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AFFINE HECKE ALGEBRAS AND ORTHOGONAL POLYNOMIALS

by **I.G. MACDONALD**

INTRODUCTION

Orthogonal polynomials in one variable have a long history, going back at least to the 18th century, and a vast literature. The orthogonal polynomials of my title, however, are polynomials in several variables, and are of more recent vintage. They may be regarded as extrapolations and generalizations of Weyl's formula for the characters of a compact Lie group, and the combinatorial infrastructure of such a group (root system, Weyl group) correspondingly plays a preponderant role. At the present time it appears that an appropriate framework for the study of these polynomials is provided by the notion of an affine root system. To each affine root system of rank r, reduced or not, there corresponds a family (in fact, two families) of orthogonal polynomials in r variables. However, in an effort to keep things as simple as possible, we shall in this account restrict ourselves to one type of affine root systems (see (2.1) below).

1. ORTHOGONAL POLYNOMIALS

Let R be an irreducible reduced root system. For the most part we shall adhere to Bourbaki's notation [B]. Thus we shall denote by :

R^+ the set of positive roots relative to a fixed basis of R,

P the weight lattice of R,

P^+ the cone of dominant weights,

Q the root lattice of R,

Q^+ the cone spanned by the positive roots,

W_0 the Weyl group of R.

We have $Q \subset P$ (but $Q^+ \not\subset P^+$), and P/Q is a finite group. Let m be the smallest positive even integer such that $mP \subset Q$, let q be an indeterminate and let

$$K = \mathbf{Q}(q^{\frac{1}{m}}).$$

Let $A = K[P]$ be the group algebra of P over K. For each $\lambda \in P$, let e^λ denote the corresponding element of A (so that $e^\lambda e^\mu = e^{\lambda+\mu}$, $(e^\lambda)^{-1} = e^{-\lambda}$, and e^0 is the identity element of A). The Weyl group W_0 acts on P, hence on $A : w(e^\lambda) = e^{w\lambda}$. Let $A_0 = A^{W_0}$ denote the subalgebra of W_0-invariants.

Since each W_0-orbit in P meets P^+ exactly once, it follows that the *orbit-sums*

$$m_\lambda = \sum_{\mu \in W_0\lambda} e^\mu, \tag{1.1}$$

where $\lambda \in P^+$ and $W_0\lambda$ is the W_0-orbit of λ, form a K-basis of A_0.

We shall now define a scalar product on A. If $f \in A$, say $f = \sum_\lambda f_\lambda e^\lambda$, with coefficients $f_\lambda \in K$, let

$$\overline{f} = \sum_\lambda f_\lambda e^{-\lambda}$$

and let $[f]_1$ $(= f_0)$ denote the *constant term* of f. Now let k be a non-negative integer and define, for $f, g \in A$

$$\langle f, g \rangle_k = \frac{1}{|W_0|} [f \, \overline{g} \, \Delta_k]_1, \tag{1.2}$$

where

$$\Delta_k = \prod_{\alpha \in R} \prod_{i=0}^{k-1} (1 - q^i e^\alpha). \tag{1.3}$$

This scalar product is symmetric and non-degenerate.

On P^+ we have a partial order defined by

$$\lambda \geq \mu \quad \textit{if and only if} \quad \lambda - \mu \in Q^+. \tag{1.4}$$

With this explained, we can state the following existence theorem [M4] :

THEOREM 1.5.— *There exists a unique K-basis $(P_\lambda)_{\lambda \in P^+}$ of A_0 such that*

a) $P_\lambda = m_\lambda + \sum_{\mu < \lambda} a_{\lambda\mu} m_\mu$, *with coefficients* $a_{\lambda\mu}$ *rational functions of* q *and* q^k,

b) $\langle P_\lambda, P_\mu\rangle_k = 0$ *if* $\lambda \neq \mu$.

At this stage we shall say nothing about the proof of 1.5 (see § 6 below). We shall only remark that since the ordering (1.4) is not a total ordering (unless the rank of R is 1), the existence of the P_λ satisfying a) and b) is not immediately obvious. Indeed, we can determine P_λ uniquely to satisfy a) and

c) $\langle P_\lambda, m_\mu\rangle_k = 0$ *whenever* $\mu < \lambda$,

and this would imply b) when either $\lambda > \mu$ or $\lambda < \mu$, but not when λ and μ are incomparable.

In particular, when $k = 0$ (so that $\Delta_k = 1$), P_λ reduces to the orbit-sum m_λ, and when $k = 1$, the polynomial P_λ is given by Weyl's character formula. In the limiting case $q \to 1$, the P_λ are the "Jacobi polynomials" of Heckman and Opdam ([H], [HO], [O1], [O2]).

THEOREM 1.6 [C1].— *We have*

$$\langle P_\lambda, P_\lambda\rangle_k = \prod_{\alpha\in R^+}\prod_{i=1}^{k-1} \frac{1 - q^{\langle\lambda+k\rho,\alpha^\vee\rangle+i}}{1 - q^{\langle\lambda+k\rho,\alpha^\vee\rangle-i}}$$

for all $\lambda \in P^+$, *where* $\alpha^\vee$ *is the coroot of* α, *and*

$$\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha.$$

This formula was conjectured in [M4] and verified there in various particular cases. In the limiting case $q \to 1$, it was proved for all root systems R by Opdam [O3], and in full generality by Cherednik [C1].

