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EIGENVALUE DISTRIBUTION OF RANDOM OPERATORS AND MATRICES

by **Leonid PASTUR**

1. INTRODUCTION

The study of the eigenvalue distribution of various classes of selfadjoint operators is one of the oldest and most popular branches of the spectral theory. Recall, for instance, the well-known problem of H. Weyl on the high energy asymptotic behaviour of the counting function

$$\mathcal{N}_\Lambda(\lambda) = \#\{\lambda_i \le \lambda\} \tag{1}$$

of eigenvalues λ_i of a selfadjoint boundary value problem for an elliptic operator A_Λ in a compact domain $\Lambda \subset \mathbb{R}^d$. According to H. Weyl, in the simplest but important case of $A_\Lambda = -\Delta$,

$$\mathcal{N}_\Lambda(\lambda) = c_d|\Lambda|\lambda^{d/2} + o(\lambda^{d/2}), \qquad \lambda \to \infty, \tag{2}$$

where $|\Lambda|$ is the volume of Λ and c_d depends only on the dimensionality of the space.

The study of the subsequent terms on this and similar asymptotic formulae for various cases has produced a wide variety of beautiful and important results and has revealed many deep interconnections of the spectral theory with geometry, topology, ergodic theory, various analytic and asymptotic methods, etc.

Let me remind now that one of the main Weyl's motivations for studying this problem was justification of Rayleigh and Jeans derivation of the formula for the spectral distribution of the black body radiation and of Debye derivation of the formula for the specific heat of a crystal. These problems that played an important role in the development of quantum mechanics, can be formulated as problems of constructing of the thermodynamics of the ideal Bose gas. One of the main mathematical physics concepts in statistical mechanics is the concept of the thermodynamic limit, designed in order to study the bulk properties of macroscopically large systems. From this point of view it is rather natural to consider some sequence Λ_k of compact domains expanding into the whole $\mathbb{R}^d$ as $k \to \infty$, to prove the existence for each $\lambda \in \mathbb{R}$ of the limit

$$\lim_{k\to\infty} N_{\Lambda_k}(\lambda) \equiv N(\lambda), \tag{3}$$

where

$$N_\Lambda(\lambda) = |\Lambda|^{-1}\mathcal{N}_\Lambda(\lambda), \tag{4}$$

to check the independence of this limit of a sequence $\{\Lambda_k\}$ for a sufficiently broad family of sequences and after that to study the nondecreasing function (3) for various ranges of λ (in particular, for $\lambda \to \infty$) and other parameters.

In the case of the Laplacian and a sufficiently regular sequence of domains this problem can be reduced to the problem of finding the high energy asymptotic of $N_\Lambda(\lambda)$ for a fixed Λ. This is why H. Weyl was rather interested in the proof of the independence of the leading term in (2) of the shape of a domain and considered this result as one of his main achievements.

The problem of studying $N_\Lambda(\lambda)$, known as the integrated density of states (IDS), can also be formulated in the discrete case, i.e. for a sequence of $n \times n$ matrices with $n \to \infty$. Here the analogue of (4) is

$$N_n(\lambda) = \#\{\lambda_i \le \lambda\}n^{-1} \tag{5}$$

In the case when a matrix is the restriction to a finite set $\Lambda \subset \mathbb{Z}^d$ of the Toeplitz operator the problem was considered by Grenander and Szegö [1].

In this paper I am going to discuss three classes of random differential and matrix operators for which the above formulated problem can be studied rather completely. The first class includes differential and finite-difference operators with random coefficients. The second class includes random matrices with independent and identically distributed entries. Thus, the main difference between these two classes is that operators of the first class have nonzero entries only for a finite and Λ-independent number of diagonals adjacent to the principal one, while for operators (matrices) of the second class all entries have, roughly speaking, the same order of magnitude. This difference turns out to be rather serious and results in different forms of problems, answers and techniques. Therefore it is somewhat surprising that there exists operators that are in a certain sense interpolating between these two classes. This is our third class.

