Phi(w\lambda,k;h) \tag{2.20}$$

extends to a holomorphic function on

$$\mathfrak{h}^* \times K \times U \tag{2.21}$$

with U a small W-invariant tubular neighborhood of A in H. Moreover it satisfies

$$\widetilde{F}(w\lambda, k; h) = \widetilde{F}(\lambda, k; wh) = \widetilde{F}(\lambda, k; h) \tag{2.22}$$

for all $w \in W$ and $(\lambda, k, h) \in \mathfrak{h}^* \times K \times U$.

This result is due to Opdam [O2]. For a proof see also [HS, Part I, Ch 4].

REMARK 2.9.– Under the assumption

$$(\lambda, \alpha^\vee) \notin \mathbb{Z} \quad \forall \alpha \in R \tag{2.23}$$

the asymptotically free solutions $\Phi(w\lambda, k; \cdot)$ with $w \in W$ are a basis for the solution space of the hypergeometric system (2.4) on A_+. Being invariant under W the system (2.4) can be considered as a system of differential equations on the quotient $W\backslash H \cong \mathbb{C}^n$. The fundamental group $\Pi_1(W\backslash H_{\mathrm{reg}})$ of the regular orbit space is the affine braid group associated with R. In the above basis it can be checked that the monodromy of the hypergeometric system yields a representation of the affine Hecke algebra with quadratic relations

$$(T_j + 1)(T_j - e^{2\pi i(k_{\frac{1}{2}\alpha_j} + k_{\alpha_j})}) = 0 \tag{2.24}$$

and with central character $s = e^{2\pi i\lambda}$. See [HS, Part I, Ch 4] for more details.

The next result is also due to Opdam [O4].

THEOREM 2.10.– *For all $(\lambda, k) \in \mathfrak{h}^* \times K$ we have*

$$\widetilde{F}(\lambda, k; 1) = \widetilde{c}(\rho(k), k). \tag{2.25}$$

Outside the zeros of the entire function $\widetilde{c}(\rho(k), k)$ on K we put

$$F(\lambda, k; \cdot) = \widetilde{c}(\rho(k), k)^{-1} \widetilde{F}(\lambda, k; \cdot) \tag{2.26}$$

This solution of (2.4) is called the *hypergeometric function* associated with R. In the rank one case of Example 2.6 it is just the Gauss hypergeometric function.

3. THE KNIZHNIK-ZAMOLODCHIKOV CONNECTION

In the previous section we have seen how Dunkl operators commuting with the action of W give rise to the system of hypergeometric differential equations. There is a more direct way to rewrite the eigenvalue problem for the Dunkl operators as an integrable connection of Knizhnik-Zamolodchikov (or short KZ) type. This is due to Matsuo [Mat] and Cherednik [Ch1, Ch2]. Our exposition is inspired by [O5, Section 3] and [Lo].

Let Ω^p (and $\mathcal{O} = \Omega^0$) denote the sheaf of holomorphic p-forms on H_{reg}, and likewise $\Omega^p \otimes \mathbb{C}[W]$ the sheaf of holomorphic p-forms with values in the trivial bundle on H_{reg} with fiber $\mathbb{C}[W]$. For $\lambda \in \mathfrak{h}^*$ write $d\lambda$ for the translation invariant 1-form on H corresponding to λ. For $\alpha \in R$ let $\varepsilon_\alpha \in \text{End}(\mathbb{C}[W])$ be defined by

$$\varepsilon_\alpha(w) = -\text{sign}(w^{-1}\alpha)w \quad \text{for } w \in W. \tag{3.1}$$

It is clear that these endomorphisms satisfy

$$\varepsilon_\alpha + \varepsilon_{-\alpha} = 0 \qquad \forall \alpha \in R \tag{3.2}$$

$$w\varepsilon_\alpha w^{-1} = \varepsilon_{w\alpha} \quad \forall w \in W,\ \forall \alpha \in R \tag{3.3}$$

with $w \in W$ acting on $\mathbb{C}[W]$ by left multiplication.

DEFINITION 3.1.– *Fix $\lambda \in \mathfrak{h}^*$ and $k \in K$. The (trigonometric)* KZ-connection *associated with R is the connection $\nabla(\lambda, k) : \mathcal{O} \otimes \mathbb{C}[W] \to \Omega^1 \otimes \mathbb{C}[W]$ on the trivial bundle over H_{reg} with fiber $\mathbb{C}[W]$ given by*

$$\nabla(\lambda, k) = d \otimes 1 - e(\lambda) + \tfrac{1}{2} \sum_{\alpha>0} k_\alpha \left(\frac{1+e^{-\alpha}}{1-e^{-\alpha}} d\alpha \otimes (1-s_\alpha) + d\alpha \otimes s_\alpha \varepsilon_\alpha \right)$$

with $e(\lambda)$ the map sending $f \otimes w$ to $f dw\lambda \otimes w$.