Finally, observe that when $\lambda = 0$ (so that $P_\lambda = 1$), the formula (1.6) gives the constant term of Δ_k. This was the subject of earlier conjectures [M3], which at the time of Cherednik's paper had been settled affirmatively for all R with the exception of E_6, E_7 and E_8.

2. THE AFFINE ROOT SYSTEM AND EXTENDED AFFINE WEYL GROUP

As before, let R be a reduced irreducible root system, spanning a real vector space E of dimension r, and let $\langle x, y\rangle$ be a positive definite scalar product on E

invariant under the Weyl group W_0 of R. For each $\alpha \in R$, let

$$\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle},$$

so that

$$R^\vee = \{\alpha^\vee : \alpha \in R\}$$

is the dual root system. Let $Q^\vee$ denote the root lattice and $P^\vee$ the weight lattice of $R^\vee$.

We shall regard each $\alpha \in R$ as a linear function on E : $\alpha(x) = \langle \alpha, x \rangle$ for $x \in E$. Also let δ denote the constant function 1 on E. Then

$$S = S(R) = \{\alpha + n\delta : \alpha \in R,\, n \in \mathbf{Z}\} \tag{2.1}$$

is the *affine root system* associated with R. The elements of S are affine-linear functions on E, called *affine roots*.

For each $a \in S$, let H_a denote the affine hyperplane on which a vanishes,and let s_a denote the orthogonal reflection in this hyperplane. The *affine Weyl group* W_S is the group of affine isometries of E generated by these reflections. For each $\alpha \in R$, the mapping $s_\alpha \circ s_{\alpha+\delta}$ takes $x \in E$ to $x + \alpha^\vee$, so that

$$\tau(\alpha^\vee) = s_\alpha \circ s_{\alpha+\delta}$$

is translation by $\alpha^\vee$. It follows that W_S contains a subgroup of translations isomorphic to $Q^\vee$, and we have

$$W_S = W_0 \ltimes \tau(Q^\vee) \tag{2.2}$$

(semi-direct product).

The *extended affine Weyl group* is

$$W = W_0 \ltimes \tau(P^\vee). \tag{2.3}$$

It acts on E as a discrete group of isometries, and hence by transposition on functions on E. As such, it permutes the affine roots $a \in S$.

Let $\alpha_1, \cdots, \alpha_r$ be a set of simple roots (or basis) of R, let R^+ (resp. R^-) be the set of positive (resp. negative) roots determined by this basis, and let $\varphi \in R^+$ be the highest root. Correspondingly, the affine roots $a_0, a_1, , \cdots, a_r$, where

$$a_0 = -\varphi + \delta \quad , \quad a_i = \alpha_i \quad (1 \le i \le r)$$

form a set of simple roots for S. Let

$$C = \{x \in E : a_i(x) > 0 \quad (0 \le i \le r)\},$$

so that C is an open r-simplex bounded by the hyperplanes H_{a_i} $(0 \le i \le r)$. Then W_S is generated by the reflections $s_i = s_{a_i}$ $(0 \le i \le r)$, subject to the relations

$$s_i^2 = 1, \tag{2.4}$$

$$s_i\, s_j\, s_i \cdots = s_j\, s_i\, s_j \cdots \tag{2.5}$$

whenever $i \ne j$ and $s_i\, s_j$ has finite order m_{ij} in W_S, there being m_{ij} terms on either side of (2.5). In other words, W_S is a Coxeter group on the generators s_i.

The connected components of $E - \cup_{a \in S} H_a$ are open simplexes, each congruent to C; and each such component is of the form wC for a unique $w \in W_S$.

An affine root $a \in S$ is *positive* (resp. *negative*) relative to C if $a(x) > 0$ (resp. $a(x) < 0$) for all $x \in C$. Let S^+ (resp. S^-) denote the set of positive (resp. negative) affine roots. Then $S^- = -S^+$, and $S = S^+ \cup S^-$. Explicitly, we have

$$S^+ = \{\alpha + (n + \chi(\alpha))\,\delta : \alpha \in R\,,\, n \in \mathbf{N}\} \tag{2.6}$$

where χ is the characteristic function of R^- (*i.e.* $\chi(\alpha) = 0$ if $\alpha \in R^+$, and $\chi(\alpha) = 1$ if $\alpha \in R^-$).

We now define a length function on the extended group W : if $w \in W$, let

$$\ell(w) = \mathrm{Card}(S^+ \cap w^{-1}\, S^-), \tag{2.7}$$

the number of positive affine roots made negative by w. Equivalently, $\ell(w)$ is the number of hyperplanes H_a separating C from wC. From (2.3), any element of W is uniquely of the form $w\tau(\lambda)$, where $w \in W_0$ and $\lambda \in P^\vee$, and it follows from the description (2.6) of S^+ that

$$\ell(w\tau(\lambda)) = \sum_{\alpha \in R^+} |\langle \lambda, \alpha\rangle + \chi(w\alpha)| \tag{2.8}$$

where as above χ is the characteristic function of R^-.