2. DIFFERENTIAL AND FINITE-DIFFERENCE OPERATORS WITH RANDOM COEFFICIENTS

It is clear that if we are going to prove the existence of the limit (3), we should impose certain conditions on the coefficients of the respective operators. Indeed, consider, for instance, the Schrödinger operator H_Λ defined in Λ by the operation

$$-\Delta + q(x) \tag{6}$$

and some (say Dirichlet) boundary conditions on $\partial\Lambda$. Then it is easy to see that if $q(x) \to \infty$ as $|x| \to \infty$, then $N(\lambda) \equiv 0$ and if $q(x) \to 0$ as $|x| \to \infty$ then $N(\lambda) = N_0(\lambda)$, where

$$N_0(\lambda) = c_d \lambda^{d/2} \tag{7}$$

is, according to (2), the integrated density of states of $-\Delta$. Thus, to obtain a nontrivial result (neither 0 nor $N_0(\lambda)$) we should consider a nonzero potential $q(x)$ that does not grow and does not decay at infinity. Moreover, it is clear that $q(x)$ should behave rather "regularly" at infinity not to produce too irregular oscillations of $N_\Lambda(\lambda)$ for large Λ.

The simplest nontrivial case is that of a periodic potential. The respective Schrödinger operator describes the electron motion in an ideal solid. A rather general class of potentials for which there exists the limit (3) is given by realizations (sample functions) of metrically transitive (ergodic) random fields in $\mathbb{R}^d$ which model disordered solids. This means that we consider not a single operator H_Λ with some particular $q(x)$ but rather a family $H_\Lambda(\omega)$ of operators corresponding to potentials of the form

$$q(x,\omega) = Q(T_x\omega), \tag{8}$$

where $Q(\omega)$, $\omega \in \Omega$, is a measurable function on a probability space Ω in which a group T_x, $x \in \mathbb{R}^d$, of measure-preserving and metrically transitive automorphisms acts. A simple example of (8) in the discrete case, i.e. for the finite-difference analogue of (6)

$$-\Delta_{\text{disc}} + q(x), \qquad x \in \mathbb{Z}^d, \tag{9}$$

is a family of independent identically distributed (i.i.d.) random variables. This operator is a discretized model for electron motion in completely disordered solids, as amorphous substances, disordered alloys, etc.

The existence of the limit (3) with probability 1 and its nonrandomness for the Schrödinger operator with potentials having the form (8) and bounded below was given by myself in early seventies. Later this problem was considered by several authors and the results which are rather close to optimal were obtained (see, e.g., books [2,3] and references therein). For example, in the case of the operator (9) the IDS exists for any metrically transitive field (8) on $\mathbb{Z}^d$ which is finite with probability 1.

Let us discuss now some properties of the IDS, restricting ourselves mainly to the case of the Schrödinger operators (6) and (9) (for the respective proofs and references see, e.g., books [2,3]).

(i) **Smoothness.** In the discrete case (i.e. for (9)) and in the continuous one-dimensional case (i.e. for (6) with $d = 1$) the IDS is always continuous. This can be proved on the basis of simple ergodic arguments. A more refined technique based on the notion of the Lyapunov exponent yields the log-Hölder property of the IDS:

$$|N(\lambda_1) - N(\lambda_2)| \leq \text{const}\,|\log|\lambda_1 - \lambda_2||^{-1}.$$

There are counter examples showing that this estimate is optimal for the whole class of metrically transitive potentials. However, if we consider the discrete operator (9) with i.i.d. random potential whose distribution function is

$F(dq) = f(q)dq$, $\sup_{q\in\mathbb{R}} f(q) \le f_0 < \infty$, then, according to Wegner

$$N(d\lambda) = \rho(\lambda)d\lambda, \qquad \sup_{\lambda\in\mathbb{R}} \rho(\lambda) \le f_0.$$

(Here and below I denote by the same symbol $N(\cdot)$ the nondecreasing function and the measure that are related in the obvious way.)

Further, assuming some smoothness of $f(q)$, one can deduce much stronger smoothness of $\rho(\lambda)$, up to its C^∞ or real analyticity. On the other hand, it is known that if for $d = 1$ an i.i.d. random potential takes two values, say 0 and q_0, then if q_0 is large enough the IDS has a singular continuos component [2,3].