PROPOSITION 3.2.– *The KZ-connection commutes with the diagonal action of W on $\Omega^p \otimes \mathbb{C}[W]$, with the action on the first factor being the natural one, and on the second factor given by left multiplication.*

Proof: Using (3.2) one can rewrite the formula for $\nabla(\lambda, k)$ as

$$\nabla(\lambda, k) = d \otimes 1 - e(\lambda) + \tfrac{1}{4} \sum_{\alpha\in R} k_\alpha \left(\frac{1+e^{-\alpha}}{1-e^{-\alpha}} d\alpha \otimes (1-s_\alpha) + d\alpha \otimes s_\alpha \varepsilon_\alpha \right)$$

and the W-equivariance follows by direct verification using (3.3). □

For $h \in H_{\text{reg}}$ let us denote $\Omega^p_{Wh} = \bigoplus_w \Omega^p_{wh}$ for the multigerms of p-forms at the orbit Wh. We have a natural isomorphism of vector bundles over $W\backslash H_{\text{reg}}$

$$\Omega^p_{Wh} \xrightarrow{\cong} (\Omega^p_{Wh} \otimes \mathbb{C}[W])^W \tag{3.4}$$

given by $\omega \mapsto \sum w\omega \otimes w$. By the previous proposition the KZ-connection descends to a connection on the vector bundle $(\Omega^p_{Wh} \otimes \mathbb{C}[W])^W$, and via the isomorphism (3.4) transfers into a connection

$$d(\lambda, k) : \Omega^p_{Wh} \to \Omega^{p+1}_{Wh}. \tag{3.5}$$

Using the explicit formula for $\nabla(\lambda, k)$ and the isomorphism (3.4) one has

$$d(\lambda, k) = d - d\lambda + \tfrac{1}{2} \sum_{\alpha > 0} k_\alpha \left(\frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} d\alpha \otimes (1 - s_\alpha) - d\alpha \otimes s_\alpha \right)$$

or equivalently

$$d(\lambda, k) = d - d\big(\lambda + \rho(k)\big) + \sum_{\alpha > 0} k_\alpha (1 - e^{-\alpha})^{-1} d\alpha \otimes (1 - s_\alpha). \tag{3.6}$$

CONCLUSION 3.3.– *Fix $h \in H_{\text{reg}}$ and let $v(\xi)$ for $\xi \in \mathfrak{h}$ be the unique W-invariant vector field around Wh which is equal to $\partial(\xi)$ around h. Then the covariant derivative $d_{v(\xi)}(\lambda, k) \in \operatorname{End}(\mathcal{O}_{Wh})$ of the connection (3.5) along $v(\xi)$ (in a local coordinate around $Wh \in W\backslash H_{\text{reg}}$ coming from a local coordinate around h) is given by*

$$d_{v(\xi)}(\lambda, k) = T(\xi, k) - \lambda(\xi) \tag{3.7}$$

with $T(\xi, k)$ the Dunkl operator of Definition 1.1.

COROLLARY 3.4.– *The KZ-connection is integrable.*

Proof: The formula for the curvature $R(\lambda, k)$ of $d(\lambda, k)$ is given by (cf [De, Chapter I, §2])

$$R(\lambda, k)\big(v(\xi), v(\eta)\big) = [d_{v(\xi)}(\lambda, k), d_{v(\eta)}(\lambda, k)] - d_{[v(\xi), v(\eta)]}(\lambda, k)$$

and therefore $R(\lambda, k) = 0$ by Theorem 1.2. □

REMARK 3.5.– The proof of Matsuo [Mat] of the integrability of the KZ-connection proceeds in a similar way as the "first proof" of Theorem 1.2. Another proof of the integrability (by a more elegant calculation) has been given by Looijenga [Lo], whereas

Opdam [O5] derives the integrability from the integrability of the hypergeometric system (2.4).

4. HYPERGEOMETRIC SHIFT OPERATORS

In the degenerate Hecke algebra $\mathbb{H}(R_+, k)$ of Definition 1.4 we denote by

$$J = \sum_w S\mathfrak{h} \otimes (1 - w) \tag{4.1}$$

the left ideal in $\mathbb{H}(R_+, k)$ generated by the elements $1 \otimes (1 - w) \in 1 \otimes \mathbb{C}[W]$ for all $w \in W$. For a left module V for $\mathbb{H}(R_+, k)$ the subspace $V^W = \{v \in V; w(v) = v\ \forall w\}$ is called the space of spherical vectors in V. Clearly $h_1 \equiv h_2 \bmod J$ for $h_1, h_2 \in \mathbb{H}(R_+, k)$ implies $h_1(v) = h_2(v)\ \forall v \in V^W$, $\forall V$.

PROPOSITION 4.1.– *If we write*

$$\pi(k) = \prod_{\alpha \in R_+^0} (\alpha^\vee + \tfrac{1}{2}k_{\frac{1}{2}\alpha} + k_\alpha) \in S\mathfrak{h} \subset \mathbb{H}(R_+, k) \tag{4.2}$$

then we have in $\mathbb{H}(R_+, k)$ the relation

$$w \cdot \pi(k) \equiv \varepsilon(w)\pi(k) \bmod J \quad \forall w \in W. \tag{4.3}$$

Here $\varepsilon : W \to \{\pm 1\}$ is the sign character.