Now W, unlike W_S, is in general not a Coxeter group (unless $P^\vee = Q^\vee$) and may contain elements $\ne 1$ of length zero. Let

$$\Omega = \{w \in W : \ell(w) = 0\}. \tag{2.9}$$

The elements of Ω stabilize the simplex C, and hence permute the simple affine roots $a_0, \cdots, a_r$. For each $w \in W$, there exists a unique $w' \in W_S$ such that $wC = w'C$, and hence w factorizes uniquely as $w = w'v$ with $w' \in W_S$ and $v \in \Omega$. Consequently we have

$$W = W_S \rtimes \Omega \tag{2.10}$$

(semidirect product). From (2.2), (2.3) and (2.10), it follows that $\Omega \cong W/W_S \cong P^\vee/Q^\vee$, hence is a finite abelian group.

We regard each weight $\mu \in P$, like each root $\alpha \in R$, as a linear function on $E : \mu(x) = \langle \mu, x \rangle$ for $x \in E$. If $w \in W$, then $w\mu$ is an affine-linear function on $E : (w\mu)(x) = \langle \mu, w^{-1}x \rangle$. Suppose that $w = v\tau(\lambda)$, where $v \in W_0$ and $\lambda \in P^\vee$. Then we have

$$w\mu = (v\tau(\lambda))\mu = v\mu - \langle \lambda, \mu \rangle\, \delta. \tag{2.11}$$

3. THE BRAID GROUP

The *braid group* B of W is the group with generators $T(w)$, $w \in W$, and relations

$$T(v)\, T(w) = T(vw) \tag{3.1}$$

whenever $\ell(vw) = \ell(v) + \ell(w)$. We shall denote $T(s_i) = T(s_{a_i})$ by T_i $(0 \leq i \leq r)$, and $T(\omega)$ $(\omega \in \Omega)$ simply by ω. Then B is generated by $T_0, \cdots, T_r$ and Ω subject to the following relations :

a) the counterparts of (2.5), namely the *braid relations*

$$T_i\, T_j\, T_i \cdots = T_j\, T_i\, T_j \cdots \tag{3.2}$$

with m_{ij} terms on either side ;

b) the relations

$$\omega\, T_i\, \omega^{-1} = T_j \tag{3.3}$$

for $\omega \in \Omega$, where $\omega\, a_i = a_j$.

Let λ be a dominant weight for $R^\vee$, and define

$$Y^\lambda = T(\tau(\lambda))$$

where as before $\tau(\lambda)$ is translation by λ. From (2.8) and (3.1) it follows that

$$Y^\lambda Y^\mu = Y^{\lambda+\mu} \tag{3.4}$$

if λ, μ are both dominant. If now λ is any element of $P^\vee$, we can write $\lambda = \mu - \nu$, with $\mu, \nu \in P^\vee$ both dominant, and we define

$$Y^\lambda = Y^\mu (Y^\nu)^{-1}. \tag{3.5}$$

In view of (3.4), this definition is unambiguous. The elements Y^λ, $\lambda \in P^\vee$, form a commutative subgroup of B, isomorphic to $P^\vee$.

3.6.— *Let $\lambda \in P^\vee$, $1 \le i \le r$.*

i) *If $\langle \lambda, \alpha_i \rangle = 0$, then $T_i Y^\lambda = Y^\lambda T_i$.*

ii) *If $\langle \lambda, \alpha_i \rangle = 1$, then $Y^\lambda = T_i Y^{s_i \lambda} T_i$.*

Proof : i) Suppose first that λ is dominant, and let $w = s_i \tau(\lambda) = \tau(\lambda) s_i$. From (2.8) we have $\ell(w) = \ell(\tau(\lambda)) + 1$ and hence $T_i Y^\lambda = T(w) = Y^\lambda T_i$. If now λ is not dominant, we can write $\lambda = \mu - \nu$ with μ, ν both dominant and $\langle \mu, \alpha_i \rangle = \langle \nu, \alpha_i \rangle = 0$.

ii) Again suppose first that λ is dominant, and let $p = \ell(\tau(\lambda))$. Then $\pi = \lambda + s_i \lambda$ is dominant and $\langle \pi, \alpha_i \rangle = 0$. Let $w = \tau(\lambda) s_i \tau(\lambda) = s_i \tau(\pi)$. We have, using (2.8), $\ell(\pi) = 2p - 2$, $\ell(w) = 2p - 1$, $\ell(\tau(\lambda) s_i) = p - 1$. Hence

$$T_i Y^\pi = T(w) = T(\tau(\lambda) s_i)\, T(\tau(\lambda)) = Y^\lambda T_i^{-1} Y^\lambda$$

giving $Y^\lambda = T_i Y^{s_i \lambda} T_i$ as required. Finally, if λ is not dominant, we can write $\lambda = \mu - \nu$ with μ, ν both dominant, $\langle \mu, \alpha_i \rangle = 1$, $\langle \nu, \alpha_i \rangle = 0$.