Smoothness of the IDS, being of considerable interest in itself, plays an important role in the proofs of the Anderson localization, i.e. of the presence of a point component in the spectrum of operators (6) and (9).

It should be emphasized that most of the known results on the smoothness of the IDS and other spectral properties of the random operator are proved for the discrete case (9). The continuous case is much more technically difficult and much less studied so far.

(ii) **Explicit formulae.** The IDS is found explicitly for the 1-dimensional continuous case with the Markov random potential taking two values and for the multidimensional discrete case with the Cauchy-distributed or certain quasi periodic potential. In the latter cases the IDS is an analytic function in some strip $|\operatorname{Im}\lambda| \le$ const. These formulae provide a considerable amount of quantitative information on the behaviour of the IDS on various parts of the spectrum.

(iii) **Asymptotic behaviour.**

(a) High energies. Let the potential $q(x)$ in (6) be a metrically transitive field on $\mathbb{R}^d$ such that

$$\mathbb{E}\{|q(0)|^{p+1}\} < \infty\ ,$$

where p is the smallest even number greater than $d/2$. Then

$$N(\lambda) = N_0(\lambda)(1 + o(1)), \qquad \lambda \to \infty, \tag{10}$$

where $N_0(\lambda)$ is the IDS for $-\Delta$ and is given by (7). This is an analogue of the Weyl asymptotic (2) containing the leading term only. The naive perturbation theory yields for the subsequent term in the one-dimensional case

$$N(\lambda) = N_0(\lambda) - (4\pi\lambda)^{-1} \int_0^\infty B(x) \cos 2\sqrt{\lambda} x \, dx. \tag{11}$$

Here we assume that $\mathbb{E}\{q(x)\} = 0$ and set $\mathbb{E}\{q(x)q(0)\} = B(x)$. This formula can be justified in many interesting cases. But the proof of the multidimensional analogue of (11) is still absent.

However, the problem of its justification seems rather important, because it is the simplest nontrivial case of the semi-classical asymptotics for the random Schrödinger operator. In the last decade physicists have found a lot of beautiful results by using various versions of the perturbation theory and the semi-classical approximation. These results are known as the weak localization theory. Their mathematical meaning is completely unexplored.

(b) Low energies. This is an asymptotic region that has no analogues in the conventional spectral theory but turns out to be rather rich and interesting for random operators. Let me mention here several typical results.

According to the quantum mechanical ideology, the low energy part of the spectrum should depend strongly on the particular potential. Nevertheless there are several rather well defined types of asymptotic behaviour of the IDS at low energies.

The simplest one corresponds to the unbounded potential. Consider, for instance, a potential whose probability distribution $F(dq)$ is such that

$$\lim_{q \to -\infty} \frac{\log F(q)}{\log F(q+a)} = 1, \qquad \forall a \in \mathbb{R}. \tag{12}$$

Then for the operators (6) and (9) ,

$$\log N(\lambda) = \log F(\lambda)(1 + o(1)), \qquad \lambda \to -\infty. \tag{13}$$

In the discrete case, i.e. for (9), the proof of (13) is fairly simple and uses in fact elementary variational arguments. In the continuous case, i.e. for (6), the proof of (13) is more complicated, although it also can be carried out in the variational terms. It was given by myself for the Gaussian random potential and for the so-called Poisson potential

$$q(x) = \sum_j u(x - x_j) \tag{14}$$

where $u(x)$ is a nonpositive function with a compact support and $\{x_j\}$ are the Poisson random points in $\mathbb{R}^d$.

It is believed that the asymptotic relation (13) is valid in the continuous case for a rather wide class of random potentials satisfying some sufficiently weak continuity condition.

Another type of asymptotic behaviour takes place for a random potential bounded from below, in particular for (14) with nonnegative $u(x)$. Here the spectrum is $\mathbb{R}^+$ and

$$\log N(\lambda) = -c_d n \lambda^{-d/2}(1 + o(1)), \qquad \lambda \to 0_+, \tag{15}$$

where c_d depends only on d and n is the concentration of the Poisson points.

This asymptotic was suggested by I. Lifshitz and was proved by several authors including myself. The proof is based on the Wiener integral technique and an important ingredient of the proof are the deep results by Donsker and Varadhan on the large deviations for the Wiener process, or, in other words, on the infinite dimensional Laplace method.