Proof: Writing $\pi(k) = (\alpha_i^\vee + \frac{1}{2}k_{\frac{1}{2}\alpha_i} + k_{\alpha_i})\pi_i(k)$ with $\pi_i(k) \in S\mathfrak{h}$ invariant under s_i, relation (4.3) follows from Proposition 1.5(2). □

Let $\Delta \in \mathbb{C}[H]$ denote the Weyl denominator

$$\Delta = \prod_{\alpha \in R_+^0} (e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}) \tag{4.4}$$

which transforms under W according to the sign character ε. It is well known that every element of $\mathbb{C}[H]$ transforming under W by ε is of the form Δf with $f \in \mathbb{C}[H]^W$. Now we write as in (2.1)

$$T(\pi(k), k) = \sum_w D(w, \pi(k), k) \otimes w \tag{4.5}$$

with $D(w, \pi(k), k) \in \mathbb{D}[H_{\text{reg}}]$.

DEFINITION 4.2.– *The fundamental* shift operators $G_+(k)$, $G_-(k+1) \in \mathbb{D}[H_{\text{reg}}]$ *are defined by*

$$G_+(k) = \sum_w \Delta^{-1} D\big(w, \pi(k), k\big) \tag{4.6}$$

$$G_-(k+1) = |W|^{-1} \sum_{w,v} \varepsilon(w) v\big(D(w, \pi(k), k)\Delta\big). \tag{4.7}$$

Here $1 \in K$ *is the multiplicity parameter defined by* $1_\alpha = 1$ *if* $\alpha \in R^0$ *and* $1_\alpha = 0$ *if* $\alpha \in R \backslash R^0$.

It is clear that for $f \in \mathbb{C}[H]^W$ we have

$$G_+(k)f = \Delta^{-1} T\big(\pi(k), k\big) f \tag{4.8}$$

$$G_-(k+1)f = |W|^{-1} \sum_v vT\big(\pi(k), k\big)\Delta f. \tag{4.9}$$

From the results of Section 1 and Proposition 4.1 it follows that both $G_+(k)$ and $G_-(k+1)$ preserve the space $\mathbb{C}[H]^W$. In particular it follows that $G_+(k), G_-(k+1) \in \mathbb{D}[H_{\text{reg}}]^W$. Finally from (1.6) it is easy to see that (for $k_\alpha \geq 0\ \forall \alpha$)

$$\big(G_+(k)f, g\big)_{k+1} = \big(f, G_-(k+1)g\big)_k \tag{4.10}$$

for all $f, g \in \mathbb{C}[H]^W$.

For $\mu \in P_+$ let $m_\mu \in \mathbb{C}[H]^W$ be the orbit sum defined by

$$m_\mu = \sum_{\nu \in W\mu} e^\nu. \tag{4.11}$$

Since P_+ is a strict fundamental domain for the action of W on P the m_μ for $\mu \in P_+$ are a basis of $\mathbb{C}[H]^W$.

DEFINITION 4.3.– *For* $k_\alpha \geq 0\ \forall \alpha$ *let* $P(\mu, k)$ *for* $\mu \in P_+$ *be the basis of* $\mathbb{C}[H]^W$ *characterized by*

(1) $P(\mu, k) = m_\mu + \cdots$

(2) $\big(P(\mu, k), m_\nu\big)_k = 0\ \forall \nu \in P_+$ *with* $\nu < \mu$.

Here the dots represent lower order terms $\sum a_{\mu\nu} m_\nu$ *(sum over* $\nu \in P_+$ *with* $\nu < \mu$*). The* $P(\mu, k)$ *are called the* Jacobi polynomials *associated with* R.

Now consider the action of the degenerate Hecke algebra $\mathbb{H}(R_+, k)$ on $\mathbb{C}[H]$ via Dunkl operators and the usual action of W. It is easy to see that for $\mu \in P_+$ (in the notation of the proof of Theorem 1.2)

$$\text{span}\{E(\nu, k); \nu \in W\mu\} \tag{4.12}$$

is an (in fact irreducible) module for $\mathbb{H}(R_+, k)$ with central character equal to $\mu+\rho(k)$. The Jacobi polynomial $P(\mu, k)$ is the suitably normalized (namely by asymptotics) spherical vector in this module. Therefore it satisfies the hypergeometric system

$$D(p,k)P(\mu,k) = p\big(\mu+\rho(k)\big)P(\mu,k) \quad \forall p \in S\mathfrak{h}^W. \tag{4.13}$$

Using (4.10) and arguing as in the proof of Theorem 1.2 one easily obtains the following result.