3.7.— *B is generated by $T_1, \cdots, T_r$ and the Y^λ, $\lambda \in P^\vee$.*

4. THE AFFINE HECKE ALGEBRA

As in § 1, let $K = \mathbf{Q}(q^{\frac{1}{m}})$ and let $t = q^{-\frac{k}{2}}$, where k is a non-negative integer. The *Hecke algebra* H of W is the quotient of the group algebra $K[B]$ of the braid group by the ideal generated by the elements $(T_i - t)(T_i + t^{-1})$ $(0 \le i \le r)$. For each $w \in W$, we denote the image of $T(w)$ in H by the same symbol $T(w)$. It is well-known (but requires proof, see e.g. [B], ch. IV, § 2, Ex. 23) that the $T(w)$ form a K-*basis* of H. Thus H is generated by T_i $(0 \le i \le r)$ and Ω subject to the relations (3.2), (3.3) and

$$(4.1) \qquad (T_i - t)(T_i + t^{-1}) = 0 \qquad (0 \le i \le r).$$

The following formula, due to Lusztig [L], is fundamental for what follows.

4.2. *Let* $\lambda \in P^\vee$, $1 \le i \le r$. *Then*

$$Y^\lambda T_i - T_i Y^{s_i\lambda} = \frac{(t - t^{-1})(Y^\lambda - Y^{s_i\lambda})}{(1 - Y^{-\alpha_i^\vee})}.$$

Proof : If this is true for λ and for μ, a simple calculation shows that it is true for $\lambda + \mu$ and $-\lambda$. Hence we may assume that λ is a fundamental weight, so that $\langle \lambda, \alpha_i \rangle = 0$ or 1. If $\langle \lambda, \alpha_i \rangle = 0$, then (4.2) reduces to (3.6) i), and if $\langle \lambda, \alpha_i \rangle = 1$, it follows from (3.6) ii) and (4.1).

Remark : Since $s_i\lambda = \lambda - p\alpha_i^\vee$, where $p = \langle \lambda, \alpha_i \rangle \in \mathbf{Z}$, it follows that the right-hand side in (4.2) is a polynomial in the Y's.

From (3.7) and (4.2) it follows (*cf.* [L]) that

4.3. *The elements* $T(w) Y^\lambda$ (resp. *the elements* $Y^\lambda T(w)$), *where* $w \in W_0$ *and* $\lambda \in P^\vee$, *form a* K-*basis of* H.

Let $A^\vee = K[P^\vee]$ be the group algebra of $P^\vee$ over K. For each $f \in A^\vee$, say $f = \sum f_\lambda e^\lambda$, let

$$f(Y) = \sum f_\lambda Y^\lambda \in H,$$

and let $A^\vee(Y)$ denote the subalgebra of H generated by the Y^λ, $\lambda \in P^\vee$. From (4.3) we have $A^\vee(Y) \cong A^\vee$ and

$$(4.4) \qquad H \cong A^\vee \otimes_K H_0,$$

where H_0 is the Hecke algebra of the finite Weyl group W_0, generated by $T_1, \cdots, T_r$ subject to the braid relations (3.2) (with $i, j \ne 0$) and the Hecke relations (4.1) (with $i \ne 0$).

Let $A_0^\vee = (A^\vee)^{W_0}$ be the subalgebra of W_0-invariants of $A^\vee$.

4.5. *The centre of H is $A_0^\vee(Y)$.*

Proof : It follows from (4.2) that if $f \in A^\vee$ is W_0-invariant, then $f(Y)$ commutes with $T_1, \cdots, T_r$ and hence by (4.3) lies in the centre of H. The reverse inclusion may be proved by a specialization argument (let $q \to 1$).

If $a = \alpha + n\delta \in S$, we define

$$e^a = e^{\alpha + n\delta} = q^{-n}\, e^\alpha$$

(*i.e.* we define e^δ to be q^{-1}). Likewise, if $\mu \in P$ and $w \in W$, where $w = v\,\tau(\lambda)$ as in (2.11), we have

$$e^{w\mu} = q^{\langle \lambda, \mu \rangle}\, e^{v\mu}.$$

The following proposition, due to Cherednik [C1], is a key result.

4.6.— *The Hecke algebra H acts on $A = K[P]$ as follows :*

$$\begin{aligned} &T_i\, e^\mu = t e^{s_i\mu} + (t - t^{-1})(1 - e^{a_i})^{-1}(e^\mu - e^{s_i\mu}) && (0 \le i \le r) \\ &\omega\, e^\mu = e^{\omega\mu} && (\omega \in \Omega). \end{aligned}$$

Moreover, this representation is faithful.

We shall sketch a proof. If V is any H_0-module, we can form the induced H-module :

$$\begin{aligned} \mathrm{ind}_{H_0}^H(V) &= H \otimes_{H_0} V \\ &\cong A^\vee \otimes_K V \end{aligned} \tag{4.7}$$

by (4.4). Suppose in particular that V is 1-dimensional and that $T_i v = tv$ for $v \in V$, $1 \le i \le r$. From (4.7) the induced module may be identified with $A^\vee$, and by (4.2) the action of T_i $(1 \le i \le r)$ on $A^\vee$ is given by :

$$T_i\, e^\lambda = t\, e^{s_i\lambda} + (t - t^{-1})(1 - e^{-\alpha_i^\vee})^{-1}\,(e^\lambda - e^{s_i\lambda}) \tag{4.8}$$

where $\lambda \in P^\vee$. The operators T_i defined by (4.8) therefore define a representation of H_0 on $A^\vee$, and it is not difficult to show that this representation is faithful. Now H_0 depends only on t and the Weyl group W_0, not on the root system $R^\vee$. We may therefore replace $R^\vee$ by R and $A^\vee$ by A, and the basis $\alpha_1^\vee, \cdots \alpha_r^\vee$ of $R^\vee$ by the opposite basis $-\alpha_1, \cdots, -\alpha_r$ of R. This gives us the operators T_i of (4.6) for $1 \leq i \leq r$, and they will by our construction automatically satisfy the braid relations (3.2) and Hecke relations (4.1). But then, if we define T_0 as in (4.6), all the relations (4.2) and (4.3) will be satisfied, and we have a representation of H on A.