It is natural to study the subsequent terms in the asymptotic formulae (13) and (15), in particular, the preexponential factor. This factor was found rigorously only in some one-dimensional cases.

Let me mention one more type of the asymptotic behaviour of the IDS. Consider the elliptic operator

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right), \tag{16}$$

where $a_{ij}(x)$ are random fields of the form (8). The spectrum of this operator is also $\mathbb{R}^+$ and we are interested in the behaviour of $N(\lambda)$ for $\lambda \to 0_+$. The change of the variables $\lambda = \varepsilon^2$, $x_i = \xi_i \varepsilon^{-1}$ and $\varepsilon^d N(\varepsilon^2) = \nu(\varepsilon)$ reduces our problem to the study of (16) in which $a_{ij}(x)$ are replaced by the fastly oscillating coefficients $a_{ij}(\xi/\varepsilon)$. This problem is the objective of the homogenization theory which has been developed in the last decades.

A rather general result which is suitable for the spectral theory has been proved by S. Kozlov. By using this result, one can prove that for the random operator (16) (cf. (7) and (10))

$$N(\lambda) = c(a)\lambda^{d/2}(1 + o(1)), \qquad \lambda \to 0_+,$$

where $c(a)$ is a rather complicated functional of a_{ij}'s. Its explicit form is known in the 1-dimensional case and some 2-dimensional examples.

3. RANDOM MATRICES

Let us consider the symmetric $n \times n$ random matrices $V^{(n)}$ with the entries

$$V_{ij}^{(n)} = n^{-1/2} W_{ij}, \qquad i, j = 1, \ldots, n \tag{17}$$

where W_{ij} are independent (except the symmetry condition $W_{ij} = W_{ji}$) random variables such that

$$\mathbb{E}\{W_{ij}\} = 0, \qquad \mathbb{E}\{W_{ij}W_{kl}\} = v^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{18}$$

The simplest and best known case of such W_{ij}'s corresponds to the Gaussian ones where their joint distribution can be written as

$$Z_n^{-1} \exp(-n \operatorname{Tr} W^2/4a^2) dW, \qquad dW = \prod_1^n dW_{ii} \prod_{i<j} dW_{ij}, \tag{19}$$

where Z_n is the normalization constant. This matrix ensemble was introduced by Wigner to describe some properties of the spectra of the energy levels of heavy nuclei.

There are many beautiful results for this matrix ensemble (see e.g. books [4,5] and references therein). In particular, if $\lambda_1, \ldots, \lambda_n$ are eigenvalues of the respective random matrix and $N(\lambda)$ is specified by (5), then with probability 1,

$$\lim_{n\to\infty} N_n(\lambda) \equiv N(\lambda) = \int_{-\infty}^{\lambda} \rho(\mu) d\mu, \tag{20}$$

where

$$\rho(\lambda) = \frac{1}{2\pi v^2} \begin{cases} \sqrt{4v^2 - \lambda^2}, & |\lambda| \le 2v \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

This is the well known Wigner or semicircle law. Wigner and several other authors have extended these results to more general cases. The optimal result proved by myself [6] is that if we replace in (20) convergence with probability 1 by convergence in probability, then (20) and (21) are valid under conditions that W_{ij}'s for $i < j$ are identically distributed and satisfy (18).

A somewhat more general form of this condition which is an analogue of the Lindeberg condition well known in the probability theory turns out to be necessary and sufficient [5,6]. Thus, the semicircle law (20) and (21) is as universal a form of the limiting eigenvalue distribution of random matrices with independent entries as the normal distribution is universal for the normalized sums of independent variables.