PROPOSITION 4.4.– *There exist polynomials η_+ and η_- on $\mathfrak{h}^* \times K$ such that*

$$G_+(k)P(\mu,k) = \eta_+\big(\mu+\rho(k),k\big)P\big(\mu-\rho(1),k+1\big) \tag{4.14}$$

$$G_-(k)P(\mu,k) = \eta_-\big(\mu+\rho(k),k\big)P\big(\mu+\rho(1),k-1\big) \tag{4.15}$$

for all $\mu \in P_+$.

PROPOSITION 4.5.– *We have (with $N = \#R^0_+$)*

$$\eta_+(\lambda,k) = \pi(\lambda,-k) = (-1)^N\widetilde{c}(-\lambda,k)/\widetilde{c}(-\lambda,k+1) \tag{4.16}$$

$$\eta_-(\lambda,k) = \pi(\lambda,k-1) = \widetilde{c}(\lambda,k-1)/\widetilde{c}(\lambda,k). \tag{4.17}$$

Proof: Indeed we have for $\mu \in P_+$

$$\begin{aligned} G_+(k)P(\mu,k) &= \Delta^{-1}T\big(\pi(k),k\big)\big)m_\mu + \cdots \\ &= \Delta^{-1}T\big(\pi(k),k\big)\big)e^{w_0\mu} + \cdots \\ &= (-1)^N\pi(\widetilde{w_0\mu},k)e^{w_0\mu+\rho(1)} + \cdots \\ &= \pi(-\widetilde{w_0\mu},-k)P\big(\mu-\rho(1),k+1\big) + \cdots \end{aligned}$$

using Remark 1.3 and the equality $\Delta = (-1)^N e^{-\rho(1)}\prod_{\alpha\in R^0_+}(1-e^\alpha)$. Since $\widetilde{w_0\mu} = w_0\big(\mu+\rho(k)\big)$ and $\pi(-w_0\lambda,k) = \pi(\lambda,k)$ relation (4.16) follows. Indeed $\pi(\lambda,k) = \widetilde{c}(\lambda,k)/\widetilde{c}(\lambda,k+1)$ is just a trivial identity by the functional equation $\Gamma(z+1) = z\Gamma(z)$ and the duplication formula $\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2})$.

For computing $\eta_-(\lambda,k)$ one first observes (say for μ regular)

$$G_-(k+1)P(\mu,k+1) = \sum_v vT\big(\pi(k),k\big)\Delta E(\mu,k+1)$$

using (4.8), (4.10) and $P(\mu,k+1) = \sum_v vE(\mu,k+1)$. Then a similar computation $\big(\Delta E(\mu,k+1) = E\big(\mu+\rho(1),k\big) + \cdots\big)$ yields (4.17). □

COROLLARY 4.6.– *For $\mu \in P_+$ and $k \in K$ with $(k-1)_\alpha \geq 0$ $\forall \alpha$ we have (with $\lambda = \mu + \rho(k)$)*

$$(4.18) \qquad \frac{\big(P(\mu,k),P(\mu,k)\big)_k}{\big(P(\mu+\rho(1),k-1),P(\mu+\rho(1),k-1)\big)_{k-1}} = (-1)^N \frac{\widetilde{c}(\lambda,k-1)\widetilde{c}(-\lambda,k)}{\widetilde{c}(\lambda,k)\widetilde{c}(-\lambda,k-1)}.$$

Proof: Just use (4.10) with k replaced by $k-1$, and substitute $f = P\big(\mu+\rho(1),k-1\big)$ and $g = P(\mu,k)$. □

Let $L \subset K$ be the lattice

$$(4.19) \qquad L = \{k \in K; k_\alpha \in \mathbb{Z} \wedge k_{\frac{1}{2}\alpha} \in 2\mathbb{Z}\ \forall \alpha \in R^0\}.$$

Now the theory of shift operators can be generalized, and shifts over arbitrary $l \in L$ can be established. The outcome is that formula (4.18) holds equally well with $k-1$ replaced by $k-l$ and N by $\sum\limits_{\alpha>0} l_\alpha$. This enables one to compute the norm $\big(P(\mu,k),P(\mu,k)\big)_k$ inductively, and the final result takes the following form.

THEOREM 4.7.– *For $\mu \in P_+$ and $k \in K$ with $k_\alpha \geq 0$ $\forall \alpha$ we have (with $\lambda = \mu+\rho(k)$ and $(\cdot,\cdot)_k$ given by (1.5))*

$$(4.20) \qquad \big(P(\mu,k),P(\mu,k)\big)_k = \frac{c^*(-\lambda,k)}{\widetilde{c}(\lambda,k)},$$

with $\widetilde{c}(\lambda,k)$ given by (2.19) and $c^*(\lambda,k)$ by

$$(4.21) \qquad c^*(\lambda,k) = \prod_{\alpha>0} \frac{\Gamma\big(-\lambda(\alpha^\vee) - \frac{1}{2}k_{\frac{1}{2}\alpha} - k_\alpha + 1\big)}{\Gamma\big(-\lambda(\alpha^\vee) - \frac{1}{2}k_{\frac{1}{2}\alpha} + 1\big)}.$$

REMARK 4.8.– Shift operators were introduced by Opdam [O1, O2], and their application to the norm computation of the Jacobi polynomials is also due to him [O3]. The particular case of (4.20) with $\mu = 0$ is the constant term conjecture of Macdonald [Ma], but the only known proof of this conjecture (which works in a uniform way for all root systems) is the one that proves (4.20) at the same time. A complete proof of Theorem 4.7 along the above lines is given in [HS, Part I, Section 3]. A somewhat different proof can be found in [O5].