4.9. *Let $f \in A_0^\vee$. Then $f(Y)$ maps A_0 into A_0.*

This follows without difficulty from 4.5 and 4.6.

From 4.6 we have an action of $A^\vee = K[P^\vee]$ on $A = K[P]$, with e^λ ($\lambda \in P^\vee$) acting as Y^λ. Except in simple cases it appears not to be possible to make this action explicit, *i.e.* we cannot calculate $Y^\lambda e^\mu$ explicitly. However, it is possible to calculate the "leading term" of $Y^\lambda e^\mu$, in a sense now to be described, and it will appear that this will be sufficient for our purposes.

For this purpose we shall define a partial ordering on the weight lattice P which extends that defined by (1.4). If $\lambda \in P$, let λ^+ denote the unique dominant weight in the W_0-orbit of λ, and define ($\lambda, \mu \in P$)

4.10. $\lambda \geq \mu$ *if and only if either* i) $\lambda^+ > \mu^+$, *or* ii) $\lambda^+ = \mu^+$, *and* $\mu - \lambda \in Q^+$.

Thus, in a given W_0-orbit, the antidominant weight is highest.

Next, for $\mu \in P$, let

$$\rho(\mu) = \frac{1}{2} \sum_{\alpha \in R^+} \varepsilon(\langle \mu, \alpha^\vee \rangle)\, \alpha \tag{4.11}$$

where $\varepsilon(x) = 1$ if $x > 0$ and $\varepsilon(x) = -1$ if $x \leq 0$; and let

$$\mu^* = \mu + k\rho(\mu). \tag{4.12}$$

Then we have

4.13. *If $\lambda \in P^\vee$, $\mu \in P$,*

$$Y^\lambda e^\mu = q^{\langle \lambda, \mu^* \rangle} e^\mu + \text{ lower terms},$$

where by lower terms is meant a linear combination of the exponentials e^ν such that $\nu < \mu$.

5. CHEREDNIK'S SCALAR PRODUCT

The symmetric scalar product $\langle f, g\rangle_k$ on A defined in § 1 is not suitable in the present context. With k a non-negative integer as before, let

$$S(k) = \{a \in S : 0 < a(x) < k \text{ for all } x \in C\}.$$

Explicitly (see (2.6)) $S(k)$ consists of the affine roots $a = \alpha + (n + \chi(\alpha))\delta$ for $\alpha \in R$ and $n = 0, 1, \cdots, k-1$. Now define

$$C_k = \prod_{a \in S(k)} (e^{\frac{a}{2}} - e^{-\frac{a}{2}}) \tag{5.1}$$

and if $f \in A$, say $f = \sum f_\lambda e^\lambda$ with coefficients $f_\lambda \in K$, let

$$\widetilde{f} = \sum \widetilde{f}_\lambda e^{-\lambda},$$

where $\widetilde{f}_\lambda$ is the image of f_λ under the automorphism $q \mapsto q^{-1}$ of K. Thus we have $(e^a)^\sim = e^{-a}$ for all $a \in S$. We now define

$$(f, g)_k = [f\,\widetilde{g}\,C_k]_1 \tag{5.2}$$

for $f, g \in A$, where as in § 1 the square brackets denote the constant term. This scalar product (due to Cherednik) is non-degenerate and hermitian (relative to the involution $q \mapsto q^{-1}$ of K), because the product (5.1) defining C_k contains an even number of terms, and therefore $\widetilde{C}_k = C_k$.

The advantage of this scalar product is contained in the following proposition [C1] :

5.3.— *Let $w \in W$. Then the adjoint of $T(w)$ for the scalar product* (5.2) *is $T(w)^{-1}$, i.e. we have*

$$(T(w)f, g)_k = (f, T(w)^{-1} g)_k$$

for all $f, g \in A$. In particular, the adjoint of Y^λ ($\lambda \in P^\vee$) is $Y^{-\lambda}$, and the adjoint of $u(Y)$, where $u \in A^\vee$, is $\widetilde{u}(Y)$.

Proof : It is enough to show that the adjoint of T_i (resp. $\omega \in \Omega$) is T_i^{-1} (resp. ω^{-1}), and this is verified directly from the definitions.

Finally, when restricted to $A_0 = A^{W_0}$, the scalar product (5.2) is closely related to the symmetric scalar product (1.2). Namely :

5.4.— *For all* $f, g \in A_0$ *we have*

$$(f,g)_k = c_k \langle f, g^\iota\rangle_k,$$

where $g^\iota = (\overline{g})^\sim$, *and* c_k *depends only on* k (and not on f, g).