The latter results were obtained by constructing certain recursion relations in n for $n^{-1}\,\mathrm{Tr}(V^{(n)} - z)^{-1}$, $\mathrm{Im}\, z \neq 0$ and by studying their asymptotic behaviour for $n \to \infty$. This technique allow us to consider the more general matrix ensemble

$$H_{ij}^{(n)} = h_i\delta_{ij} + n^{-1/2}W_{ij}, \qquad i,j = 1,\dots,n \tag{22}$$

with the same W_{ij}'s and h_i's admitting the existence of the limit

$$\lim_{n\to\infty} \#\{h_i \le \lambda\}n^{-1} \equiv N_0(\lambda) \tag{23}$$

for each λ which is a continuity point of $N_0(\lambda)$. This limit may be called the unperturbed integrated density of states. Consider the Stieltjes transform

$$r_o(z) = \int(\mu - z)^{-1}N_0(d\mu), \qquad \mathrm{Im}\, z \neq 0$$

of $N_0(d\mu)$. According to [6], if

$$r_n(z) = \int(\mu - z)^{-1}N_n(d\mu),$$

then the limit $r(z) = \lim_{n\to\infty} r_n(z)$ exists in probability for all $\mathrm{Im}\, z \neq 0$ and can be found as a unique solution of the functional equation

$$r(z) = r_o(z + v^2 r(z)) \tag{24}$$

in the class of functions, which are analytic for all $\operatorname{Im} z \neq 0$ and such that $\operatorname{Im} r(z) \cdot \operatorname{Im} z > 0$, $\operatorname{Im} z \neq 0$.

The IDS that corresponds to (24) is known as the deformed semicircle law. By using Eq. (24) one can prove the existence of the density of states, its boundedness, the location of the support of $\rho(\lambda)$, its asymptotic behaviour near end points of the support etc.

It is noteworthy that the above results for the ensemble (22) can be regarded as a limiting case of more general result obtained by Marchenko and myself (see [6]). Indeed, consider the following ensemble of $n \times n$ matrices

$$H^{(n)} = h^{(n)} + \sum_{i=1}^{m} \tau_i(\cdot\,, q_i) q_i\,, \tag{25}$$

where $h^{(n)}$ is an arbitrary matrix having the limiting IDS $N_0(d\lambda)$, τ_i and q_i are independent identically distributed random variables and unit vectors in $\mathbb{R}^n$ respectively and the distribution of the latter is "close enough" to the uniform distribution. Then, if $n \to \infty$, $m \to \infty$, $m/n \to c < \infty$, there exists the limiting IDS $N(d\lambda)$ of the ensemble (25) and the Stieltjes transforms $r(z)$ and $r_o(z)$ of $N(d\lambda)$ and $N_0(d\lambda)$ are related as

$$r(z) = r_o\left(z + c\int_{-\infty}^{\infty} \tau(1 + \tau r(z))^{-1}\sigma(d\tau)\right)\,, \tag{26}$$

where $\sigma(d\tau)$ is the distribution of τ_i. It is easy to see that if we redenote in (26) $z + c\mathbb{E}\{\tau\}$ by z and after that perform the limiting transition $c \to \infty$, $\mathbb{E}\{\tau^2\} \to 0$, $c\mathbb{E}\{\tau^2\} \to v^2$, then we arrive at (24).

One can regard (24) and (26) as a complete solution of the eigenvalue problem for the ensembles (22), (25) and (27).

As I have mentioned above, the initial physical motivation for these problems was provided by nuclear physics. In the last decades the random matrix ensembles have been used in many other branches of quantum physics. In particular, the problem of the study of the quantum kicked rotator that is an archetype model in quantum chaology is related to the study of the eigenvalues distribution of random matrices

$$V_{ij}^{(n,b)} = (2b_n + 1)^{-1/2}\phi((i-j)/b_n)W_{ij}\,. \tag{27}$$

Here $b_n \to \infty$ as $n \to \infty$ and $\phi(t) = \phi(-t) \in \mathbb{R}$ is a piecewise continuous and bounded function with a compact support and

$$\int \phi^2(t)dt = 1. \tag{28}$$

In particular, if $\phi(t) \equiv 1$ and $2b_n + 1 = n$, we obtain (17) and if $\phi(t)$ is the indicator of the interval $(-1, 1)$, then (27), (28) defines the so-called band matrices that have nonzero i.i.d. entries only inside the "band" of a width $2b_n + 1$ around the principal diagonal of $V^{(n,b)}$.