5. HARMONIC ANALYSIS ON A

We shall think of A as a linear space via the isomorphism $\exp : \mathfrak{a} \to A$ (with log as its inverse). Let da denote the Haar measure on A normalized by requiring $A/\exp(Q^\vee)$ to have volume 1. In this section we want to study the *hypergeometric Fourier transform* $\mathfrak{F}$ defined by

$$\mathfrak{F}f(\lambda) = |W|^{-1} \int_A f(a)\widetilde{F}(-\lambda, k; a)\delta(k; a)da \tag{5.1}$$

for f a suitable function on A invariant under W, and the weight function $\delta(k; \cdot)$ on A given by

$$\delta(k; \cdot) = \prod_{\alpha>0} \left| e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha} \right|^{2k_\alpha} . \tag{5.2}$$

The hypergeometric Fourier transform reduces for $k = 0$ to the Euclidean Fourier transform $f \mapsto \widehat{f}(\lambda) = \int f(a)a^{-\lambda}da$, and for particular values of k (as in Remark 2.3) to the spherical Fourier transform of Harish-Chandra (up to a factor $\widetilde{c}(\rho(k), k)$). The line of arguments will be similar to the one in the context of semisimple groups [Hel, BS, MW], but particular adaptations to the present situation are sometimes necessary [O5, HO2, O6]. Analogous arguments work in the setting of harmonic analysis for the affine Hecke algebras [HO3].

It is clear that $\widetilde{c}(\rho(k), k) > 0$ for all $k \in K$ with $k_\alpha \geq 0$ $\forall \alpha$. Let K_+ be the connected component of $\{k \in K; k_\alpha \in \mathbb{R}\ \forall \alpha, \widetilde{c}(\rho(k), k) \neq 0\}$ containing $\{k \in K; k_\alpha \geq 0\ \forall \alpha\}$.

PROPOSITION 5.1.– *For real $k \in K$ the condition $k \in K_+$ is equivalent to $\delta(k; \cdot)$ being locally integrable on A.*

Proof: It is easy to compute $\widetilde{c}(\rho(k), k)$ for each of the irreducible root systems case by case. For R reduced and $k_\alpha = k$ $\forall \alpha$ one has

$$\widetilde{c}(\rho(k), k) = \prod_{i=1}^{n} \frac{\Gamma(k)}{\Gamma(d_i k)}$$

with $d_1 \leq d_2 \leq \cdots \leq d_n$ the primitive degrees of R. For R of type BC_n with k_s, k_m, k_ℓ the multiplicities of the short, medium and long roots respectively one gets

$$\widetilde{c}(\rho(k), k) = \prod_{i=1}^{n} \frac{\Gamma(k_s + (i-1)k_m + k_\ell)}{\Gamma(2(k_s + (i-1)k_m + k_\ell)} \cdot \frac{\Gamma(k_m)}{\Gamma(ik_m)}.$$

For R of type F_4 one finds

$$\tilde{c}(\rho(k),k) = \frac{\Gamma(k_s)}{\Gamma(2k_s)} \cdot \frac{\Gamma(k_s)}{\Gamma(3k_s)} \cdot \frac{\Gamma(k_\ell)}{\Gamma(2k_\ell)} \cdot \frac{\Gamma(k_\ell)}{\Gamma(3k_\ell)} \times$$

$$\frac{\Gamma(k_s+k_\ell)}{\Gamma(4(k_s+k_\ell))} \cdot \frac{\Gamma(3(k_s+k_\ell))}{\Gamma(6(k_s+k_\ell))} \cdot \frac{\Gamma(2k_s+k_\ell)}{\Gamma(2(2k_s+k_\ell))} \cdot \frac{\Gamma(k_s+2k_\ell)}{\Gamma(2(k_s+2k_\ell))}$$

and for R of type G_2 the outcome is

$$\tilde{c}(\rho(k),k) = \frac{\Gamma(k_s)}{\Gamma(2k_s)} \cdot \frac{\Gamma(k_\ell)}{\Gamma(2k_\ell)} \cdot \frac{\Gamma(k_s+k_\ell)}{\Gamma(3(k_s+k_\ell))}.$$

With these explicit formulas it is easy to check the proposition case by case (see [BHO, Section 2]). □

Assume from now on that $k \in K_+$. For $f \in C_c^\infty(A)^W$ the Fourier transform $\mathfrak{F}f$ is well defined and entire on $\mathfrak{h}^*$ (by Theorem 2.8).