Proof : From the definitions of C_k and Δ_k it follows that

$$\frac{C_k}{\Delta_k} = q^{-\frac{Nk^2}{2}} \prod_{\alpha\in R^+} \frac{1-q^k e^\alpha}{1-e^\alpha},$$

where $N = \mathrm{Card}(R^+)$. Now (5.4) follows from the identity [M2]

$$\sum_{w\in W_0} \prod_{\alpha\in R^+} \frac{1-q^k e^{w\alpha}}{1-e^{w\alpha}} = W_0(q^k) \tag{5.5}$$

where $W_0(q^k)$ is the Poincaré polynomial (*loc. cit.*) of W_0. (Explicitly, $c_k = q^{-\frac{Nk^2}{2}} W_0(q^k)$.)

Let

$$\pi_k = \prod_{\alpha\in R^+} (te^{\frac{\alpha}{2}} - t^{-1} e^{-\frac{\alpha}{2}}). \tag{5.6}$$

Then the same identity (5.5) can be used to prove

5.7.— *For all* $f, g \in A_0$ *we have*

$$(\pi_k f, \pi_k g)_k = q^{-Nk} c_k \langle f, g^\iota\rangle_{k+1}.$$

6. ORTHOGONAL POLYNOMIALS AGAIN

If $f \in A^\vee$, say $f = \sum f_\lambda e^\lambda$, and $\mu \in P$, we define

$$f(\mu) = \sum f_\lambda q^{\langle\lambda,\mu\rangle}, \tag{6.1}$$

thus regarding f as a K-valued function on P.

As already remarked in § 1, the polynomials $P_\lambda \in A_0$, where $\lambda \in P^+$, are uniquely determined by the two conditions :

i) $P_\lambda = m_\lambda +$ lower terms,
ii) $(P_\lambda, m_\mu)_k = 0$ for $\mu \in P^+$, $\mu < \lambda$.
(We can replace $\langle P_\lambda, m_\mu \rangle_k$ by $(P_\lambda, m_\mu)_k$ in ii) by virtue of (5.4).)

Now let $f \in A_0^\vee$ and consider $f(Y)\, m_\mu$, where $\mu \in P^+$. Since

$$m_\mu = e^{w_0\mu} + \text{lower terms}$$

for the ordering (4.10), where w_0 is the longest element of W_0, it follows from (4.13) and the definition (6.1) that

$$f(Y)\, m_\mu = f((w_0\mu)^*)\, e^{w_0\mu} + \text{lower terms.}$$

Now $f(Y)\, m_\mu$ is W_0-symmetric, by (4.9), and since $(w_0\mu)^* = w_0(\mu + k\rho)$, we have $f((w_0\mu)^*) = f(\mu + k\rho)$. Hence

(6.2) $$f(Y)\, m_\mu = f(\mu + k\rho)\, m_\mu + \text{lower terms.}$$

By (5.3) the adjoint of $f(Y)$ is $\widetilde{f}(Y)$. Hence if $\mu \in P^+$, $\mu < \lambda$, we have

$$(f(Y)\, P_\lambda, m_\mu)_k = (P_\lambda, \widetilde{f}(Y)\, m_\mu)_k,$$

which is zero by (6.2) and the definition of P_λ. It follows that $f(Y)\, P_\lambda$ is a scalar multiple of P_λ, namely (by (6.2) again)

(6.3) $$f(Y)\, P_\lambda = f(\lambda + k\rho)\, P_\lambda$$

for all $f \in A_0^\vee$. Thus the P_λ diagonalize the action of $A_0^\vee$ on A_0, and it follows from (6.3) that they are pairwise orthogonal :

(6.4) $$\langle P_\lambda, P_\mu \rangle_k = 0 \quad \text{if} \quad \lambda \neq \mu.$$

For if $f \in A_0^\vee$, we have

$$\begin{aligned} f(\lambda + k\rho)(P_\lambda, P_\mu)_k &= (f(Y)\, P_\lambda, P_\mu)_k \\ &= (P_\lambda, \widetilde{f}(Y)\, P_\mu)_k \\ &= (P_\lambda, \widetilde{f}(\mu + k\rho)\, P_\mu)_k \\ &= f(\mu + k\rho)(P_\lambda, P_\mu)_k \end{aligned}$$

by (6.3) and (5.3). Since $\lambda \neq \mu$, we can choose $f \in A_0^\vee$ so that $f(\lambda+k\rho) \neq f(\mu+k\rho)$. Hence $(P_\lambda, P_\mu)_k = 0$ and therefore also $\langle P_\lambda, P_\mu\rangle_k = 0$ by (5.4). This proves (1.5).

Next, for each $\lambda \in P$, there is a unique element $E_\lambda \in A$ satisfying the two conditions :

i) $E_\lambda = e^\lambda +$ lower terms,

ii) $(E_\lambda, e^\mu)_k = 0$ for all $\mu < \lambda$.