The IDS of this matrix ensemble have been found recently by several groups of authors [8-10]. The most general result was proved in [10]. It asserts that the IDS of (27) is given by the semicircle law (20), (21) if $\lim_{n\to\infty} b_n n^{-1}$ equals 0 or 1/2. If, however, $0 < \lim_{n\to\infty} b_n n^{-1} < 1/2$, then the limiting eigenvalue distribution exists but it is not the semicircle law. Its Stieltjes transform can be found as a unique solution of some non-linear integral equation. The situation can be "corrected" in a sense that the semicircle law can still be obtained if we consider a certain periodic function $\phi(t)$ in (27).

The proof of these results is based on a certain new approach. It consists in deriving an infinite system of linear equations for the moments of the diagonal matrix elements of the resolvent $(V^{(n.b)} - z)^{-1}$ and in asymptotic solution of this system for b_n, $n \to \infty$. This method proves to be rather efficient and general and allows us to consider a wide variety of related problems (see review [11]). In particular it can be proved that the deformed semicircle law (24) is valid for any not necessarily diagonal "unperturbed" matrix $h^{(n)} = \{h_{ij}\}_{i,j=1}^n$ in (22) for which the IDS exists, i.e. if $\mu_1, \ldots, \mu_n$ are the eigenvalues of $h^{(n)}$, then we only need to assume that the limit (5) exists.

In the next section we consider other problems that can be solved by the same approach.

I have discussed matrix ensembles that share with the Gaussian ensemble (19) the property of statistical independence of all functionally independent entries. All these ensembles have the same IDS described by the semicircle law (21). There is another generalization of (19) which is defined by the probability distribution of the form

$$Z_n^{-1}\exp(-n\,\mathrm{Tr}\,V(W))dW\,, \tag{29}$$

where $V(t)$ is a real-valued function that grows at infinity faster than $a\log|t|$ $\forall\ a>0$. The Gaussian ensemble (19) corresponds obviously to $V(t)=t^2/4a^2$. Polynomials of an even degree $p>2$ appear in quantum field theory, statistical mechanics of random surfaces, combinatorics, etc. (see [7] and references therein).

The IDS of these ensembles differs from the semicircle law. Generically its support consists of $p/2$ intervals with square root zeros of the density of states at each endpoint. However, by varying $V(t)$ (e.g. the coefficients in a polynomial $V(t)$), one can obtain a variety of degenerated cases with smaller number of intervals and other behaviour of $\rho(\lambda)$ at their end points. For instance, if $V(t)=|t|^\alpha$, $\alpha>1$, then

$$\rho(\lambda)=v_\alpha(\lambda B_\alpha^{-1})B_\alpha^{-1}\,, \tag{30}$$

where

$$v_\alpha(t)=v_\alpha(-t)=\frac{2\alpha}{\pi}\begin{cases}\displaystyle\int_{|t|}^1\tau^{\alpha-1}(\tau^2-t^2)^{-1/2}d\tau, & |t|\le 1\\ 0, & |t|>1\end{cases}$$

$$B_\alpha=\left(2\alpha\pi^{-1}\int_0^1 t^\alpha(1-t^2)^{-1/2}dt\right)^{1/\alpha}. \tag{31}$$

There is a beautiful approach to the study of the ensembles (29). It is based on the possibility of expressing the expectation of any unitary invariant of a random matrix belonging to the ensemble (29) via polynomials orthogonal on $\mathbb{R}$ with respect to the weight $\exp(-nV(\lambda))$ [4,7]. Physicists have found a lot of very interesting properties of the eigenvalue distribution of the ensemble (29) by using this approach. However, rigorous proofs are not numerous here (see however paper [15] in which the rigorous derivation of the form of the density of states for some class of even polynomials in (29) is given). One of the reasons is that rigorous study should be based on precise asymptotic formulae for the respective orthogonal polynomials. The best recent result is for the case of $V(t) = |t|^\alpha$ only [13]. The corresponding asymptotic formulae allow us to obtain rigorously the form (30), (31) of the density of states of the ensemble (29) with the same $V(t)$ [14]. However, these formulae are not precise enough to calculate other quite important characteristics of random matrices. Besides, almost nothing is known rigorously in case of a nonmonomial $V(t)$, especially for a nonconvex one, e.g. for $V(t) = at^2 + bt^4$ with $a < 0$ and $b > 0$. The corresponding sufficiently precise asymptotic formulae for the orthogonal polynomials, which should be in many respect analogous to semiclassical formulae of quantum mechanics, would be of great use and importance in spectral theory.