DEFINITION 5.2.– *Given $a \in A$ let C_a denote the convex hull of Wa, and let the support function H_a on $\mathfrak{a}^*$ be given by $H_a(\lambda) = \sup\{\lambda(\log b); b \in C_a\}$. An entire function F on $\mathfrak{h}^*$ is said to have Paley-Wiener type a if $\forall N \in \mathbb{N}\, \exists C > 0$ such that*

$$|F(\lambda)| \le C(1+|\lambda|)^{-N} e^{H_a(-\mathrm{Re}(\lambda))} \quad \forall \lambda \in \mathfrak{h}^*$$

(so F is rapidly decreasing on subspaces of the form $\lambda_0 + i\mathfrak{a}^*$ for $\lambda_0 \in \mathfrak{a}^*$). The space of functions on $\mathfrak{h}^*$ of Paley-Wiener type a is denoted by $PW(a)$, and we also write $PW = \cup_{a\in A} PW(a)$.

The first step is to obtain uniform (both in λ and a) estimates for $\widetilde{F}(\lambda, k; a)$ of the following form. Given $D \subset A$ compact and $p \in S\mathfrak{h}$ then $\exists C > 0$, $N \in \mathbb{N}$ such that

$$|\partial(p)\widetilde{F}(\lambda, k; a)| \le C(1+|\lambda|^N) e^{\max\{\mathrm{Re}[w\lambda(\log a)]; w \in W\}} \tag{5.3}$$

for all $\lambda \in \mathfrak{h}^*$, $a \in D$. Such an estimate was derived in [O5, Section 6] using the KZ-connection in case $k_\alpha \ge 0\ \forall\alpha$. The extension of (5.7) to K_+ (and even all of K) follows with the help of hypergeometric shift operators [O6, Theorem 2.5]. Using (5.3) the following result can be obtained [O6, Theorem 4.1].

THEOREM 5.3.– *If $f \in C_c^\infty(A)^W$ has support in C_a for some $a \in A$ then $\mathfrak{F}f \in PW(a)$.*

For technical reasons we have to impose the following additional restrictions on $k \in K_+$:

$$k^0_\alpha = \tfrac{1}{2}k_{\frac{1}{2}\alpha} + k_\alpha > -\tfrac{1}{2} \wedge k_{\frac{1}{2}\alpha} \in 2\mathbb{N} \text{ if } \tfrac{1}{2}\alpha, \alpha \in R. \tag{5.4}$$

So for R reduced this condition is vacuous. As a candidate for inversion introduce the *wave packet operator* $\mathfrak{J}$ on PW by

$$\mathfrak{J}F(a) = \int_{\lambda_0 + i\mathfrak{a}^*} F(\lambda)\Phi(\lambda, k; a)\frac{d\mu(\lambda)}{\tilde{c}(-\lambda, k)}. \tag{5.5}$$

Here $a \in A_+$. For all $\varepsilon > 0$ and $a_0 \in A_+$ the series $\Phi(\lambda, k; a)$ converges uniformly for $\mathrm{Re}\lambda(\alpha^\vee) < 1 - \varepsilon\ \forall\alpha > 0$ and $a \in a_0 A_+$, as shown by Gangolli. Moreover $\lambda_0 \in \mathfrak{a}^*$ satisfies $\lambda_0(\alpha^\vee) < \min(1, \frac{1}{2}(k_{\frac{1}{2}\alpha} + 1), k^0_\alpha)\ \forall\alpha \in R^0_+$. Finally $d\mu(\lambda)$ denotes Lebesgue measure on $i\mathfrak{a}^*$ (or its translates $\lambda_0 + i\mathfrak{a}^*$) normalized such that $i\mathfrak{a}^*/2\pi i P$ has volume 1 (this normalization is called the regular normalization for our choice of Haar measure da on A). Using the duplication formula one can rewrite (2.19) as

$$\tilde{c}(\lambda, k) = \prod_{\alpha \in R^0_+} \frac{2^{-k_{\frac{1}{2}\alpha}}\Gamma(\lambda(\alpha^\vee))\Gamma(\lambda(\alpha^\vee) + \frac{1}{2})}{\Gamma(\lambda(\alpha^\vee) + \frac{1}{2}(k_{\frac{1}{2}\alpha} + 1))\Gamma(\lambda(\alpha^\vee) + k^0_\alpha)}. \tag{5.6}$$

Hence it follows from our restriction on λ_0 and the Cauchy integral formula that the integral (5.5) is independent of λ_0.

The next step is to rewrite (5.5) in a different form. Under the hypothesis

$$k^0_\alpha \geq 0\ \forall\alpha \in R^0 \tag{5.7}$$

one can directly take $\lambda_0 = 0$ in (5.5) and get for $F \in PW^W$

$$\mathfrak{J}F(a) = |W|^{-1}\int_{i\mathfrak{a}^*} F(\lambda)\tilde{F}(\lambda, k; a)\frac{d\mu(\lambda)}{\tilde{c}(\lambda, k)\tilde{c}(-\lambda, k)}. \tag{5.8}$$

Then one can proceed as in [O5, Section 8 and 9] to prove the Paley-Wiener theorem, the inversion formula and the Plancherel theorem for the hypergeometric Fourier transform.