If $f \in A^\vee$, it follows from (5.3) that

$$(f(Y)\,E_\lambda, e^\mu)_k = (E_\lambda, \widetilde{f}(Y)\,e^\mu)_k,$$

which is zero if $\mu < \lambda$, by (4.13). Hence $f(Y)\,E_\lambda$ is a scalar multiple of E_λ, namely (by (4.13) again)

$$f(Y)\,E_\lambda = f(\lambda^*)\,E_\lambda. \tag{6.5}$$

Thus the E_λ diagonalize the action of $A^\vee$ on A, and the same argument as in (6.4) shows that they are pairwise orthogonal :

$$(E_\lambda, E_\mu)_k = 0 \quad \text{if} \quad \lambda \neq \mu. \tag{6.6}$$

One shows next that if $\lambda \in P$ is such that $\lambda \neq s_i\,\lambda$, then $T_i\,E_\lambda$ is a linear combination of E_λ and $E_{s_i\,\lambda}$, with coefficients that can be explicitly computed. From this and (6.5), it follows that for each $\lambda \in P^+$, the K-subspace $A(\lambda)$ of A spanned by the E_μ, $\mu \in W_0\lambda$, is stable under the action of H.

Consider now the operators

$$U^+ = \sum_{w\in W_0} t^{\ell(w)}\,T(w),$$
$$U^- = \sum_{w\in W_0} (-t)^{-\ell(w)}\,T(w)$$

on A, where $\ell(w)$ is the length of $w \in W_0$. We have

$$(T_i - t)\,U^+ = U^+(T_i - t) = 0, \tag{6.7}$$

$$(T_i + t^{-1})\,U^- = U^-(T_i + t^{-1}) = 0 \tag{6.8}$$

for $1 \le i \le r$. From (6.7) it follows that $U^+ f$ is W_0-symmetric for all $f \in A$ (but $U^- f$ is *not* W_0-skew, unless $q = 1$). In particular, if $\lambda \in P^+$, then $U^+ E_\lambda$ is a scalar multiple of P_λ (because it has the same defining properties). Hence $P_\lambda \in A(\lambda)$, say

$$P_\lambda = \sum_{\mu \in W_0 \lambda} a_{\lambda\mu} \, E_\mu$$

and the coefficients $a_{\lambda\mu} \in K$ can be calculated explicitly : in fact

$$a_{\lambda\mu} = \prod_{\substack{\alpha \in R^+ \\ \langle \mu, \alpha^\vee \rangle > 0}} \frac{1 - q^{\langle \mu^*, \alpha^\vee \rangle - k}}{1 - q^{\langle \mu^*, \alpha^\vee \rangle}}.$$

Next define, again for $\lambda \in P^+$,

$$Q_\lambda = U^- \, E_\lambda.$$

If λ is not regular (*i.e.* if $\langle \lambda, \alpha_i^\vee \rangle = 0$ for some i), then $Q_\lambda = 0$. We have $Q_\lambda \in A(\lambda)$, say

$$Q_\lambda = \sum_{\mu \in W_0 \lambda} b_{\lambda\mu} \, E_\mu$$

and as in the case of P_λ, the coefficients $b_{\lambda\mu}$ can be calculated explicitly. In this way $(P_\lambda, P_\lambda)_k$ and $(Q_\lambda, Q_\lambda)_k$ can each be expressed in terms of $(E_\lambda, E_\lambda)_k$, and we obtain (λ dominant and regular) :

$$\frac{(Q_\lambda, Q_\lambda)_k}{(P_\lambda, P_\lambda)_k} = q^{-Nk} \prod_{\alpha \in R^+} \frac{1 - q^{\langle \lambda + k\rho, \alpha^\vee \rangle + k}}{1 - q^{\langle \lambda + k\rho, \alpha^\vee \rangle - k}} \tag{6.9}$$

where as before $N = \mathrm{card}(R^+)$.

7. CALCULATION OF $\langle P_\lambda, P_\lambda \rangle_k$

We shall now prove the scalar product formula (1.6) by induction on k, the case $k = 0$ (or $k = 1$) being trivial. The formula (6.9) is one of the two ingredients in the proof, and we shall now briefly sketch the other one. From now on we shall write $P_{\lambda,k}$ and $Q_{\lambda,k}$ in place of P_λ and Q_λ, to stress the dependence on the parameter k.

As in § 5, let

$$\pi_k = \prod_{\alpha \in R^+} \left(t \, e^{\frac{\alpha}{2}} - t^{-1} \, e^{-\frac{\alpha}{2}} \right).$$

Then it follows from the definition (4.6) of T_i that

$$(7.1) \qquad (T_i + t^{-1})\, \pi_k\, f = s_i(\pi_k)\, (T_i - t)\, f \qquad (1 \le i \le r)$$

for $f \in A$. We use this formula to prove

7.2. *Let $f \in A$. Then $(T_i + t^{-1})\, f = 0$ for $1 \le i \le r$ if and only if $f \in \pi_k A_0$.*

Proof : Suppose that $(T_i + t^{-1})\, f = 0$, $1 \le i \le r$. Then (7.1) shows that $g = \pi_k^{-1} f$ is killed by each $T_i - t$, hence is W_0-symmetric. Hence if w_0 is the longest element of W_0 we have $\pi_k^{-1} f = w_0(\pi_k^{-1} f)$, *i.e.*

$$w_0(\pi_k)\, f = \pi_k\, w_0(f).$$

Now π_k and $w_0(\pi_k)$ are coprime. Hence π_k divides f in A, *i.e.* we have $g \in A_0$ and hence $f \in \pi_k A_0$. Conversely, if $f = \pi_k\, g$ with $g \in A_0$, (7.1) shows that $(T_i + t^{-1})\, f = 0$.