In the last three years a new wave of activity in this field has been initiated by the progress in the study of models of two-dimensional gravity and the string theory [12]. This progress being translated into the random matrix theory language consists in establishment of the form of the behaviour of the mean prelimit density of states $\rho_n(\lambda) = \mathbb{E}\{n^{-1} \sum_1^n \delta(\lambda - \lambda_i)\}$ of the ensemble (29) with the special polynomials $V(\lambda)$ for $n \to \infty$ and λ tending simultaneously to an end point of the support of the limiting density of states $\rho(\lambda) = \lim_{n\to\infty} \rho_n(\lambda)$. These studies revealed many beautiful connections of the random matrix theory with the theory of integrable systems, the spectral theory of Jacobi matrices, the theory of orthogonal polynomials and have raised a lot of problems that are of great interest for both mathematical and theoretical physics.

4. "INTERPOLATING" FAMILIES OF RANDOM OPERATORS

The two classes of random operators that were considered in preceding sections have different origin and have been studied independently and by rather different techniques. Therefore it seems rather natural to look for interconnections between these classes. In the present section we are going to consider three families of random operators belonging to the first class. However, these families depend on certain parameters (we call them the interaction radius, the dimensionality of space and the number of components, respectively) in such a way that the limits of the IDS of the corresponding random operators for infinite values of these parameters coincide with the deformed semicircle law which is typical for the second class.

Our first family consists of the operators $H^{(R)} = h + V^{(R)}$ acting in $l^2(\mathbb{Z}^d)$ and defined by the matrices

$$H^{(R)}(x,y) = h(x-y) + R^{-d/2}\phi((x-y)R^{-1})W(x,y), \quad x,\, y \in \mathbb{Z}^d, \tag{32}$$

where $h(-x) = \overline{h(x)}$, $h(x) \in l^1(\mathbb{Z}^d)$, $R < \infty$, $\phi(t)$, $t \in \mathbb{R}^d$, is a piecewise continuous function with compact support and $W(x,y)$ are independent (except the symmetry condition $W(x,y) = W(y,x)$) and identically distributed random variables such that (cf. (18))

$$\mathbb{E}\{W(x,y)\} = 0\,, \qquad \mathbb{E}\{W(x,y)W(s,t)\} = v^2(\delta_{xs}\delta_{yt} + \delta_{xt}\delta_{ys})\,. \tag{33}$$

The matrix (32) defines the d-dimensional finite difference operator. In particular, if $h(x) = 0$, $|x| \neq 1$, $R = 1$ and $\phi(t) = 0$ for $|t| > 1$, then this operator is of the second order. However, it differs from the discrete Schrödinger operator (6) in that it has random off-diagonal entries.

The random operator $H^{(R)}$, defined by (32), is a metrically transitive operator in the terminology of book [3]. In particular, it admits the nonrandom IDS $N^{(R)}(\lambda)$ defined by (3) and (4) in which $H_\Lambda^{(R)}$ is a restriction of $H^{(R)}$ to a finite "box" $\Lambda \in \mathbb{Z}^d$. We have proved recently [16] that there exists the limit $\lim\limits_{R\to\infty} N^{(R)}(\lambda)$ and that its Stieltjes transform satisfies equation (24), where $r_o(z)$ is the same transform of the IDS of the Toeplitz operator $h(x-y)$ and can be easily expressed via the symbol of this operator [1,3]. Thus, $\lim\limits_{R\to\infty} N^{(R)}(\lambda)$ coincides with the deformed semicircle law. In particular, if the nonrandom part of (32) is absent (i.e. $h(x) \equiv 0$), then this limit coincides with the semicircle law (20), (21).