If (5.7) is not valid one still moves λ_0 back to 0 at the cost of picking up residues caused by the factor $\tilde{c}(-\lambda, k)^{-1}$ in (5.5). To understand what happens write

$$\tilde{c}(\lambda, k) = \prod_{\alpha \in R^0_+} \frac{\lambda(\alpha^\vee) + k^0_\alpha}{\lambda(\alpha^\vee)} \prod_{\alpha \in R^0_+} \frac{2^{-k_{\frac{1}{2}\alpha}}\Gamma(\lambda(\alpha^\vee) + 1)\Gamma(\lambda(\alpha^\vee) + \frac{1}{2})}{\Gamma(\lambda(\alpha^\vee) + \frac{1}{2}(k_{\frac{1}{2}\alpha} + 1))\Gamma(\lambda(\alpha^\vee) + k^0_\alpha + 1)}. \tag{5.9}$$

Because of our restrictions $k \in K_+$ and (5.4) it so happens that the entire residue calculation is determined by the first factor in the right hand side of (5.9). This factor is the c-function for the Yang particle system which was studied in [HO2]. In order to state the result we need a definition.

DEFINITION 5.4.– *For $L \subset \mathfrak{a}^*$ an affine subspace put $R^0_L = \{\alpha \in R^0; L(\alpha^\vee) =$ constant$\}$. The property of L being* residual *is defined by induction on the codimension of L. By definition $\mathfrak{a}^*$ itself is residual. An affine subspace L of $\mathfrak{a}^*$ of positive codimension is called residual if there exists a residual subspace M of $\mathfrak{a}^*$ with $L \subset M$ and $\dim M = \dim L + 1$ such that*

$$(5.10) \qquad \#\{\alpha \in R^0_L \backslash R^0_M; L(\alpha^\vee) = k^0_\alpha\} \geq \#\{\alpha \in R^0_L \backslash R^0_M; L(\alpha^\vee) = 0\} + 1.$$

A residual point is also called a distinguished *point. Being residual or distinguished is a notion invariant under W. For L residual let c_L be the point of L closest to the origin, and put*

$$(5.11) \qquad L_{\text{temp}} = c_L + i(L - c_L) \subset \mathfrak{h}^*$$

for the tempered form *of L.*

It is easy to see that L is residual (for R^0) if and only if c_L is distinguished (for R^0_L). So the classification of residual subspaces reduces to the classification of distinguished points (for R^0 and all its parabolic subsystems), and this was carried out in [HO2, Section 4]. It is easy to see that this classification is invariant under scaling $k \mapsto xk$ for $x \in (0, 1]$. In case $k^0_\alpha = k^0_\beta \; \forall \alpha, \beta \in R^0$ the classification of W-orbits of distinguished points is equivalent to the classification of distinguished nilpotent orbits (in the Bala-Carter classification [Ca, Chapter 5]) in a semisimple Lie algebra $\mathfrak{g}$ with root system dual to R^0. We can now formulate the analogue of (5.8) in case

$$(5.12) \qquad k^0_\alpha < 0 \; \forall \alpha \in R^0.$$

This restriction is necessary because [HO2] is restricted to this situation, but hopefully the method can be generalized to the case $k \in K_+$ (i.e. with some multiplicity parameters positive and others negative).

THEOREM 5.5.– *If $k \in K_+$ satisfies (5.4) and (5.12) then for $F \in PW^W$ we can write*

$$(5.13) \qquad \mathfrak{J}F(a) = \sum_L \int_{L_{\text{temp}}} F(\lambda)\widetilde{F}(\lambda, k; a)\gamma_L(k) \frac{\Pi'_L|\Gamma(\lambda(\alpha^\vee) + \frac{1}{2}k_{\frac{1}{2}\alpha})|}{\Pi'_L|\Gamma(\lambda(\alpha^\vee) + \frac{1}{2}k_{\frac{1}{2}\alpha} + k_\alpha)|} d\mu_L(\lambda)$$

with the sum over all residual subspaces of $\mathfrak{a}^*$. *Here* μ_L *is the Lebesgue measure on* $L_{\rm temp}$ *normalized such that the volume of* $i(L-c_L)/2\pi i(P\cap(L-c_L))$ *is equal to 1. The expression* Π'_L *means that in the product over all roots* $\alpha\in R$ *those* Γ-*factors are deleted whose arguments vanish identically on* L. *The number* $\gamma_L(k)$ *is nonnegative and rational satisfying* $\gamma_{wL}(xk)=\gamma_L(k)$ *for all* $w\in W$, $x\in(0,1]$. *Moreover* $\gamma_{\mathfrak{a}^*}=|W|^{-1}$ *and for* λ *a regular distinguished point with* $\{\beta_i\}\subset R^0$ *the* n *roots for which* $\lambda(\beta_i^\vee)-k_{\beta_i}=0$ *one has* $\gamma_\lambda(k)=|W|^{-1}\cdot[Q^\vee:\Sigma\mathbb{Z}\beta_i^\vee]^{-1}$ *if* λ *is a negative combination of the roots* β_i, *and* $\gamma_\lambda(k)=0$ *otherwise.*