7.3. *Let $\lambda \in P$ be dominant and regular* (so that $\lambda - \rho \in P^+$). *Then we have*

$$Q_{\lambda,k} = q^{\frac{Nk}{2}}\, \pi_k\, P_{\lambda-\rho,k+1}.$$

Proof : It follows from (6.8) that $(T_i + t^{-1})\, Q_{\lambda,k} = 0$ for $1 \le i \le r$, and hence by (7.2) that $Q_{\lambda,k} = \pi_k\, g$ for some $g \in A_0$. Consideration of the leading terms of $Q_{\lambda,k}$ and π_k shows that g is of the form

$$(1) \qquad g = \sum_{\mu \le \lambda-\rho} c_{\lambda\mu}\, m_\mu.$$

Now if $\mu \in P^+$, the highest exponential that occurs in $\pi_k\, m_\mu$ is $e^{w_0(\mu+\rho)}$; and since $Q_{\lambda,k}$ is a linear combination of the $E_{w\lambda}$, $w \in W_0$, it follows that

$$(\pi_k\, g\, ,\ \pi_k\, m_\mu)_k = 0$$

for all $\mu < \lambda - \rho$ in P^+, and this in turn implies (using 5.7)

$$(2) \qquad \langle g, m_\mu \rangle_{k+1} = 0.$$

From (1) and (2) it follows that g is a scalar multiple of $P_{\lambda-\rho,k+1}$, and the scalar is determined from the coefficient of $e^{w_0\lambda}$.

From (7.3) and (5.7) we obtain

$$(7.4) \qquad (Q_{\lambda,k}, Q_{\lambda,k})_k = q^{-Nk} c_k \langle P_{\lambda-\rho,k+1}, P_{\lambda-\rho,k+1}\rangle_{k+1}.$$

Together with (6.9) this gives

$$\frac{\langle P_{\lambda-\rho,k+1}, P_{\lambda-\rho,k+1}\rangle_{k+1}}{\langle P_{\lambda,k}, P_{\lambda,k}\rangle_k} = \prod_{\alpha\in R^+} \frac{1-q^{\langle\lambda+k\rho,\alpha^\vee\rangle+k}}{1-q^{\langle\lambda+k\rho,\alpha^\vee\rangle-k}}$$

from which (1.6) follows by induction on k.

Finally, once $\langle P_\lambda, P_\lambda\rangle_k$ is known, it is straightforward to calculate $(E_\lambda, E_\lambda)_k$ for any $\lambda \in P$. Let us write

$$[s] = q^{\frac{s}{2}} - q^{-\frac{s}{2}}$$

for all $s \in \mathbf{Z}$. With this notation we have

$$(7.5) \qquad (E_\lambda, E_\lambda)_k = \prod_{\alpha\in R^+} \left(\prod_{i=0}^{k-1} \frac{[\langle\lambda^*,\alpha^\vee\rangle + i]}{[\langle\lambda^*,\alpha^\vee\rangle - i - 1]}\right)^{\varepsilon(\langle\lambda^*,\alpha^\vee\rangle)}$$

for all $\lambda \in P$, where (as in (4.11)) $\varepsilon(x) = 1$ for $x > 0$, and $\varepsilon(x) = -1$ for $x \leq 0$.

8. CONCLUDING REMARKS

The proof of the scalar product formula (1.6) sketched here is somewhat different from that of Cherednik. It was inspired by recent work of Opdam [O4], who defined the non-symmetric orthogonal polynomials E_λ in the limiting case $q \to 1$ and worked out their properties, and suggested that analogous things should exist for arbitrary q.

Cherednik's proof exploited what he calls the "double affine Hecke algebra", which (as an algebra of linear operators on A) is generated by H and operators X^λ ($\lambda \in P$), where X^λ is multiplication by e^λ. In this algebra there is a symmetry as between the X's and the Y's, and very recently [C2] Cherednik has made use of this to confirm two other conjectures of the author relating to the polynomials P_λ.

In this account we have restricted ourselves to affine root systems ot the type $S(R)$ (2.1) and a single parameter k, in an attempt to avoid drowning both author and reader in a sea of technicalities. The general picture is that one can attach to any affine root system S, reduced or not, a family of symmetric orthogonal polynomials

P_λ, and another family of non-symmetric orthogonal polynomials E_λ. These depend (apart from q) on as many parameters k as there are orbits in S under the affine Weyl group W_S. For an irreducible S, the maximum number of orbits is 5, and is attained by the (non-reduced) affine root systems denoted by $C^\vee C_n$ $(n \geq 2)$ in the tables at the end of [M1]. Correspondingly, we have orthogonal polynomials P_λ, E_λ depending on q and five parameters k_i. These P_λ are precisely the polynomials defined by Koornwinder in [K], which are therefore amenable to the Hecke algebra techniques described here. In particular, Koornwinder's conjecture for the value of $\langle P_\lambda, P_\lambda \rangle$ can be shown to be correct.

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