Our second family $H^{(d)}$ of random operators acts also in $l^2(\mathbb{Z}^d)$ and is defined by the random matrix

$$H^{(d)}(x,y) = h_d(x-y) + d^{-1/2}\delta(|x-y|-1)W(x,y)\,, \tag{34}$$

where

$$h_d(x) = d^{-1/2}\sum_{i=1}^{d} h_1(x_i - y_i)\prod_{k\neq i}\delta(x_k - y_k)\,, \tag{35}$$

$h_1(x) \in l_1(\mathbb{Z})$ and the random variables $W(x,y)$ are the same as in (33). This operator also belongs to the class of metrically transitive operators, in particular it admits the IDS $N^{(d)}(\lambda)$. Now, sending d to infinity, we again obtain the limiting eigenvalue distribution coinciding with the deformed semicircle law. The form (35) of the unperturbed part of (34) is more special than that in (32). This special form is needed to guarantee the existence of the $d = \infty$ limit for the unperturbed IDS which in this case turns out to be of the Gaussian form.

The third family of operators $H^{(n)}$ acts in $l^2(\mathbb{Z}^d) \otimes \mathbb{C}_n$ and is defined by the random "block" matrix

$$H^{(n)}(\alpha, x; \beta, y) = h(x-y)\delta_{\alpha\beta} + n^{-1/2}\delta(x-y)W_{\alpha\beta}(x)\,, \tag{36}$$

where α, $\beta = 1, \dots, n$, $x, y \in \mathbb{Z}^d$, $h(x)$ is the same as in (32) and $W_{\alpha\beta}(x)$ are independent (except the symmetry condition $W_{\alpha\beta}(x) = W_{\beta\alpha}(x)$) identically distributed random variables such that (cf. (18) and (32))

$$\mathbb{E}\{W_{\alpha\beta}(x)\} = 0\,, \qquad \mathbb{E}\{W_{\alpha\beta}(x)W_{\gamma\delta}(y)\} = v^2\delta(x-y)(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})\,.$$

The IDS $N^{(n)}(\lambda)$ of $H^{(n)}$ is defined as the limit of $(n|\Lambda|)^{-1}\mathcal{N}_{\Lambda,n}(\lambda)$, where $\mathcal{N}_{\Lambda,n}(\lambda)$ is the counting function of the eigenvalues of the restriction $H_\Lambda^{(n)}$ of (36) to the finite domain $\Lambda \in \mathbb{Z}^d$.

Here we send to infinity the order n of the blocks. The resulting IDS is again the deformed semicircle law.

Operators $H^{(n)}$ were introduced by Wegner [17], who obtained the latter result at the theoretical physics level of rigour by using the perturbation theory with respect to the random part of (36).

Thus we see once again that the deformed semicircle law is a rather general and universal form of the limiting eigenvalue distribution.

The results formulated in this section were proved [16] by using the general approach that was briefly outlined in the previous section. The main technical result here is the statement that in the limit R, d, $n \to \infty$ the moments $\mathbb{E}\{\prod_{i=1}^{l} G(x_i, y_i; z_i)\}$ take the form $\prod_{i=1}^{l} \mathbb{E}\{G(x_i, y_i; z_i)\}$ for any fixed $l \geq 1$, x_i, $y_i \in \mathbb{Z}^d$ and $|\operatorname{Im} z_i| \geq \eta$, $i = 1, \dots, l$ with sufficiently large η. Here $G(x,y;z) =$

$(H^{(a)} - z)^{-1}(x, y)$ for $a = R$, d and $G(x, y; z) = n^{-1} \sum_{\alpha=1}^{n} (H^{(n)} - z)^{-1}(\alpha, x; \alpha, y)$ for $a = n$.

We prove this result by constructing a certain infinite system of equations for this family of moments and by proving that this system admits a unique solution that can be written in the factorized form in the limits R, d, $n = \infty$. The function $\mathcal{G}(x - y; z) = \lim_{R,d,n\to\infty} \mathbb{E}\{G(x, y; z)\}$ that determinates this form can be found as a unique solution of a certain nonlinear integral equation. Its solubility condition yields (24).

The reader familiar with statistical mechanics will find a close analogy of the results described above with the limit of the infinite interaction radius or the limit of the infinite dimensionality of space which are widely used in statistical mechanics giving a rigorous form of the mean field approximation. Thus, our limiting results can be regarded as analogues of the respective results in statistical mechanics.

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