REMARK 5.6.– By induction on the rank the calculation of $\gamma_L(k)$ reduces to the case that L is a distinguished point λ. For subregular λ the calculation is still manageable (but already cumbersome). For general λ the computation of $\gamma_\lambda(k)$ is hard and captures the full complexity of the residue calculation. From this perspective it is equally hard to decide whether $\gamma_\lambda(k)>0$ or $\gamma_\lambda(k)=0$. However in case $k^0_\alpha=k^0_\beta\ \forall\alpha,\beta\in R^0$ it can be deduced from the work of Kazhdan and Lusztig [KL] (as in [HO2]) that $\gamma_\lambda(k)>0$ always. The example below shows that this need no longer be true in the multiparameter setting (see [HO2, Section 2] for a conjectural explanation).

With the formulas (5.8) and (5.13) at hand one can proceed as in [O6] to prove the Paley-Wiener theorem for the hypergeometric Fourier transform.

THEOREM 5.7.– *The hypergeometric Fourier transform* $\mathfrak{F}$ *is a bijection from* $C_c^\infty(A)^W$ *onto* PW^W *with inverse equal to the wave packet operator* $\mathfrak{J}$.

For a further discussion of the Plancherel theorem see [O6]. The distinguished points λ with $\gamma_\lambda(k)>0$ are exactly those $\lambda\in\mathfrak{h}^*$ for which $\widetilde{F}(\lambda,k;\cdot)\in L^2(A,\delta(k;a)da)^W$. In the spirit of Harish-Chandra one might call these hypergeometric functions *cuspidal*.

EXAMPLE 5.8.– Let R be of type G_2 with simple roots α_s,α_ℓ (short and long respectively) and fundamental weights ω_s,ω_ℓ with $\omega_i(\alpha_j^\vee)=\delta_{ij}$ for $i,j\in\{s,\ell\}$. The restrictions $k\in K_+$ and (5.12) amount to $k_s<0$, $k_\ell<0$ and $k_s+k_\ell>-\frac{1}{3}$. Generically there are 3 (regular) distinguished points given by

$$\lambda_1=k_s\omega_s+k_\ell\omega_\ell=\rho(k)$$
$$\lambda_2=k_s\omega_s+\tfrac{1}{2}(k_\ell-k_s)\omega_\ell$$
$$\lambda_3=k_s\omega_s+(k_\ell-k_s)\omega_\ell$$

One can check that under the above conditions on k the points λ_1 and λ_2 are always cuspidal, whereas λ_3 is cuspidal for either $\frac{2}{3}k_s<k_\ell<\frac{1}{2}k_s$ or $k_s=k_\ell$ (in which case

$\lambda_3 = \lambda_2$). For R of type F_4 a similar (but more complicated) pattern arises [HO2]. One might wonder where to look for the geometry behind all this?

6. FINAL REMARKS

A good portion of the results discussed so far admits a deformation with a parameter q. This originated with the q-constant term conjectures of Macdonald [Ma1]. Shortly after the introduction of the Jacobi polynomials $P(\mu, k)$ and the computation of their norms [He1, O3] Macdonald introduced his orthogonal polynomials $P(\mu, q, t)$ with $0 < q < t < 1$ being independent parameters [Ma2]. One has (say R reduced and $k_\alpha = k\ \forall\alpha$) $\lim_{q\uparrow 1} P(\mu, q, q^k) = P(\mu, k)$ and one can think of the Macdonald polynomials as multivariable analogues of the basic hypergeometric polynomials of Askey, Ismail and Wilson [GR].

It was an exciting discovery of Cherednik to see how to construct the appropriate q-analogues of the Dunkl operators using representation theory of affine and double Hecke algebras [Ch3, Ch4, Ma3, Ma5]. As an application of the theory one gets the evaluation of the norm of the Macdonald polynomials as an explicit product of q-shifted factorials. This deformation by q is not merely another generalization for its own sake. It is a beautiful fact (due to Koornwinder for type A_n [Ma4], conjectured in precise terms by Macdonald and then proved by Cherednik [Ch4]) that the Fourier analysis on T is selfdual in the sense that the spectral parameter and the variable in the (suitably normalized) Macdonald polynomials play a symmetric role. This symmetry is destroyed in the classical limit $q \uparrow 1$.

The extension with a parameter q has also a meaning in physical terms. As shown by Ruijsenaars it can be interpreted as the relativistic variation of the quantum Calogero-Moser system [Ru].

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G.J. HECKMAN

Department of Mathematics
University of Nijmegen
P.O. Box 9010
6500 GL NIJMEGEN, The Netherlands
E-mail: heckman@sci.kun.nl